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THE BOUNDEDNESS OF CERTAIN SUBLINEAR OPERATOR IN THE WEIGHTED VARIABLE LEBESGUE SPACES

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Abstract. The main purpose of this paper is to prove the boundedness of the multidimensional Hardy type operator in weighted Lebesgue spaces with a variable exponent. As an application we prove the boundedness of certain sublinear operators on the weighted variable Lebesgue space.

Keywords: variable Lebesgue space, weights, Hardy operator, boundedness

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Introduction

It is well known that the variable exponent Lebesgue space appeared in literature for the first time already in 1931 by Orlicz [31]. In [31] the Hölder inequality for variable exponent discrete Lebesgue space was proved. Orlicz also considered the variable exponent Lebesgue space on the real line, and proved the Hölder inequality in this setting.

However, after this one paper, Orlicz abandoned the study of variable exponent Lebesgue spaces, to concentrate on the theory of the Orlicz spaces (see also [27]). Further development of this theory was connected with the theory of modular function spaces. The first systematic study of modular spaces is due to Nakano [28]. In the appendix, Nakano mentioned explicitly variable exponent Lebesgue spaces as an example of the more general spaces he considered. Somewhat later, a more explicit version of these spaces, namely modular function spaces, were investigated by many mathematicians (see Musielak [26]).

The next step in the investigation of variable exponent spaces were the papers by Sharapudinov [35] and Kováčik and Rákosník [20]. The study of these spaces has

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been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions(see [2], [32], [39]).

In this paper sufficient conditions for pairs of weights ensuring the validity of two-weight inequalities of a strong type for the multidimensional Hardy type operator are found. We also investigate the corresponding problem for the dual operator. We also give applications to the boundedness of certain sublinear operators on the weighted variable Lebesgue space.

1. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x=(x_1,\dots,x_n)$ and let Ω be a measurable subset in $\mathbb{R}^n$ and $|x|=\Bigl(\sum\limits_{i=1}^n x_i^2\Bigr)^{1/2}$. Suppose that p is a measurable function on Ω such that $1\leqslant p(x)<\infty$, $\underline{p}=\operatorname*{ess\,inf}\limits_{x\in\Omega} p(x)$, $\overline{p}=\operatorname*{ess\,sup}\limits_{x\in\Omega} p(x)$, and ω is a weight function on Ω, i.e. ω is a non-negative, almost everywhere (a.e.) positive function on Ω. For the sake of simplicity, the letter C always denotes a positive constant which may change from one step to the next.

Definition 1. By $L_{p(x),\omega}(\Omega)$ we denote the space of measurable functions f on Ω such that

$$\int_\Omega |f(x)\omega(x)|^{p(x)}\,\mathrm{d}x<\infty.$$

Under the condition $1\leqslant p(x)\leqslant\overline{p}<+\infty$, the space $L_{p(x),\omega}(\mathbb{R}^n)$ is a Banach space (see [17]) with respect to the norm

$$\|f\|_{L_{p(x),\,\omega}(\mathbb{R}^n)}=\|f\|_{L_{p(\cdot),\,\omega}(\mathbb{R}^n)}=\inf\Bigl\{\lambda>0\colon\ \int_{\mathbb{R}^n}\Bigl|\frac{f(x)\,\omega(x)}{\lambda}\Bigr|^{p(x)}\,\mathrm{d}x\leqslant 1\Bigr\}.$$

For $\omega=1$ the space $L_{p(x),\omega}(\Omega)$ coincides with the variable Lebesgue space $L_{p(x)}(\Omega)$. The following theorems are known.

