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ON THE DIAMETER OF THE BANACH-MAZUR SET

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Abstract. On every subspace of $l_\infty(\mathbb{N})$ which contains an uncountable ω-independent set, we construct equivalent norms whose Banach-Mazur distance is as large as required. Under Martin's Maximum Axiom (MM), it follows that the Banach-Mazur diameter of the set of equivalent norms on every infinite-dimensional subspace of $l_\infty(\mathbb{N})$ is infinite. This provides a partial answer to a question asked by Johnson and Odell.

Keywords: Banach-Mazur diameter, elastic Banach spaces, Martin's Maximum axiom

MSC 2010: 46B03, 46B26, 03E50

1. Introduction

It has been shown in [10] that if X is a separable infinite-dimensional Banach space and A is any positive real number, there exist two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that the Banach-Mazur distance between the Banach spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ is greater than A. It turns out that separability plays a role in the proof, and it is asked in [10] whether the result actually holds for every infinite-dimensional Banach space. The purpose of this note is to provide a partial affirmative answer to this question, which remains open in full generality.

We refer to [8] for the notation and terminology. In particular, we denote by ω_1 the first uncountable ordinal. We recall that a subset S of a Banach space X is said to be ω-independent if the equation

$$\sum_{n=0}^{\infty} \alpha_n x_n = 0$$

where the α_n's are scalars and (x_n) is an arbitrary sequence in S, implies that $\alpha_n = 0$ for all n. Every minimal system is clearly ω-independent, where we recall that minimal systems are the "X parts" of biorthogonal systems (see [8], Def. 1.1). It has been shown in [4] that if X is separable, then ω-independent subsets of X are at most countable (see also [11] for a more precise result). It follows from a recent work by S. Todorcevic and others that whether every non-separable Banach space contains an uncountable ω-independent subset is undecidable in (ZFC) (see [8], Theorem 8.24).

The purpose of this note is to use ω-independent subsets for constructing equivalent norms which are far from each other in the Banach-Mazur distance. We refer to [1] for a recent work along similar lines, where it is shown that the existence of an equivalent norm with the Mazur intersection property on every Asplund space of density character ω_1 is undecidable in (ZFC).

2. Results

The following lemma is the main technical result of this note. The proof relies on some techniques from ([8], Section 8.2).

Lemma 2.1. *Let X be a Banach space which contains an uncountable ω-independent family. Then for any $A > 0$ there is an equivalent norm on X whose dual norm satisfies: for any countable subset $(x_k^*)_{k\in\mathbb{N}}$ of the dual unit ball B_{X^*} there are $x \in X$ and $x^* \in B_{X^*}$ such that $|x_k^*(x)| \leqslant 1$ for all k and $x^*(x) > A$.*

Proof. Clearly we can assume without loss of generality that the uncountable ω-independent family is bounded. It is shown in [7] that given any $\varepsilon > 0$, every uncountable ω-independent family contains an uncountable subset $(e_i)_{i\in I}$ such that there exists a bounded subset $(e_i^*)_{i\in I}$ of X^* which satisfies $e_i^*(e_i) = 1$ for all i and $|e_i^*(e_j)| < \varepsilon$ for all $i \neq j$.

We pick $n \in \mathbb{N}$ and apply the above to $\varepsilon = n^{-2}$. We define a closed bounded balanced subset C of X as follows:

$$C = \overline{\operatorname{conv}}\left\{n^{-1}\sum_{i\in J}\eta_i e_i;\ |J| \leqslant n,\ |\eta_i| = 1\right\}.$$

Let $(x_k^*)_{k\in\mathbb{N}}$ be a sequence in X^* such that $\sup\limits_C(x_k^*) \leqslant 1$ for all k. Let

$$E_k = \{i \in I;\ |x_k^*(e_i)| > 1\}.$$

It is clear that $|E_k| \leqslant 2n$ for all k, and thus there exists $i \in I \setminus \bigcup\limits_{k\in\mathbb{N}} E_k$. We pick this index i, and we note that

$$\left|e_i^*\left(n^{-1}\sum_{j\in J}\eta_j e_j\right)\right| \leqslant n^{-1} + n^{-2} \tag{1}$$

for any set J with $|J| \leqslant n$. We let now

$$x^* = \frac{ne_i^*}{1+n^{-1}}.$$

It follows from (1) that $\sup_C(x^*) \leqslant 1$. On the other hand,

$$x^*(e_i) = \frac{n}{1+n^{-1}}$$

while $|x_k^*(e_i)| \leqslant 1$ for all $k \in \mathbb{N}$. If n is chosen in such a way that

$$\frac{n}{1+n^{-1}} > A$$

we reach our conclusion, except that the convex set C is not necessarily the unit ball of an equivalent norm. For completing the proof, we therefore consider the equivalent norm $\|\cdot\|$ whose unit ball is

$$B = \overline{C + \alpha B_X}$$

where B_X is the original unit ball and $\alpha > 0$ is properly chosen. Any sequence (x_k^*) such that $\|x_k^*\| \leqslant 1$ satisfies in particular $\sup_C(x_k^*) \leqslant 1$ for all k and we can apply the above argument. The linear form x^* is such that

$$\|x^*\| \leqslant 1 + \frac{nL\alpha}{1+n^{-1}}$$

with $L = \sup\{N(e_i^*);\ i \in I\}$, where N is the original norm. The lemma easily follows through renormalization by choosing $\alpha > 0$ small enough. □

Our main result is now easy to show.

