Ali Jaballah

Ring extensions with some finiteness conditions on the set of intermediate rings

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 1, 117–124

Persistent URL: `http://dml.cz/dmlcz/140555`

Czechoslovak Mathematical Journal, 60 (135) (2010), 117–124

RING EXTENSIONS WITH SOME FINITENESS CONDITIONS ON THE SET OF INTERMEDIATE RINGS

Ali Jaballah, Sharjah

(Received August 11, 2008)

Abstract. A ring extension $R \subseteq S$ is said to be FO if it has only finitely many intermediate rings. $R \subseteq S$ is said to be FC if each chain of distinct intermediate rings in this extension is finite. We establish several necessary and sufficient conditions for the ring extension $R \subseteq S$ to be FO or FC together with several other finiteness conditions on the set of intermediate rings. As a corollary we show that each integrally closed ring extension with finite length chains of intermediate rings is necessarily a normal pair with only finitely many intermediate rings. We also obtain as a corollary several new and old characterizations of Prüfer and integral domains satisfying the corresponding finiteness conditions.

Keywords: integral domain, intermediate ring, overring, integrally closed, Prüfer domain, residually algebraic pair, normal pair, primitive extension, a.c.c., d.c.c., minimal condition, maximal condition, affine extension, Dilworth number, width of an ordered set

MSC 2010: 13B02, 13B22, 13E15, 13E99, 13F05, 13G05

1. Introduction

Let $R \subseteq S$ be an extension of integral domains. If T is a subring of S, we assume that T has the same identity element of S. The set of subrings of S that contain R is called the set of intermediate rings in the ring extension $R \subseteq S$. We let $[R, S]$ denote this set. If K is the field of fractions of R, then an intermediate ring in the extension $R \subseteq K$ is called an overring of R.

Recently several authors have been interested in ring extensions with only finitely many intermediate rings or with only finite length chains of intermediate rings, and in integral domains that have only finitely many overrings or only finite length chains of overrings. This paper is mainly concerned with the following two finiteness conditions on the set of intermediate rings in the ring extension $R \subseteq S$:

– the ring extension $R \subseteq S$ is said to be FO if this extension *has only finitely many intermediate rings*.
– the ring extension $R \subseteq S$ is said to be FC if *each chain of distinct intermediate rings in this extension is finite*.

The two conditions have been recently introduced by Gilmer in [7] for the set of overrings of an integral domain. An integral domain is said to be FO (or FC) if the corresponding condition is satisfied for the extension $R \subseteq K$, where K is the field of fractions of R. Several characterizations of extensions $R \subseteq K$ satisfying these conditions have been established by Gilmer in [7]. Several related results can be found in [9], [11], [3] and [2]. We investigate in this paper the realization of these two conditions in the more general setting of extensions of integral domains, where the upper ring S is not necessarily the field of quotients of the ring R. We establish in Theorem 2.2 and in Theorem 3.2 of this paper several characterizations of these extensions. The characterizations obtained in Theorem 3.2 provide a generalization of Theorem 1.5 of Gilmer in [7] where the upper ring S is supposed to be the field of fractions of R. Theorem 3.2 is also a generalization of Theorem 2.1 and Corollary 2.1 of the author in [10] where each intermediate ring is supposed to be integrally closed in S. In particular, we show that if $R \subseteq S$ is an FC extension such that R is integrally closed in S, then (R, S) is necessarily a normal pair with only finitely many intermediate rings, see Corollary 3.3. We obtain as a by-product several new and old characterizations of integral domains with some finiteness conditions on the set of overrings, see Corollary 2.3 and Corollary 3.7.

In the following all rings are assumed to be commutative with identity. We let $\operatorname{Spec}(R)$ denote as usual the set of proper prime ideals of the integral domain R, and we let $\operatorname{Max}(R)$ denote the set of its maximal ideals. Let A be a set of prime ideals of R. We define the Krull dimension of A to be the maximal length of chains of prime R-ideals from A, and we will denote it by $\dim(A)$.