Theorem 1 (see [3]). *Let $1\leqslant\underline{p}\leqslant p(x)\leqslant q(y)\leqslant\overline{q}<\infty$ for all $x\in\Omega_1\subset\mathbb{R}^n$ and $y\in\Omega_2\subset\mathbb{R}^m$. If $p(x)\in C(\Omega_1)$, then the inequality*

$$\bigl\|\|f\|_{p,\Omega_1}\bigr\|_{q,\Omega_2}\leqslant\Bigl(\frac{\overline{p}}{\underline{q}}+\frac{\overline{q}-\underline{p}}{\overline{q}}\Bigr)^{1/\underline{p}}\bigl\|\|f\|_{q,\Omega_2}\bigr\|_{p,\Omega_1}$$

is valid, where $\underline{q}=\operatorname*{ess\,inf}\limits_{\Omega_2} q(x)$, $\overline{q}=\operatorname*{ess\,sup}\limits_{\Omega_2} q(x)$ and $C(\Omega_1)$ is the space of continuous functions in Ω_1 and $f\colon\ \Omega_1\times\Omega_2\to\mathbb{R}$ is any measurable function such that

$$\bigl\|\|f\|_{q,\Omega_2}\bigr\|_{p,\Omega_1}=\inf\Bigl\{\mu>0\colon\ \int_{\Omega_1}\Bigl(\frac{\|f(x,\cdot)\|_{q(\cdot),\Omega_2}}{\mu}\Bigr)^{p(x)}\,\mathrm{d}x\leqslant 1\Bigr\}<\infty.$$

Remark 1. Note that in the case $p(x) = 1$, Theorem 1 is the analog of the generalized Minkowski type inequality and was proved in [34].

Remark 2. Let v be a weight function. Analogously replacing the function f by the function fv in Theorem 1, one can have the weighted analog of Theorem 1.

In [6] the following theorem was proved.

Theorem 2. *Let $p(x)$ and $q(x)$ be bounded exponents on Ω. Then*

$$L_{q(x)}(\Omega) \hookrightarrow L_{p(x)}(\Omega)$$

if and only if $p(x) \leqslant q(x)$ for a.e. $x \in \Omega$ and there exists $\delta \in [0,1)$ such that

$$\int_\Omega \delta^{\frac{p(x)\,q(x)}{q(x)-p(x)}}\,\mathrm{d}x < \infty,$$

where $\delta^{\frac{p(x)\,q(x)}{q(x)-p(x)}} = 0$ for $p(x) = q(x)$.

Remark 3. Note that in the case $n = 1$, $x \in [0,1]$ Theorem 2 was proved in [36]. For $n > 1$ and when the Lebesgue measure of the set Ω is finite, Theorem 2 was proved in [20] (see also [33]).

2. Main results

Now we prove boundedness of the multidimensional Hardy type operator in weighted variable Lebesgue space.

Theorem 3. *Let $1 < \underline{p} \leqslant p(x) \leqslant \overline{p} < \infty$, and $p(x)$ be a function measurable on $\mathbb{R}^n$ and satisfying the condition $\int_\Omega \delta^{\frac{p(x)\,q(x)}{q(x)-p(x)}}\,\mathrm{d}x < \infty$, where $\delta \in [0,1)$. Suppose that $v(x)$ and $w(x)$ are weights on $\mathbb{R}^n$ and there exists a constant $\alpha < 1$ such that*

$$A = \sup_{t>0} \left(\int_{|y|<t} [v(y)]^{-\overline{p}'}\,\mathrm{d}y \right)^{\frac{\alpha}{\overline{p}'}} \left\| w(\cdot) \left(\int_{|y|<|\cdot|} [v(y)]^{-\overline{p}'}\,\mathrm{d}y \right)^{\frac{1-\alpha}{\overline{p}'}} \right\|_{L_{p(\cdot)}(|x|>t)} < \infty. \tag{1}$$

Then for every $f \in L_{p(x),v(x)}(\mathbb{R}^n)$ the inequality

$$\left\| \int_{|y|<|\cdot|} f(y)\,\mathrm{d}y \right\|_{L_{p(\cdot),w}(\mathbb{R}^n)} \leqslant C \, \|f\|_{L_{p(\cdot),v}(\mathbb{R}^n)}$$

is valid with $C = (1 + (\overline{p} - \underline{p})/\overline{p})^{1/\underline{p}} (1-\alpha)^{-1/\overline{p}'}\, C_1\, A$.

