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A PREDATOR-PREY MODEL WITH COMBINED DEATH AND COMPETITION TERMS

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Abstract. The existence of a positive solution for the generalized predator-prey model for two species

$$\Delta u + u(a + g(u, v)) = 0 \quad \text{in } \Omega,$$
$$\Delta v + v(d + h(u, v)) = 0 \quad \text{in } \Omega,$$
$$u = v = 0 \quad \text{on } \partial\Omega,$$

are investigated. The techniques used in the paper are the elliptic theory, upper-lower solutions, maximum principles and spectrum estimates. The arguments also rely on some detailed properties of the solution of logistic equations.

Keywords: predator-prey model, coexistence state

MSC 2010: 35J47, 35J57

1. Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the predator-prey relation of two or more species of animals residing in the same environment. Especially, pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the other. In this paper we focus on the general predator-prey model in order to better understand the competitive interactions between the two species. Specifically, we investigate the conditions needed for the coexistence of two species.

2. Literature review

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the following simple biological models:

$$
\begin{cases}
\Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\
\Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 \\
u(x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = 0,
\end{cases} \quad \text{in } \Omega, \tag{1}
$$

$$
\begin{cases}
\Delta u(x) + u(x)(a - bu(x) + cv(x)) = 0 \\
\Delta v(x) + v(x)(d - fv(x) + eu(x)) = 0 \\
u(x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = 0,
\end{cases} \quad \text{in } \Omega, \tag{2}
$$

and

$$
\begin{cases}
\Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\
\Delta v(x) + v(x)(d - fv(x) + eu(x)) = 0 \\
u(x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = 0,
\end{cases} \quad \text{in } \Omega, \tag{3}
$$

where $a, b, c, d, e, f > 0$.

Equations (1) describe the coexistence states of a competition system, while (2) those of a cooperation system, and (3) those of a predator-prey system (v being the predators, u the preys). In [6] and [7], we generalized (1), (2) and extended the results of existence or uniqueness of steady state solutions established in [1], [2], [8] and [12].

In this paper we improve the results for (3). This system describes the predator-prey interaction of two species residing in the same environment in the following manner:

$$
\begin{cases}
u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) - cv(x,t)) \\
v_t(x,t) = \Delta v(x,t) + v(x,t)(d - fv(x,t) + eu(x,t)) \\
u(x,t)|_{\partial\Omega} = v(x,t)|_{\partial\Omega} = 0,
\end{cases} \quad \text{in } \Omega \times \mathbb{R}^+, \tag{4}
$$

where Ω is a bounded domain in $\mathbb{R}^n$. Here $u(x,t)$ and $v(x,t)$ designate the population densities for the preys and predators, respectively. The positive constant coefficients in this system represent growth rates (a and d), death rates (b and f) and competition rates (c and e). Furthermore, we assume that both species are not residing on the boundary of Ω.

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (4) (see [3],

[4], [5], [12]). The positive steady state solution is simply the positive solution to the time-independent system

$$
\tag{5}
\begin{cases}
\Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\
\Delta v(x) + v(x)(d - fv(x) + eu(x)) = 0 \\
u(x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = 0.
\end{cases}
\quad \text{in } \Omega,
$$

One of the important results for the time-independent Lotka-Volterra model was obtained by Zhengyuan and Mottoni. In 1992 they published the following characterization of non-negative solutions to (5) in terms of growth rates (a, d):

Theorem 2.1 (in [12]). *There exist two functions $\gamma_0(a)$, $\mu_0(d)$ such that the set S of non-negative solutions to* (5) *is characterized as follows:*

(1) *If $a \leqslant \lambda_1$, $d \leqslant \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ with the homogeneous boundary condition (see Lemma 3.2), then $S = \{(0,0)\}$.*
(2) *If $a \leqslant \lambda_1$, $d > \lambda_1$, then $S = \{(0,0),(0,\theta_{d/f})\}$. (See Lemma 3.5 for $\theta_{d/f}$.)*
(3) *If $a > \lambda_1$, $d < \gamma_0(a)$, then $S = \{(0,0),(\theta_{a/b},0)\}$.*
(4) *If $\lambda_1 < a < \mu_0(d)$, $d > \lambda_1$, then $S = \{(0,0),(\theta_{a/b},0),(0,\theta_{d/f})\}$.*
(5) *If $a > \lambda_1$, $\gamma_0(a) < d \leqslant \lambda_1$, then $S = \{(0,0),(\theta_{a/b},0),(u^+,v^+)\}$, where (u^+,v^+) is a positive solution to* (5).
(6) *If $d > \lambda_1$, $a > \mu_0(d)$, then $S = \{(0,0),(\theta_{a/b},0),(0,\theta_{d/f}),(u^+,v^+)\}$.*