2. Finiteness conditions on $[R, S]$

In order to relate various finiteness conditions on the set $[R, S]$ of intermediate rings, we need to consider the set $[R, S]$ ordered by inclusion. Let P denote a general ordered set. A subset C of P is said to be a chain if every two distinct elements from C are comparable. A subset A of P is said to be an antichain if no two distinct elements from A are comparable. The size of the largest antichain in P is called the width $w(P)$ of P if such an antichain exists, otherwise the width is set to be ∞, see [12, p. 36]. A good connection between chains and antichains in ordered sets is given by Dilworth's Chain Decomposition Theorem: Any ordered set P of width k is the union of k chains $C_1, \ldots, C_k \subseteq P$, see [12, p. 49].

For each $x \in S \setminus R$, we call the ring $R[x]$ a simple extension of R in S. Two simple extensions $R[x]$ and $R[y]$ are considered different if and only if $R[x] \neq R[y]$. Recall that a ring extension $R \subseteq S$ is called (strongly) affine if every intermediate ring is finitely generated as a ring over R. In order to formulate our results we need to introduce the following definition:

Definition 2.1. We say that the ring extension $R \subseteq S$ is uniformly affine, UA, if there is a finite subset $G = \{g_1, g_2, \ldots, g_u\}$ of elements of S such that for every $T \in [R, S]$ there is a subset H of G such that $T = R[H]$.

Now we can give our first result on ring extensions with only finitely many intermediate rings.

Theorem 2.2. *Let $R \subseteq S$ be a ring extension. Then the following statements are equivalent.*
(1) *$[R, S]$ is finite.*
(2) *The set $\{R[x]\colon\ x \in S \setminus R\}$ of all distinct simple extensions of R in S is finite.*
(3) *The extension $R \subseteq S$ is a UA extension.*
(4) *$R \subseteq S$ is an FC extension and $w[R, S]$ is finite.*

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (3): Let $F = \{s_1, s_2, \ldots, s_n\}$ be a subset of S such that $\{R[s_1], R[s_2], \ldots, R[s_n]\}$ is the set of simple extensions of R in S, and let $T \in [R, S]$. Assume, after a suitable reordering of the elements of F, that $s_1, s_2, \ldots, s_m$ are the elements of F which are in T. We necessarily have $T = R[s_1, s_2, \ldots, s_m]$. Indeed, if $u \in T \setminus R[s_1, s_2, \ldots, s_m]$, then $R[u] \notin \{R[s_1], R[s_2], \ldots, R[s_m]\}$. Also $R[u] \notin \{R[s_{m+1}], R[s_{m+2}], \ldots, R[s_n]\}$ as otherwise $s_i \in T$ for some $i \in \{m+1, \ldots, n\}$, which contradicts the choice of $s_1, s_2, \ldots, s_m$ in F. Therefore $R \subseteq S$ is a UA extension.

(3) ⇒ (4): Since every intermediate ring is generated over R by a subset of a finite set $G = \{g_1, g_2, \ldots, g_u\}$ of S, it is easy to see that the set of intermediate rings is finite and therefore every chain and every antichain of $[R, S]$ is finite. That is, $R \subseteq S$ is a FC extension and $w[R, S]$ is finite.

(4) ⇒ (1): Let $w = w[R, S]$ be the width of the set of intermediate rings $[R, S]$ ordered by inclusion and let $C_1, C_1, \ldots, C_w$ be the chains from $[R, S]$ into which $[R, S]$ is decomposed. Then $R \subseteq S$ is a FO extension as $|[R, S]| \leqslant \sum\limits_{i=1}^{w} |C_i|$. □

In the case of the ring extension $R \subseteq K$, where K is the field of quotients of R, we obtain the following result.

Corollary 2.3. *Let R be an integral domain, and let K be the field of quotients of R. Then the following statements are equivalent.*

(1) *$[R, K]$ is finite.*

(2) *The set $\{R[x]\colon x \in K \setminus R\}$ of all distinct simple extensions of R in K is finite.*

(3) *The extension $R \subseteq K$ is a UA extension.*

(4) *R is an FC domain and $w[R, K]$ is finite.*

We recall that an ordered set P is said to satisfy the ascending (descending) chain condition, a.c.c. (d.c.c.), if and only if each ascending (descending) chain of elements of P is stationary. An ordered set P is said to satisfy the maximal (minimal) condition if and only if each nonempty subset of P has a maximal (minimal) element. It is well known that P satisfies the a.c.c. condition if and only if it satisfies the maximal condition; and that P satisfies the d.c.c. condition if and only if it satisfies the minimal condition.