P r o o f. Let

$$h(y)=\Big(\int_{|z|<|y|}[v(z)]^{-\overline{p}'}\,\mathrm{d}y\Big)^{\alpha/\overline{p}'}=\Big(\int_0^{|y|}s^{n-1}\Big(\int_{|\xi|=1}[v(s\xi)]^{-\overline{p}'}\,\mathrm{d}\sigma(\xi)\Big)\,\mathrm{d}s\Big)^{\alpha/\overline{p}'}.$$

It is obvious that $h(y)$ is radial function.

Applying Hölder's inequality for $L_{\underline{p}}(\mathbb{R}^n)$ spaces (with conjugate exponent $\underline{p}'=\underline{p}/(\underline{p}-1)=\overline{p}')$, we have

$$\begin{aligned}\Big\|\int_{|y|<|x|}f(y)\,\mathrm{d}y\Big\|_{L_{p(x),w}(\mathbb{R}^n)}&=\Big\|\int_{|y|<|x|}[f(y)h(y)v(y)][h(y)v(y)]^{-1}\Big\|_{L_{p(x),w}(\mathbb{R}^n)}\\&\leqslant\big\|\|f\,h\,v\|_{L_{\underline{p}}(|y|<|x|)}\|[h\,v]^{-1}\|_{L_{\overline{p}'}(|y|<|x|)}\big\|_{L_{p(x),w}(\mathbb{R}^n)}.\end{aligned}$$

By Remark 2, we obtain

$$\begin{aligned}&\big\|\|fhv\|_{L_{\underline{p}}(|y|<|x|)}\|[hv]^{-1}\|_{L_{\overline{p}'}(|y|<|x|)}\big\|_{L_{p(x),w}(\mathbb{R}^n)}\\&\quad=\big\|\big\|\|fhv\chi_{\{|y|<|x|\}}(x)\|[hv]^{-1}\|_{L_{\overline{p}'}(|y|<|x|)}\big\|_{L_{\underline{p}}(\mathbb{R}^n)}\big\|_{L_{p(x),w}(\mathbb{R}^n)}\\&\quad\leqslant\Big(1+\frac{\overline{p}-\underline{p}}{\overline{p}}\Big)^{1/\underline{p}}\big\|\big\|\|fhv\chi_{\{|y|<|x|\}}(x)\|[hv]^{-1}\|_{L_{\overline{p}'}(|y|<|x|)}\big\|_{L_{p(x),w}(\mathbb{R}^n)}\big\|_{L_{\underline{p}}(\mathbb{R}^n)}\\&\quad=\Big(1+\frac{\overline{p}-\underline{p}}{\overline{p}}\Big)^{1/\underline{p}}\big\|fhv\big\|\chi_{\{|y|<|x|\}}(x)\,\|[h\,v]^{-1}\|_{L_{\overline{p}'}(|y|<|x|)}\big\|_{L_{p(x),w}(\mathbb{R}^n)}\big\|_{L_{\underline{p}}(\mathbb{R}^n)}\\&\quad=\Big(1+\frac{\overline{p}-\underline{p}}{\overline{p}}\Big)^{1/\underline{p}}\big\|fhv\big\|\|[hv]^{-1}\|_{L_{\overline{p}'}(|y|<|x|)}\big\|_{L_{p(x),w}(|x|>|y|)}\big\|_{L_{\underline{p}}(\mathbb{R}^n)}.\end{aligned}$$