The work of Zhengyuan and Mottoni provides insight into the predator-prey interactions of two species operating under the conditions described in the Lotka-Volterra model. However, their results are somewhat limited by a few key assumptions. In the Lotka-Volterra model that they studied, the rate of change of densities largely depends on constant rates of reproduction, self-limitation, and competition. The model also assumes a linear relationship of the terms affecting the rate of change for both population densities.

However, in reality, the rates of change of population densities may vary in a more complicated and irregular manner than can be described by the simple predator-prey model. Therefore, in this paper we focus on the existence of the positive steady state solution of the general predator-prey model for two species,

$$
\begin{cases}
u_t(x,t) = \Delta u(x,t) + u(x,t)(a + g(u(x,t),v(x,t))) \\
v_t(x,t) = \Delta v(x,t) + v(x,t)(d + h(u(x,t),v(x,t))) \\
u(x,t)|_{\partial\Omega} = v(x,t)|_{\partial\Omega} = 0,
\end{cases}
\quad \text{in } \Omega \times \mathbb{R}^+,
$$

or, equivalently, the positive solution to

$$
\tag{6}
\begin{cases}
\Delta u(x) + u(x)(a + g(u(x),v(x))) = 0 \\
\Delta v(x) + v(x)(d + h(u(x),v(x))) = 0 \\
u|_{\partial\Omega} = v|_{\partial\Omega} = 0,
\end{cases}
\quad \text{in } \Omega,
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, a, d are positive reproduction constants, $g, h \in C^1$ designate the death and competition rates that satisfy the growth conditions $g_u < 0$, $g_v < 0$, $h_v < 0$, $h_u > 0$, $g(0,0) = h(0,0) = 0$, and there exists $c_0 > 0$ such that $a + g(u,0) \leqslant 0$ and $d + h(0,v) \leqslant 0$ for $u, v \geqslant c_0$.

We can interpret the functions g, h, g_u, g_v, h_u, and h_v as the manner in which the members of each species u and v interact among themselves and with the members of the other species.

We note that the system (5) is a specific case of (6). Hence the research presented in this paper is about the mathematical community's discussion on the existence of the steady state solution for the general predator-prey model. In our analysis we focus on the conditions required for the maintenance of the coexistence state of (6). Mathematically, our results generalize Theorem 2.1 developed by Zhengyuan and Mottoni.

3. Preliminaries

In this section we state some preliminary results which will be useful for our later arguments.

Definition 3.1 (upper and lower solutions). Consider the problem

$$\begin{cases} \Delta u + f(x,u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{7}$$

where $f \in C^{\alpha}(\overline{\Omega} \times \mathbb{R})$, $0 < \alpha < 1$ and Ω is a bounded domain in $\mathbb{R}^n$.

(A) A function $\bar{u} \in C^{2,\alpha}(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta \bar{u} + f(x,\bar{u}) \leqslant 0 \text{ in } \Omega, \\ \bar{u}|_{\partial\Omega} \geqslant 0 \end{cases}$$

is called an upper solution to (7).

(B) A function $\underline{u} \in C^{2,\alpha}(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta \underline{u} + f(x,\underline{u}) \geqslant 0 \text{ in } \Omega, \\ \underline{u}|_{\partial\Omega} \leqslant 0 \end{cases}$$

is called a lower solution to (7).

lemma 3.1 ([9]). *Let $f(x,\xi) \in C^{\alpha}(\overline{\Omega} \times \mathbb{R})$ and let $\bar{u}, \underline{u} \in C^{2,\alpha}(\overline{\Omega})$ be, respectively, upper and lower solutions to* (7) *which satisfy $\underline{u}(x) \leqslant \bar{u}(x)$, $x \in \overline{\Omega}$, where $0 < \alpha < 1$. Then* (7) *has a solution $u \in C^{2,\alpha}(\overline{\Omega})$ with $\underline{u}(x) \leqslant u(x) \leqslant \bar{u}(x)$, $x \in \overline{\Omega}$.*

Lemma 3.2 (The first eigenvalue) ([9]). *Consider the problem*

$$\begin{cases} -\Delta u + q(x)u = \lambda u \text{ in } \quad \Omega, \\ \qquad\qquad u|_{\partial\Omega} = 0, \end{cases} \tag{8}$$

where $q(x)$ is a smooth function from Ω to $\mathbb{R}$ and Ω is a bounded domain in $\mathbb{R}^n$.