Remark 2.4. It is easy to see that a ring extension $R \subseteq S$ is an FC extension if and only if a.c.c. and d.c.c. with respect to the usual set inclusion are satisfied in $[R, S]$, if and only if every nonempty collection of intermediate rings in $[R, S]$ has both a maximal element and a minimal element.

3. The integrally closed case

In this section we assume that R is integrally closed in S, and we are going to use residually algebraic pairs in our characterizations. Recall that a ring extension $R \subseteq T$ is called a *residually algebraic extension*, [5], if for each prime ideal Q of T, T/Q is algebraic over $R/(Q \cap R)$. We say that (R, S) is a *residually algebraic pair*, Definition 2.1 of [1], if for each $T \in [R, S]$, $R \subseteq T$ is a residually algebraic extension. We say that (R, S) is a *normal pair*, [4], if for each $T \in [R, S]$, T is integrally closed in S. The extension $R \subseteq S$ is called a *primitive extension* (or P-extension, see [8]) if each element u of S is a root of a polynomial $f(X) \in R[X]$ with unit content; i.e., the coefficients of $f(X)$ generate the unit ideal of R.

Residually algebraic pairs (R, S) with R integrally closed in S are necessarily normal pairs by Theorem 2.5 (vi) of [1], therefore they enjoy several nice properties. The following properties of normal pairs are going to be used in this paper.

Remark 3.1. Let (R, S) be a normal pair, let $T \in [R, S]$ be any intermediate ring, and let $\operatorname{Max}(R) = \{M_i\colon i \in I\}$ be the set of maximal ideals of R. Then the following properties hold true.

(1) (Proposition 4 of [4], and Lemma 3.1 of [1]). The intermediate ring T is the intersection of some localizations of R, more precisely, for each maximal R-ideal M_i

there is a prime R-ideal Q_i such that $T_{R\setminus M_i} = R_{Q_i}$ and $T = \bigcap\limits_{i\in I} T_{R\setminus M_i} = \bigcap\limits_{i\in I} R_{Q_i}$.

(2) (Lemma 3.1 of [1]). $\operatorname{Spec}(T) = \{PT : P \in \operatorname{Spec}(R)$ and $P \subseteq Q_i$ for some $i \in I\}$.

(3) (Lemma 3.1 of [1]) $\operatorname{Max}(T)$ is the subset of maximal elements in the set $\{Q_iT\colon i \in I\}$.

(4) Each prime ideal of T is an extension PT of a prime ideal P of R. Furthermore, for each prime Q of R such that $QT \neq T$, QT is a prime ideal of T satisfying $QT \cap R = Q$ and $T_{QT} = R_{QT\cap R} = R_Q$.

For a ring extension $R \subseteq S$, we denote by $\Phi\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ the canonical contraction map, i.e. $\Phi(P) = P \cap R$ for every P in $\operatorname{Spec}(S)$.

It is easy to see that FO extensions are necessarily FC extensions. The following result shows that in the case of integrally closed extensions of integral domains these notions are equivalent to each other and also to several other characterizations.

Theorem 3.2. *Let $R \subseteq S$ be an extension of integral domains, and let $A = A(R,S) := \{P \in \operatorname{Spec}(R)\colon P \not\subset N \cap R,\ \forall N \in \operatorname{Max}(S)\}$. If R is integrally closed in S, then the following statements are equivalent.*

(1) *$[R,S]$ is finite.*

(2) *The set $\{R[x]\colon x \in S \setminus R\}$ of all simple extensions of R in S is finite.*

(3) *The extension $R \subseteq S$ is a UA extension.*

(4) *$R \subseteq S$ is an FC extension and $w[R,S]$ is finite.*

(5) *$R \subseteq S$ is an FC extension.*

(6) *(R,S) is a residually algebraic pair such that $A \setminus \Phi(\operatorname{Max}(S))$ is finite.*

(7) *(R,S) is a residually algebraic pair such that $\operatorname{Max}(R)\setminus\Phi(\operatorname{Max}(S))$ and $\dim A$ are finite.*

(8) *a.c.c. and d.c.c. are satisfied in $[R,S]$.*

(9) *Every non-empty collection of intermediate rings in $[R,S]$ has both a maximal element and a minimal element.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) is satisfied by Theorem 2.2. Also (4) $\Rightarrow$ (5) and (5) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) are trivial. It is enough to show that (5) $\Rightarrow$ (7) $\Leftrightarrow$ (6) $\Rightarrow$ (1).