We have

$$\begin{aligned}&\|[h\,v]^{-1}\|_{L_{\overline{p}'}(|y|<|x|)}=\Big(\int_{|y|<|x|}[h(|y|)\,v(y)]^{-\overline{p}'}\,\mathrm{d}y\Big)^{1/\overline{p}'}\\&\quad=\Big(\int_0^{|x|}r^{n-1}\Big[h(r)\int_{|\eta|=1}[v(r\eta)]^{-\overline{p}'}\,\mathrm{d}\sigma(\eta)\Big]\,\mathrm{d}r\Big)^{1/\overline{p}'}\\&\quad=\Big(\int_0^{|x|}\Big[\int_0^{r}s^{n-1}\Big(\int_{|\xi|=1}[v(s\xi)]^{-\overline{p}'}\,\mathrm{d}\sigma(\xi)\Big)\,\mathrm{d}s\Big]^{-\alpha}\\&\qquad\times\Big(\int_{|\eta|=1}[v(r\eta))^{-\overline{p}'}\,\mathrm{d}\sigma(\eta)\Big)r^{n-1}\,\mathrm{d}r\Big)^{1/\overline{p}'}\\&\quad=\Big(\frac{1}{1-\alpha}\Big)^{1/\overline{p}'}\Big(\int_0^{|x|}\frac{\mathrm{d}}{\mathrm{d}r}\Big[\Big(\int_0^{r}s^{n-1}\Big(\int_{|\xi|=1}[v(s\xi)]^{-\overline{p}'}\,\mathrm{d}\sigma(\xi)\Big)\,\mathrm{d}s\Big)^{1-\alpha}\Big]\,\mathrm{d}r\Big)^{1/\overline{p}'}\\&\quad=\Big(\frac{1}{1-\alpha}\Big)^{1/\overline{p}'}\Big(\int_0^{|x|}s^{n-1}\Big(\int_{|\xi|=1}[v(s\xi)]^{-\overline{p}'}\,\mathrm{d}\sigma(\xi)\Big)\,\mathrm{d}s\Big)^{(1-\alpha)/\overline{p}'}\\&\quad=\Big(\frac{1}{1-\alpha}\Big)^{1/\overline{p}'}\Big(\int_{|y|<|x|}[v(y)]^{-\overline{p}'}\,\mathrm{d}y\Big)^{(1-\alpha)/\overline{p}'}=\Big(\frac{1}{1-\alpha}\Big)^{1/\overline{p}'}[h(x)]^{(1-\alpha)/\alpha}.\end{aligned}$$

Therefore by the condition (1), we obtain

$$\begin{aligned}
&\Big\| fhv \Big\| \|[hv]^{-1}\|_{L_{\overline{p}'}(|y|<|x|)} \Big\|_{L_{p(x),w}(|x|>|y|)} \Big\|_{L_{\underline{p}}(\mathbb{R}^n)} \\
&= \Big(\frac{1}{1-\alpha}\Big)^{1/\overline{p}'} \Big\| fhv \| [h(\cdot)]^{(1-\alpha)/\alpha} \|_{L_{p(\cdot),w}(|\cdot|>|y|)} \Big\|_{L_{\underline{p}}(\mathbb{R}^n)} \\
&\leqslant \Big(\frac{1}{1-\alpha}\Big)^{1/\overline{p}'} A \, \|fv\|_{L_{\underline{p}}(\mathbb{R}^n)}.
\end{aligned}$$

Taking $\Omega = \mathbb{R}^n$ and applying Theorem 2, we obtain

$$\|fv\|_{L_{\underline{p}}(\mathbb{R}^n)} \leqslant C_1 \, \|fv\|_{L_{p(x)}(\mathbb{R}^n)}.$$

Finally, we have

$$\Big\| \int_{|y|<|x|} f(y)\,\mathrm{d}y \Big\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant \Big(1 + \frac{\overline{p} - \underline{p}}{\overline{p}}\Big)^{1/\underline{p}} \Big(\frac{1}{1-\alpha}\Big)^{1/\overline{p}'} A \, C_1 \, \|f\|_{L_{p(x),v}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 1. $\square$

Corollary. *Let* $p(x) = p = \mathrm{const}$ *and let*

$$l = \sup_{t>0} \Big(\int_{|y|<t} [v(y)]^{-p'} \,\mathrm{d}y \Big)^{1/p'} \Big(\int_{|y|>t} [w(y)]^{p} \,\mathrm{d}y \Big)^{1/p} < \infty.$$

Then there exists a constant $\alpha \in (0,1)$ *such that the inequalities*

$$l \leqslant A \leqslant C l^{\alpha}$$

holds.