(A) *The first eigenvalue $\lambda_1(q)$, denoted simply by λ_1 when $q \equiv 0$, is simple with a positive eigenfunction φ_1.*

(B) *If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.*

Lemma 3.3 (Maximum Principles) ([9]). *Let*

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}u + \sum_{i=1}^{n} a_i(x) D_i u + a(x)u = f(x) \quad \text{in } \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$.

(M1) $\partial\Omega \in C^{2,\alpha}(0 < \alpha < 1)$;

(M2) $|a_{ij}(x)|_\alpha, |a_i(x)|_\alpha, |a(x)|_\alpha \leqslant M$ $(i,j = 1,\dots,n)$, *where $|\cdot|_\alpha$ is the α-Hölder norm;*

(M3) *L is uniformly elliptic in $\overline{\Omega}$, with ellipticity constant γ, i.e., for every $x \in \overline{\Omega}$ and every real vector $\xi = (\xi_1, \dots, \xi_n)$*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \gamma \sum_{i=1}^{n} |\xi_i|^2.$$

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of $Lu \geqslant 0 (Lu \leqslant 0)$ in Ω.

(A) *If $a(x) \equiv 0$, then $\max\limits_{\overline{\Omega}} u = \max\limits_{\partial\Omega} u$ $(\min\limits_{\overline{\Omega}} u = \min\limits_{\partial\Omega} u)$.*

(B) *If $a(x) \leqslant 0$, then $\max\limits_{\overline{\Omega}} u \leqslant \max\limits_{\partial\Omega} u^+$ $(\min\limits_{\overline{\Omega}} u \geqslant -\max\limits_{\partial\Omega} u^-)$, where $u^+ = \max(u,0)$, $u^- = -\min(u,0)$.*

(C) *If $a(x) \equiv 0$ and u attains its maximum (minimum) at an interior point of Ω, then u is identically a constant in Ω.*

(D) *If $a(x) \leqslant 0$ and u attains a nonnegative maximum (nonpositive minimum) at an interior point of Ω, then u is identically a constant in Ω.*

Lemma 3.4 ([11]). *Let $g_i(u_1, u_2) \in C^1([0,\infty) \times [0,\infty))$ and suppose that there exists a positive constant M such that for every $t \in [0,1]$, if $u = (u_1, u_2)$ is a non-negative solution of the problem*

$$\begin{cases} -\Delta u_1 = t g_1(u_1, u_2) & \text{in } \Omega, \\ -\Delta u_2 = t g_2(u_1, u_2) & \text{in } \Omega, \\ \quad u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = 0, \end{cases} \tag{9}$$

then

$$u_1 \leqslant M, u_2 \leqslant M.$$

Assume that

(1) *either $g_1(0,0) > \lambda_1$, $g_2(0,0) \neq \lambda_1$ or $g_1(0,0) \neq \lambda_1$, $g_2(0,0) > \lambda_1$,*

(2)

$$\frac{\partial g_1}{\partial u_1}(u_1, 0) \leqslant 0 \quad (u_1 \geqslant 0), \quad \frac{\partial g_1}{\partial u_1}(u_1, 0) \text{ is not identically zero } (u_1 \in [0, b))$$
$$\frac{\partial g_2}{\partial u_2}(0, u_2) \leqslant 0 \quad (u_2 \geqslant 0), \quad \frac{\partial g_2}{\partial u_2}(0, u_2) \text{ is not identically zero } (u_2 \in [0, b)),$$

where b is any fixed positive number,

(3) *$(u_1^*, 0)$, $(0, u_2^*)$ is any nontrivial non-negative solution with $\lambda_1(-g_2(u_1^*, 0)) < 0$, $\lambda_1(-g_1(0, u_2^*)) < 0$.*

Then there is a solution $u_1 > 0$, $u_2 > 0$ of (7) for $t = 1$.

We also need some information on the solutions of the following logistic equations.