Theorem 2.2. *Let X be a closed subspace of $l_\infty(\mathbb{N})$ which contains an uncountable ω-independent family. Then the diameter of the set of equivalent norms on X with respect to the Banach-Mazur distance is infinite.*

Proof. Let N be the norm on X which is induced by the canonical norm of $l_\infty(\mathbb{N})$. We denote by (p_k^*) the restrictions to X of the coordinate functionals on $l_\infty(\mathbb{N})$. Let T be any isomorphism between X equipped with the norm provided by Lemma 2.1 and X equipped with N. We may and do assume that T has norm 1 and then $\|T^*(x^*)\| \leqslant N(x^*)$ for all $x^* \in X^*$. Applying Lemma 2.1 to $x_k^* = T^*(p_k^*)$ provides $x \in X$ such that $N(T(x)) \leqslant 1$ but $\|x\| > A$. Therefore T^{-1} has norm greater than A. This concludes the proof since A is arbitrary. □

Let us recall that according to [10], a Banach space X is elastic if there exists $K \in \mathbb{R}$ such that when X is equipped with an arbitrary equivalent norm, then X with this new norm K-embeds into X. Isometrically universal spaces for a given density character are clearly elastic (with $K = 1$). Our proof shows that when X is renormed via Lemma 2.1 all its embeddings into $l_\infty(\mathbb{N})$ have large norms, and thus it yields:

Corollary 2.3. *Let X be a closed subspace of $l_\infty(\mathbb{N})$ which contains an uncountable ω-independent family. Then X is not elastic.*

Before stating the next corollary, which is the main motivation for this work, we recall that Martin's Maximum Axiom (MM) states that the intersection of ω_1 dense open subsets of any Čech-complete space P in the class M is dense in P, where M is the largest possible class of Čech-complete spaces for which this transfinite version of Baire's lemma can hold. The class M, which is identified in [3], contains in particular all Čech-complete spaces with the countable chain condition. Martin's Maximum is thus provably the strongest version of Martin's axiom consistent with ZFC. With this notation, we now have:

Corollary 2.4 (MM). *Let X be an infinite dimensional closed subspace of $l_\infty(\mathbb{N})$. Then the diameter of the set of equivalent norms on X with respect to the Banach-Mazur distance is infinite.*

Proof. If X is separable, this corollary is Johnson-Odell's theorem [10], which is of course a result from ZFC. If X is not separable, it is shown in [13] that under (MM) the space X contains an uncountable minimal system, and thus in particular an uncountable ω-independent family. It suffices now to apply Theorem 2.2 to reach the conclusion. □

We note that the argument also shows that under (MM), no non separable subspace of $l_\infty(\mathbb{N})$ is elastic.

Corollary 2.4 is clearly not the final satisfactory result one could expect. Let us therefore conclude this work with some questions.

Question 1. Is Corollary 2.4 a result from ZFC? It is certainly so for "decent" subspaces of $l_\infty(\mathbb{N})$. Indeed, it is shown in [2] (and in ZFC) that if a subspace X of $l_\infty(\mathbb{N})$ contains a weak* analytic subset which is not norm-separable then it has a quotient space which does not linearly embed into $l_\infty(\mathbb{N})$, and this implies the existence of renormings for which the space is far from subspaces of $l_\infty(\mathbb{N})$ (see Corollary III.3 in [2]). A similarity with Lemma 2.1 is that a topological assumption replaces the geometric information on linear independence. This applies in particular to weak* analytic subspaces of $l_\infty(\mathbb{N})$ (i.e. representable spaces, in the sense of [6]). This applies more generally to subspaces of $l_\infty(\mathbb{N})$ which belong to the projective hierarchy in the weak* topology, provided a suitable determinacy axiom is assumed ([5]).

Although an affirmative answer to Question 1 looks plausible, it should be noticed that one would need anyway to follow different lines. Indeed, what the above actually shows is, under (MM), that for every non separable subspace X of $l_\infty(\mathbb{N})$ and every $A > 0$ there is an equivalent norm on X such that the Banach-Mazur distance from X equipped with that norm to every isometric subspace of $l_\infty(\mathbb{N})$ is greater than A. This stronger statement fails if the Continuum Hypothesis (CH) is assumed, since Kunen's $C(K)$ space (see [12]) constructed under (CH) is isometric to a subspace of $l_\infty(\mathbb{N})$ when equipped with any equivalent norm, as shown in [9].

Question 2. The above comment motivates the following: is the Banach-Mazur diameter of the set of equivalent norms on Kunen's space infinite? Is Kunen's space elastic? We refer to [13] for more references and information on similar spaces, which the above questions concern as well.

Question 3. If there is an equivalent norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ is not isometric to a subspace of $l_\infty(\mathbb{N})$, does it follow that there exist equivalent norms on X whose Banach-Mazur distance to isometric subspaces of $l_\infty(\mathbb{N})$ is arbitrarily large? The above proof shows that the answer to this question is affirmative under (MM), since then both the statements amount to saying that X is not separable (see Theorem 8.24 in [8]). However, it is natural to wonder if it can be decided in ZFC. An affirmative answer would probably request a geometric argument, comparable to Lemma 2.1, which would use ω_1-polyhedra instead of ω-independent families (see Theorem 8.19 in [8]).

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