The theorem below is proved analogously.

Theorem 4. *Let* $1 < \underline{p} \leqslant p(x) \leqslant \overline{p} < \infty$, *and* $p(x)$ *be a function measurable on* $\mathbb{R}^n$ *and satisfying the condition* $\int_\Omega \delta^{\frac{p(x)\,q(x)}{q(x)-p(x)}} \,\mathrm{d}x < \infty$, *where* $\delta \in [0,1)$. *Suppose that* $v(x)$ *and* $w(x)$ *are weights on* $\mathbb{R}^n$ *and there exists a constant* $\beta < 1$ *such that*
(1)

$$B = \sup_{t>0} \Big(\int_{|y|>t} [v(y)]^{-\overline{p}'} \,\mathrm{d}y \Big)^{\frac{\beta}{\overline{p}'}} \Big\| w(\cdot) \Big(\int_{|y|<|\cdot|} [v(y)]^{-\overline{p}'} \,\mathrm{d}y \Big)^{\frac{1-\beta}{\overline{p}'}} \Big\|_{L_{p(\cdot)}(|x|<t)} < \infty.$$

Then for every $f \in L_{p(x),v(x)}(\mathbb{R}^n)$ *the inequality*

$$\Big\| \int_{|y|<|\cdot|} f(y)\,\mathrm{d}y \Big\|_{L_{p(\cdot),w}(\mathbb{R}^n)} \leqslant C \, \|f\|_{L_{p(\cdot),v}(\mathbb{R}^n)}$$

is valid with $C = (1 + (\overline{p} - \underline{p})/\overline{p})^{1/\underline{p}} (1-\beta)^{-1/\overline{p}'} \, C_1 \, B$.

Example. Let $n=1$. We denote by $C^1(\mathbb{R})$ the space of continuously differentiable functions on $\mathbb{R}$. Let $p(x)\in C^1(\mathbb{R})$ and $w(x)=|x|^{(1-\alpha)/\overline{p}'-\alpha/\overline{p}'\cdot q(x_t)/q(t)}\cdot A^{q(x_t)/q(t)}\cdot |(\mathrm{d}q(x_t)/\mathrm{d}t)\ln At^{-\alpha/\overline{p}'}-\alpha/\overline{p}'\cdot q(x_t)/t|^{1/p(t)}$, $v(x)=1$, where $x_t\in(0,\infty)$. Then the pair (v,w) satisfies the condition (2) of Theorem 3.

Remark 4. Note that Theorem 3 and Theorem 4 in the case $n=1$, $1<\underline{p}\leqslant p(x)\leqslant\overline{p}<\infty$ for $x\in[0,1]$ and $p(x)=p=\mathrm{const}$ for $x>1$ were proved in [9] while for multidimensional Hardy type operators they were proved in [11]. Also, Theorem 3 in the case $n=1$, $1<\underline{p}\leqslant p(x)\leqslant\overline{p}<\infty$, for $x\in(0,\infty)$ and $\int_1^\infty[p(x)-\underline{p}]\,\mathrm{d}x<\infty$ was proved in [1]. In particular, for the multidimensional Hardy operator the $L_{p(x}$-boundedness was proved in [5]. Further development in the direction of the boundedness of the Hardy operator was given in the papers [15], [7] and [23]. A two-weighted criterion for the Hardy operator at $x\in[0,1]$ was proved in [19]. In the case $p(x)=p=\mathrm{const}$ at $x\in(0,\infty)$, various variants of Theorem 3 for classical Lebesgue spaces were proved in [4], [10], [21], [24], [30], etc.

Sufficient conditions for general weights ensuring the validity of the two-weight strong type inequalities for some sublinear operator are given in the following theorem.