(5) $\Rightarrow$ (7): Let $\operatorname{Max}(R) = \{M_i\colon i \in I\}$ be the set of maximal ideals of R. We first note that if (R,S) is a normal pair, then

$$A(R,S) = \{P \in \operatorname{Spec}(R)\colon P \not\subset Q_i,\ \forall\, i \in I\},$$

where Q_i is the prime ideal of R defined in Remark 3.1 for which $S_{R\setminus M_i} = R_{Q_i}$.

Indeed, by Remark 3.1, $\operatorname{Max}(S)$ is the subset of maximal elements in the set $\{Q_iS\colon i \in I\}$, hence let J be the subset of I such that $\operatorname{Max}(S) = \{Q_iS\colon i \in J\}$. For

$P \in \mathrm{Spec}(R)$ we have

$$\begin{aligned} & P \not\subset Q_i, \quad \forall\, i \in I \\ \Leftrightarrow\ & P \not\subset Q_i, \quad \forall\, i \in J \\ \Leftrightarrow\ & P \not\subset Q_i S \cap R, \quad \forall\, i \in J \\ \Leftrightarrow\ & P \not\subset N \cap R, \quad \forall\, N \in \mathrm{Max}(S) \\ \Leftrightarrow\ & P \in A(R,S). \end{aligned}$$

Therefore $\{P \in \mathrm{Spec}(R)\colon\ P \not\subset Q_i,\ \forall\, i \in I\} = A(R,S)$.

Now if $R \subseteq S$ is an FC extension, then $[R,S]$ satisfies d.c.c. and $R \subseteq S$ is a primitive extension by [7, Proposition 1.1]. Thus (R,S) is a residually algebraic pair by [6, Theorem 6.5.7]; that is, (R,S) is a normal pair as R is integrally closed in S by [1, Theorem 2.5]. Finally, (R,S) is a normal pair such that $\mathrm{Max}(R) \setminus \Phi(\mathrm{Max}(S))$ and $\dim A$ are finite by [10, Theorem 2.1].

(7) $\Leftrightarrow$ (6) $\Rightarrow$ (1) by Theorem 2.1 of [10] as in this case (R,S) is a normal pair and $A = \{P \in \mathrm{Spec}(R)\colon\ P \not\subset Q_i,\ \forall\, i \in I\}$ as in the proof of (5) $\Rightarrow$ (7). $\square$

From the equivalence of (5), (6) and (1) in Theorem 3.2, we obtain

Corollary 3.3. *Let $R \subseteq S$ be an FC ring extension such that R is integrally closed in S. Then (R,S) is a normal pair with only finitely many intermediate rings.*

From the equivalence of (4) and (5) in Theorem 3.2, we obtain

Corollary 3.4. *Let $R \subseteq S$ be a ring extension such that R is integrally closed in S. If every chain of intermediate rings is finite, then also every antichain of intermediate rings is finite.*

Remark 3.5. The converse of Corollary 3.4 is not true in general as it is enough to consider the extension $R \subseteq K$, where R is a valuation domain of infinite dimension. This extension has $w[R,S] = 1$, but is not FC.

Remark 3.6. If (R,S) is assumed to be a normal pair, then $A = \{P \in \mathrm{Spec}(R)\colon$ $P \not\subset Q_i,\ \forall\, i \in I\}$ as in the proof of Theorem 3.2. The statements (1), (5), (6), and (7) of Theorem 3.2 are the same as Theorem 2.1 of [10].