Lemma 3.5 ([10]). *Consider the problem*

$$\begin{cases} \Delta u + u f(u) = 0 & \text{in } \Omega, \\ \quad u|_{\partial\Omega} = 0, \quad u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leqslant 0$ for $u \geqslant c_0$, and Ω is a bounded domain in $\mathbb{R}^n$.

If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary condition.

We denote the unique positive solution in Lemma 3.5 as θ_f. The main property of this positive solution is that θ_f is increasing as f is increasing.

Especially, for $a > \lambda_1$, $b > 0$, we denote by $\theta_{a/b}$ the unique positive solution of

$$\begin{cases} \Delta u + u(a - bu) = 0 & \text{in } \Omega, \\ \quad u|_{\partial\Omega} = 0, \quad u > 0. \end{cases}$$

Hence, $\theta_{a/b}$ is increasing as $a > 0$ is increasing.

We consider

$$\begin{aligned} &\Delta u + u(a + g(u,v)) = 0 \quad \text{in } \Omega, \\ &\Delta v + v(d + h(u,v)) = 0 \quad \text{in } \Omega, \\ &u = v = 0 \text{ on } \partial\Omega, \end{aligned} \tag{10}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, a, d are positive constants, $g, h \in C^1$ are such that $g_u < 0$, $g_v < 0$, $h_v < 0$, $h_u > 0$, $g(0,0) = h(0,0) = 0$, and there exists $c_0 > 0$ such that $a + g(u,0) \leqslant 0$ and $d + h(0,v) \leqslant 0$ for $u, v \geqslant c_0$.

First, we see that the two species cannot coexist when the reproduction capacities are not strong enough.

Theorem 4.1. *Suppose $a \leqslant \lambda_1$, $d \leqslant \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary condition.*

Then $u = v \equiv 0$ is the only nonnegative solution to (10).

Proof. Let (u,v) be a nonnegative solution to (10). By the Mean Value Theorem, there are $\tilde{u}$, $\tilde{v}$ such that

$$g(0,v) = g(0,v) - g(0,0) = g_v(0,\tilde{v})v, \quad h(u,0) = h(u,0) - h(0,0) = h_u(\tilde{u},0)u.$$

Hence, (10) implies that

$$\begin{aligned} &\Delta u + u(a + g(u,v) - g(0,v) + g_v(0,\tilde{v})v) \\ &\qquad = \Delta u + u(a + g(u,v) - g(0,v) + g(0,v) - g(0,0)) \\ &\qquad = \Delta u + u(a + g(u,v)) = 0 \quad \text{in } \Omega, \Delta v + v(d + h(u,v) - h(u,0) + h_u(\tilde{u},0)u) \\ &\qquad = \Delta v + v(d + h(u,v) - h(u,0) + h(u,0) - h(0,0)) \\ &\qquad = \Delta v + v(d + h(u,v)) = 0 \quad \text{in } \Omega. \end{aligned}$$

Hence,

$$\begin{aligned} &\Delta u + u(a + g(u,v) - g(0,v) + \sup(g_v)v) \geqslant 0 \text{ in } \Omega, \\ &\Delta v + v(d + h(u,v) - h(u,0) + \sup(h_u)u) \geqslant 0 \text{ in } \Omega. \end{aligned}$$

Multiplying both side by $\sup(h_u)\varphi_1$, we have

$$\begin{aligned} \sup(h_u)\varphi_1\Delta u + \sup(h_u)\varphi_1 u(a + g(u,v) - g(0,v) + \sup(g_v)v) \geqslant 0 \text{ in } \Omega, \\ -\sup(g_v)\varphi_1\Delta v - \sup(g_v)\varphi_1 v(d + h(u,v) - h(u,0) + \sup(h_u)u) \geqslant 0 \text{ in } \Omega, \end{aligned}$$

where $\varphi_1 > 0$ is the first eigenfunction of $-\Delta$ with homogeneous boundary condition corresponding to λ_1. So,

$$\begin{aligned}\int_\Omega -\sup(h_u)\varphi_1\Delta u\,\mathrm{d}x &\leqslant \int_\Omega [(g(u,v)-g(0,v))\sup(h_u)u\\ &\quad+\sup(g_v)\sup(h_u)uv + a\sup(h_u)u]\varphi_1\,\mathrm{d}x,\\ \int_\Omega \sup(g_v)\varphi_1\Delta v\,\mathrm{d}x &\leqslant \int_\Omega [-\sup(g_v)(h(u,v)-h(u,0))v\\ &\quad-\sup(g_v)\sup(h_u)uv - d\sup(g_v)v]\varphi_1\,\mathrm{d}x.\end{aligned}$$