Theorem 5. *Let $1<\underline{p}\leqslant p(x)\leqslant\overline{p}<\infty$ for $x\in\mathbb{R}^n$ and $\int_{\mathbb{R}^n}\delta^{\frac{p(x)\,q(x)}{q(x)-p(x)}}\,\mathrm{d}x<\infty$, where $\delta\in[0,1)$, and let T be a sublinear operator acting boundedly from $L_{p(x)}(\mathbb{R}^n)$ to $L_{p(x)}(\mathbb{R}^n)$ such that, for any $f\in L_1(\mathbb{R}^n)$ with compact support and $x\notin\operatorname{supp}f$,*

$$|Tf(x)|\leqslant C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y, \tag{3}$$

where $C>0$ is independent of f and x.

Moreover, let $v(x)$ and $w(x)$ are weight functions on $\mathbb{R}^n$ satisfying the conditions

$$A_1=\sup_{t>0}\left(\int_{|y|<t}[v(y)]^{-\overline{p}'}\,\mathrm{d}y\right)^{\frac{\alpha}{\overline{p}'}}\left\|\frac{w(\cdot)}{|\cdot|^n}\left(\int_{|y|<|\cdot|}[v(y)]^{-\overline{p}'}\,\mathrm{d}y\right)^{\frac{1-\alpha}{\overline{p}'}}\right\|_{L_{p(\cdot)}(|\cdot|>t)}<\infty, \tag{4}$$

$$B=\sup_{t>0}\left(\int_{|y|>t}[v(y)|y|^n]^{-\overline{p}'}\,\mathrm{d}y\right)^{\frac{\beta}{\overline{p}'}}\left\|w(\cdot)\left(\int_{|y|>|\cdot|}[v(y)|y|^n]^{-\overline{p}'}\,\mathrm{d}y\right)^{\frac{1-\beta}{\overline{p}'}}\right\|_{L_{p(\cdot)}(|\cdot|<t)}<\infty. \tag{5}$$

There exists $M>0$ such that

$$\sup_{\frac14|x|<|y|\leqslant4|x|}w(y)\leqslant M\operatorname*{ess\,inf}_{x\in\mathbb{R}^n}v(x). \tag{6}$$

Then there exists a positive constant C, independent of f, such that for all $f \in L_{p(x),v}(\mathbb{R}^n)$

$$\|Tf\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant C\|f\|_{L_{p(x),v}(\mathbb{R}^n)}.$$

Proof. For $k \in \mathbb{Z}$ we define $E_k = \{x \in \mathbb{R}^n\colon\ 2^k < |x| \leqslant 2^{k+1}\}$, $E_{k,1} = \{x \in \mathbb{R}^n\colon |x| \leqslant 2^{k-1}\}$, $E_{k,2} = \{x \in \mathbb{R}^n\colon\ 2^{k-1} < |x| \leqslant 2^{k+2}\}$, $E_{k,3} = \{x \in \mathbb{R}^n\colon\ |x| > 2^{k-1}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in\mathbb{Z}}$ is equal to 3.

Given $f \in L_{p(x),v}(\mathbb{R}^n)$, we write

$$\begin{aligned}|Tf(x)| &= \sum_{k\in\mathbb{Z}} |Tf(x)|\,\chi_{E_k}(x) \leqslant \sum_{k\in\mathbb{Z}} |Tf_{k,1}(x)|\,\chi_{E_k}(x) + \sum_{k\in\mathbb{Z}} |Tf_{k,2}(x)|\,\chi_{E_k}(x)\\ &\quad + \sum_{k\in\mathbb{Z}} |Tf_{k,3}(x)|\,\chi_{E_k}(x) = T_1f(x) + T_2f(x) + T_3f(x),\end{aligned}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_k}$, $i = 1, 2, 3$.