In the case of the ring extension $R \subseteq K$, where K is the field of quotients of R, we have $S_{R\setminus M_i} = K_{R\setminus M_i} = R_{\{0\}}$ and the ideals Q_i defined by Remark 3.1 (1) are all equal to the $\{0\}$-ideal. Hence $A = \{P \in \mathrm{Spec}(R)\colon\ P \not\subset \{0\},\ \forall\, i \in I\} = \mathrm{Spec}(R)\setminus\{0\}$. Thus we obtain the following result.

Corollary 3.7. *Let R be an integrally closed domain, and let K be the field of quotients of R. Then the following statements are equivalent:*

(1) *R is an FO domain.*

(2) *The set $\{R[s]\colon\ s \in K\}$ of all distinct simple extensions of R in its quotient field is finite.*

(3) *The extension $R \subseteq K$ is a UA extension.*

(4) *R is an FC domain and $w[R, K]$ is finite.*

(5) *R is an FC domain.*

(6) *R is a Prüfer domain with finite spectrum.*

(7) *R is a Prüfer domain such that $\operatorname{Max}(R)$ and $\dim R$ are finite.*

(8) *a.c.c. and d.c.c. are satisfied in $[R, K]$.*

(9) *Every non-empty collection of overrings of R has both a maximal element and a minimal element.*

Remark 3.8. The statements (1), (5), (6), and (7) of Corollary 3.7 are the same as Theorem 1.5 of [7], which in turn is a generalization of Corollary 2.1 of [10].

Theorem 3.9. *Let $R \subseteq S$ be an extension of integral domains, and let R' be the integral closure of R in S. If $R \subseteq S$ is an FC extension then the following statements hold true.*

(1) *(R, S) is a residually algebraic pair.*

(2) *(R', S) is a normal pair.*

(3) *$[R', S]$ is finite; i.e. $R' \subseteq S$ is an FO extension.*

Proof. (1) Let R' be the integral closure of R in the ring S. Then each chain of intermediate rings in $[R', S]$ is also finite, hence (R', S) is a residually algebraic pair by Theorem 3.2. This is equivalent to (R, S) being a residually algebraic pair by Remark 2.2 of [1].

(2) (R', S) is a residually algebraic pair and also a normal pair as each residually algebraic pair (R, S) with R integrally closed in S is also a normal pair.

(3) (R', S) is also an FO extension by Theorem 3.2 as it is already an FC extension. □

References

[1] *A. Ayache and A. Jaballah*: Residually algebraic pairs of rings. Math. Z. *225* (1997), 49–65.

[2] *A. Badawi and A. Jaballah*: Some finiteness conditions on the set of overrings of a φ-ring. Houston J. Math. *34* (2008), 397–408.

[3] *M. B. Nasr, A. Jaballah*: Counting intermediate rings in normal pairs. Expo. Math. *26* (2008), 163–175.

[4] *E. D. Davis*: Overrings of commutative rings. III: Normal pairs. Trans. Amer. Math. Soc. *182* (1973), 175–185.
[5] *D. Dobbs and M. Fontana*: Universally incomparable ring homomorphisms. Bull. Aust. Math. Soc. *29* (1984), 289–302.
[6] *M. Fontana, J. A. Huckaba, I. J. Papick*: Prüfer Domains. Marcel Dekker, New York, 1997.
[7] *R. Gilmer*: Some finiteness conditions on the set of overrings of an integral domain. Proc. Am. Math. Soc. *131* (2003), 2337–2346.
[8] *R. Gilmer and J. Hoffman*: A characterization of Prüfer domains in terms of polynomials. Pacific J. Math. *60* (1975), 81–85.
[9] *A. Jaballah*: A lower bound for the number of intermediary rings. Commun. Algebra *27* (1999), 1307–1311.
[10] *A. Jaballah*: Finiteness of the set of intermediary rings in normal pairs. Saitama Math. J. *17* (1999), 59–61.
[11] *A. Jaballah*: The number of overrings of an integrally closed domain. Expo. Math. *23* (2005), 353–360.
[12] *Bernd S. W. Schröder*: Ordered Sets: an Introduction. Birkhäuser, Boston, 2003.

Author's address: A. Jaballah, Department of Mathematics, College of Sciences, University of Sharjah, P.O. Box 27272, Sharjah, U.A.E., e-mail: `ajaballah@sharjah.ac.ae`.