Hence, by Green's Identity, we have

$$\begin{aligned}\int_\Omega \sup(h_u)\lambda_1\varphi_1 u\,\mathrm{d}x &\leqslant \int_\Omega [(g(u,v)-g(0,v))\sup(h_u)u\\ &\quad+\sup(g_v)\sup(h_u)uv + a\sup(h_u)u]\varphi_1\,\mathrm{d}x,\\ \int_\Omega -\sup(g_v)\lambda_1\varphi_1 v\,\mathrm{d}x &\leqslant \int_\Omega [-\sup(g_v)(h(u,v)-h(u,0))v\\ &\quad-\sup(g_v)\sup(h_u)uv - d\sup(g_v)v]\varphi_1\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\Omega \sup(h_u)(\lambda_1-a)u\varphi_1 - \sup(g_v)(\lambda_1-d)v\varphi_1\,\mathrm{d}x\\ &\quad\leqslant \int_\Omega [(g(u,v)-g(0,v))\sup(h_u)u - \sup(g_v)(h(u,v)-h(u,0))v]\varphi_1\,\mathrm{d}x.\end{aligned}$$

Since the left hand side is nonnegative by the assumption and the right hand side is nonpositive by the monotonicity of g, h, we conclude that $u = v \equiv 0$. □

Theorem 4.2. *Let $u \geqslant 0$, $v \geqslant 0$ be a solution to* (10). *If $a \leqslant \lambda_1$, then $u \equiv 0$.*

Proof. Proceeding as in the proof of Theorem 4.1, we obtain

$$0 \leqslant \int_\Omega (\lambda_1 - a)u\varphi_1\,\mathrm{d}x \leqslant \int_\Omega [g(u,v)-g(0,v)+\sup(g_v)v]u\varphi_1\,\mathrm{d}x \leqslant 0,$$

and so, $u \equiv 0$. □

In order to prove further results, we will need the following lemma.

Lemma 4.3. *Let $u \geqslant 0$, $v \geqslant 0$ be a solution of the problem*

$$(11) \qquad \begin{cases} -\Delta u = tu(a + g(u,v)) \text{ in } \Omega, \\ -\Delta v = tv(d + h(u,v)) \text{ in } \Omega, \\ u_{\partial\Omega} = v_{\partial\Omega} = 0, \end{cases}$$

where $t \in [0,1]$. Then

(1)
$$u \leqslant M_1, \quad v \leqslant M_2,$$

where $M_1 = -a/\sup(g_u)$, $M_2 = -(\sup(h_u)M_1 + d)/\sup(h_v)$.

(2) *For $t = 1$,*
$$u \leqslant \theta_{a+g(\cdot,0)}, \quad v \geqslant \theta_{d+h(0,\cdot)}$$

if $v > 0$ in Ω.

P r o o f. (1) Since $g(0,0) = 0$, by the Mean Value Theorem we have

$$g(u,0) = g(u,0) - g(0,0) \leqslant \sup(g_u)u,$$

and so,
$$\frac{g(u,0)}{\sup(g_u)} \geqslant u.$$

Hence,

$$\Delta\Big(-\frac{a}{\sup(g_u)} - u\Big) + t\Big(-\frac{a}{\sup(g_u)} - u\Big)g(u,v) = -\Delta u - tug(u,v) - t\,\frac{a}{\sup(g_u)}g(u,v)$$
$$= tua - t\,\frac{g(u,v)}{\sup(g_u)}a \leqslant tau - ta\,\frac{g(u,0)}{\sup(g_u)} \leqslant 0$$

by the monotonicity of g. Since $g(u,v) \leqslant 0$, by the Maximum Principle we conclude