First we shall estimate $\|T_1f\|_{L_{p(x),\omega_2}}$. Note that for $x \in E_k$, $y \in E_{k,1}$ we have $|y| < 2^{k-1} \leqslant \frac{1}{2}|x|$. Moreover, $E_k \cap \operatorname{supp} f_{k,1} = \emptyset$ and $|x-y| \geqslant |x| - \frac{1}{2}|x| = \frac{1}{2}|x|$. Hence by (3)

$$\begin{aligned}|T_1f(x)| &\leqslant C\sum_{k\in\mathbb{Z}}\Big(\int_{\mathbb{R}^n}\frac{|f_{k,1}(y)|}{|x-y|^n}\,\mathrm{d}y\Big)\chi_{E_k} \leqslant C\int_{|y|<\frac{1}{2}|x|}\frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y\\ &\leqslant C\int_{|y|<|x|}\frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y \leqslant 2^n\,C\,|x|^{-n}\int_{|y|<|x|}|f(y)|\,\mathrm{d}y\end{aligned}$$

for any $x \in E_k$. Hence we have

$$\begin{aligned}\|T_1f(x)\|_{L_{p(x),w}(\mathbb{R}^n)} &\leqslant 2^n\,C\,\||\cdot|^{-n}\int_{|y|<|\cdot|}|f(y)|\,\mathrm{d}y\|_{L_{p(x),w}(\mathbb{R}^n)}\\ &= \Big\|\int_{|y|<|\cdot|}|f(y)|\,\mathrm{d}y\Big\|_{L_{p(x),\ |x|^{-n}\,w}(\mathbb{R}^n)}.\end{aligned}$$

By the condition (4) and Theorem 1 we obtain

$$\|T_1f(x)\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant C_1\,\|f\|_{L_{p(x),v}(\mathbb{R}^n)} \tag{7}$$

where $C_1 > 0$ is independent of f and $x \in \mathbb{R}^n$.

Next we estimate $\|T_3f\|_{L_{p(x),w}(\mathbb{R}^n)}$. It is obvious that for $x \in E_k$, $y \in E_{k,3}$ we have $|y| > 2\,|x|$ and $|x-y| \geqslant |y| - |x| \geqslant |y| - \frac{1}{2}|y| = \frac{1}{2}|y|$. Since $E_k \cap \operatorname{supp} f_{k,3} = \emptyset$ for $x \in E_k$ by (2), we have

$$|T_3f(x)| \leqslant C\int_{|y|>2|x|}\frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y \leqslant 2^n\,C\int_{|y|>2|x|}\frac{|f(y)|}{|y|^n}\,\mathrm{d}y.$$

Hence we obtain

$$\|T_3 f(x)\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant 2^n C \bigg\| \int_{|y|>2|\cdot|} |f(y)|\,|y|^{-n}\,\mathrm{d}y \bigg\|_{L_{p(x),w}(\mathbb{R}^n)}$$
$$\leqslant 2^n\, C \bigg\| \int_{|y|>|\cdot|} |f(y)|\,|y|^{-n}\,\mathrm{d}y \bigg\|_{L_{p(x),w}(\mathbb{R}^n)}.$$

By the condition (5) and Theorem 2 we obtain

$$\|T_3 f(x)\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant C_2\, \|f\|_{L_{p(x),v}(\mathbb{R}^n)} \tag{8}$$

where $C_2 > 0$ is independent of f and $x \in \mathbb{R}^n$.

Finally, we estimate $\|T_2 f\|_{L_{p(x),w}(\mathbb{R}^n)}$. By the $L_{p(x)}(\mathbb{R}^n)$ boundedness of T and condition (6) we have