$$u \leqslant M_1 = -\frac{a}{\sup(g_u)}.$$

Since $h(0,0) = 0$, by the Mean Value Theorem we have

$$h(0,v) = h(0,v) - h(0,0) \leqslant \sup(h_v)v,$$

and so,
$$\frac{h(0,v)}{\sup(h_v)} \geqslant v.$$

Hence,

$$\begin{aligned}
&\Delta\Big(-\frac{\sup(h_u)M_1+d}{\sup(h_v)}-v\Big)+t\Big(-\frac{\sup(h_u)M_1+d}{\sup(h_v)}-v\Big)h(0,v)\\
&=-\Delta v-tvh(0,v)-t\,\frac{\sup(h_u)M_1h(0,v)}{\sup(h_v)}-t\,\frac{dh(0,v)}{\sup(h_v)}\\
&=tvd+tvh(u,v)-tvh(0,v)-t\,\frac{\sup(h_u)M_1h(0,v)}{\sup(h_v)}-t\,\frac{dh(0,v)}{\sup(h_v)}\\
&\leqslant td\Big(v-\frac{h(0,v)}{\sup(h_v)}\Big)+tv\sup(h_u)u-t\sup(h_u)M_1\frac{h(0,v)}{\sup(h_v)}\\
&\leqslant 0.
\end{aligned}$$

Since $h(0,v)\leqslant 0$, by the Maximum Principle we conclude

$$v\leqslant M_2=-\frac{\sup(h_u)M_1+d}{\sup(h_v)}.$$

(2) If $a\leqslant\lambda_1$ or $d\leqslant\lambda_1$, then by Theorem 4.2 the results are obvious.
Suppose $a>\lambda_1$ and $d>\lambda_1$. Since

$$\Delta u+u(a+g(u,0))\geqslant\Delta u+u(a+g(u,v))=0 \text{ in } \Omega,$$

u is a lower solution to

$$\begin{cases}\Delta u+u(a+g(u,0))=0 \text{ in } \Omega,\\ u|_{\partial\Omega}=0.\end{cases}$$

We can take M large enough such that $M>u$ on $\overline{\Omega}$ and $u=M$ is an upper solution to

$$\begin{cases}\Delta u+u(a+g(u,0))=0 \text{ in } \Omega,\\ u|_{\partial\Omega}=0.\end{cases}$$

Since

$$\begin{cases}\Delta u+u(a+g(u,0))=0 \text{ in } \Omega,\\ u|_{\partial\Omega}=0\end{cases}$$

has a unique positive solution $\theta_{a+g(\cdot,0)}$, by the upper-lower solution method we conclude $u\leqslant\theta_{a+g(\cdot,0)}$ in Ω.

Since

$$\begin{cases}\Delta v+v(d+h(0,v))\leqslant\Delta v+v(d+h(u,v))=0 \text{ in } \Omega,\\ v|_{\partial\Omega}=0,\end{cases}$$

$v > 0$ is an upper solution to

$$\begin{cases} \Delta v + v(d + h(0, v)) = 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Since $v > 0$, for $n \in \mathbb{N}$ large enough $\theta_{d+h(0,\cdot)}/n < v$ in Ω. Since

$$\begin{aligned} \Delta\Big(\frac{\theta_{d+h(0,\cdot)}}{n}\Big) &+ \frac{\theta_{d+h(0,\cdot)}}{n}\Big(d + h\Big(0, \frac{\theta_{d+h(0,\cdot)}}{n}\Big)\Big) \\ &= \frac{1}{n}\Big[\Delta\theta_{d+h(0,\cdot)} + \theta_{d+h(0,\cdot)}\Big(d + h\Big(0, \frac{\theta_{d+h(0,\cdot)}}{n}\Big)\Big)\Big] \\ &\geqslant \frac{1}{n}[\Delta\theta_{d+h(0,\cdot)} + \theta_{d+h(0,\cdot)}(d + h(0, \theta_{d+h(0,\cdot)}))] = 0, \end{aligned}$$

$\theta_{d+h(0,\cdot)}/n$ is a lower solution to

$$\begin{cases} \Delta v + v(d + h(0, v)) = 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Therefore, by the uniqueness of the solution and the upper-lower solution method, we conclude $\theta_{d+h(0,\cdot)} \leqslant v$. □

Theorem 4.4. *There exist two functions $M(a), N(d)\colon [\lambda_1, \infty) \to \mathbb{R}$ such that*

(A) *if $a > \lambda_1$, $d \leqslant M(a)$, then all possible nonnegative solutions to* (10) *are* $(0, 0)$ *and* $(\theta_{a+g(\cdot,0)}, 0)$,

(B) *if $\lambda_1 < a \leqslant N(d)$, $d > \lambda_1$, then all possible nonnegative solutions to* (10) *are* $(0, 0)$, $(\theta_{a+g(\cdot,0)}, 0)$ *and* $(0, \theta_{d+h(0,\cdot)})$,

(C) *if $a > \lambda_1$, $M(a) < d < \lambda_1$, then all possible nonnegative solutions to* (10) *are* $(0, 0)$, $(\theta_{a+g(\cdot,0)}, 0)$ *and a positive solution $u^+ > 0$, $v^+ > 0$,*

(D) *if $d > \lambda_1$, $a > N(d)$, then all possible nonnegative solutions to* (10) *are* $(0, 0)$, $(\theta_{a+g(\cdot,0)}, 0)$, $(0, \theta_{d+h(0,\cdot)})$ *and a positive solution $u^+ > 0$, $v^+ > 0$.*

P r o o f. For $a \geqslant \lambda_1$, let

$$M(a) = \lambda_1(-h(\theta_{a+g(\cdot,0)}, 0)) \quad \text{and} \quad N(d) = \lambda_1(-g(0, \theta_{d+h(0,\cdot)})).$$

(A) Suppose $d \leqslant M(a)$. Let $\bar{u} \geqslant 0$, $\overline{v} \geqslant 0$ be a solution to (10). If $\overline{v} > 0$ in Ω, then $\lambda = d$ is the smallest eigenvalue of the problem

$$\begin{cases} -\Delta v + v(-h(\bar{u}, \overline{v})) = \lambda v & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

By the monotonicity of h and Lemma 4.3 we have

$$-h(\bar{u},\overline{v}) > -h(\theta_{a+g(\cdot,0)},0),$$

and so

$$d = \lambda_1(-h(\bar{u},\overline{v})) > \lambda_1(-h(\theta_{a+g(\cdot,0)},0)) = M(a),$$

which is a contradiction to $d \leqslant M(a)$. Hence, $\overline{v} \equiv 0$. Therefore, we conclude that if $a > \lambda_1$ and $d \leqslant M(a)$, then all possible nonnegative solutions to (10) are $(0,0)$ and $(\theta_{a+g(\cdot,0)},0)$.

(B) Suppose $\lambda_1 < a \leqslant N(d)$ and $d > \lambda_1$. Let $u \geqslant 0$, $v \geqslant 0$ be a solution to (10) with $v > 0$ in Ω. If $u > 0$ in Ω, then $\lambda = 0$ is the smallest eigenvalue of the problem

$$\begin{cases} -\Delta w - w(a + g(u,v)) = \lambda w \quad \text{in } \Omega, \\ w|_{\partial\Omega} = 0. \end{cases}$$

Since

$$-(g(u,v)+a) > -(g(0,\theta_{d+h(0,\cdot)})+a),$$

from Lemma 4.3 and the monotonicity of g, using Lemma 3.2 we have

$$0 > \lambda_1(-g(0,\theta_{d+h(0,\cdot)}) - a) = N(d) - a.$$

This contradicts $a \leqslant N(d)$. Hence $u = 0$, so all possible nonnegative solutions to (10) are $(0,0)$, $(0,\theta_{d+h(0,\cdot)})$ and $(\theta_{a+g(\cdot,0)},0)$.

(C) Suppose $a > \lambda_1$ and $M(a) < d < \lambda_1$. Let $u \geqslant 0$, $v \geqslant 0$ be a solution to (10) in which one component is zero. Then $u = 0$, $v = 0$ or $u = \theta_{a+g(\cdot,0)}$, $v = 0$. Since $\lambda_1(-h(\theta_{a+g(\cdot,0)},0) - d) = M(a) - d < 0$, by the combination of lemmas 3.4 and 4.3 there is a positive solution to (10) $u^+ > 0$, $v^+ > 0$.

(D) Suppose $a > N(d)$ and $d > \lambda_1$. Let $u \geqslant 0$, $v \geqslant 0$ be a solution to (10) in which one component is zero. Then since

$$a > N(d) = \lambda_1(-g(0,\theta_{d+h(0,\cdot)})) > \lambda_1(-g(0,0)) = \lambda_1(0) = \lambda_1,$$

from Lemma 3.2 and the monotonicity of g, we have $u = 0$, $v = 0$ or $u = 0$, $v = \theta_{d+h(0,\cdot)}$ or $u = \theta_{a+g(\cdot,0)}$, $v = 0$. Since $\lambda_1(-g(0,\theta_{d+h(0,\cdot)}) - a) = N(d) - a < 0$, by the combination of Lemmas 3.4 and 4.3 there is a positive solution to (10), $u^+ > 0$, $v^+ > 0$. $\square$

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