$$\|T_2 f(x)\|_{L_{p(x),w}(\mathbb{R}^n)} = \bigg\| \sum_{k\in\mathbb{Z}} |Tf_{k,2}(x)|\,\chi_{E_k}(x)w(x) \bigg\|_{L_{p(x)}(\mathbb{R}^n)}$$
$$= \bigg\| \sum_{k\in\mathbb{Z}} |Tf_{k,2}(x)|\chi_{E_k}(x)w(x) \bigg\|_{L_{p(x)}(\mathbb{R}^n)} \leqslant \sum_{k\in\mathbb{Z}} \|Tf_{k,2}\,w\|_{L_{p(x)}(\mathbb{R}^n)}$$
$$\leqslant \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} w(x)\, \|Tf_{k,2}\|_{L_{p(x)}(\mathbb{R}^n)} \leqslant \|T\|_{L_{p(\cdot)}(\mathbb{R}^n)} \sum_{k\in\mathbb{Z}} \sup_{x\in E_k} w(x)\, \|f_{k,2}\|_{L_{p(x)}(\mathbb{R}^n)}$$
$$= \|T\|_{L_{p(\cdot)}(\mathbb{R}^n)} \sum_{k\in\mathbb{Z}} \sup_{y\in E_k} w(y)\, \|f\|_{L_{p(x)}(E_{k,2})},$$

where $\|T\|_{L_{p(\cdot)}(\mathbb{R}^n)}$ is the norm of the operator T in $L_{p(x)}(\mathbb{R}^n)$. Since for $x \in E_{k,2}$ we have $2^{k-1} < |x| \leqslant 2^{k+2}$, we obtain by condition (a)

$$\sup_{y\in E_k} w(y) \leqslant \sup_{y\in E_{k,2}} w(y) \leqslant \sup_{\frac{1}{4}|x|<|y|\leqslant 4|x|} \leqslant M\, v(x) \quad \text{for a.e.} \quad x \in E_{k,2}.$$

Therefore

$$\|T_2 f(x)\|_{L_{p(x),w}(\mathbb{R}^n)} \leqslant \|T\|_{L_{p(\cdot)}(\mathbb{R}^n)}\, M \sum_{k\in\mathbb{Z}} \|f\,v\|_{L_{p(x)}(E_{k,2})} \leqslant C_3\, \|f\|_{L_{p(x),v}(\mathbb{R}^n)}, \tag{9}$$

where $C_3 = 3\,\|T\|_{L_{p(\cdot)}(\mathbb{R}^n)}\, M$, since the multiplicity of the covering $\{E_{k,2}\}_{k\in\mathbb{Z}}$ is equal to 3.

Combining the inequalities (7),(8) and (9) we obtain the proof of Theorem 3. $\square$

Remark 5. Note that the condition (3) was introduced by Soria and Weiss in [37]. Many interesting operators in harmonic analysis, such as the Calderon-Zigmund singular integral operators, Hardy-Littlewood maximal operators, Fefferman'ssingular integrals, Ricci-Stein's oscillatory singular integrals, Bochner-Riesz means and so on satisfy the condition (3). In the case $p(x) = p = \text{const}$ for the classical Lebesgue spaces Theorem 5 was proved in [38](see also [12] and [22]). Also, for classical Lebesgue spaces Edmunds and Kokilashvili [8] and Guliyev [13] found new sufficient conditions on weights for the Calderon-Zigmund singular integral operator whenever the weight functions are radial monotone functions. Further, for singular integral operators defined on homogeneous groups, an analog of Theorem 5 was proved in [16] and [14].

Remark 6. In [6] Diening was proved boundedness of the Hardy-Litlewood maximal operators provided the exponent function is constant outside of a fixed ball. Further, Nekvinda [29] has shown that this hypothesis can be weakened. Let $1 < \underline{p} \leqslant p(x) \leqslant \overline{p} < \infty$ for $|x| \leqslant 1$ and $|p(x) - p(y)| \leqslant K/\ln 1/|x-y|$, $0 < |x-y| \leqslant \frac{1}{2}$, and $p(x) = p = \text{const}$ for $0 < |x-y| \geqslant 1$. Further, suppose that T is either the Calderon-Zigmund singular integral operator or the Hardy-Littlewood maximal operator. Then for radial increasing weights functions Theorem 5 was proved in [11]. In [18] boundedness of the maximal operator in weighted variable spaces on a metric measure space is proved.

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