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Lutz Volkmann

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A BOUND ON THE k-DOMINATION NUMBER OF A GRAPH

Lutz Volkmann, Aachen

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Abstract. Let G be a graph with vertex set $V(G)$, and let $k \geqslant 1$ be an integer. A subset $D \subseteq V(G)$ is called a *k-dominating set* if every vertex $v \in V(G) - D$ has at least k neighbors in D. The k-domination number $\gamma_k(G)$ of G is the minimum cardinality of a k-dominating set in G. If G is a graph with minimum degree $\delta(G) \geqslant k+1$, then we prove that

$$\gamma_{k+1}(G) \leqslant \frac{|V(G)| + \gamma_k(G)}{2}.$$

In addition, we present a characterization of a special class of graphs attaining equality in this inequality.

Keywords: domination, k-domination number, P_4-free graphs

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Let G be a finite and simple graph with vertex set $V(G)$. The *neighborhood* $N_G(v) = N(v)$ of a vertex $v \in V(G)$ is the set of vertices adjacent to v, and the number $d_G(v) = d(v) = |N(v)|$ is the *degree* of the vertex v. By $n(G) = n$, $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *order*, the *maximum degree* and the *minimum degree* of the graph G, respectively. If $A \subseteq V(G)$, then $G[A]$ is the graph induced by the vertex set A. Denote by $\alpha(G)$ the *independence number* and by $\omega(G)$ the *clique number* of a graph G, respectively. We denote by K_n the *complete graph* of order n and by $K_{r,s}$ the *complete bipartite graph* with partite sets X and Y such that $|X| = r$ and $|Y| = s$. Next assume that G_1 and G_2 are two graphs with disjoint vertex sets. The *corona* $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i-th vertex of G_1 is adjacent to every vertex of the i-th copy of G_2. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* $G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

A set $D \subseteq V(G)$ is a *k-dominating set* of G if every vertex of $V(G) - D$ has at least $k \geqslant 1$ neighbors in D. The *k-domination number* $\gamma_k(G)$ of G is the cardinality of a minimum k-dominating set. If D is a k-dominating set of G with $|D| = \gamma_k(G)$, then we say that D is a $\gamma_k(G)$-set.

In [6] and [7], Fink and Jacobson introduced the concept of k-domination. The case $k = 1$ leads to the classical *domination number* $\gamma(G) = \gamma_1(G)$. Bounds on the k-domination number can be found, for example, in [2], [3], [4], [5], [9], [12], [13], [14]. Now we prove the following relation between the $(k+1)$-domination and the k-domination numbers.

Theorem 1. *If G is a graph and k an integer such that $1 \leqslant k \leqslant \delta(G) - 1$, then*

$$\gamma_{k+1}(G) \leqslant \frac{n(G) + \gamma_k(G)}{2}.$$

Proof. Let S be a $\gamma_k(G)$-set, and let A be the set of isolated vertices in the subgraph induced by the vertex set $V(G) - S$. Then the subgraph induced by $V(G) - (S \cup A)$ contains no isolated vertices. If D is a minimum dominating set of $G[V(G) - (S \cup A)]$, then the well-known inequality of Ore [10] implies

$$|D| \leqslant \frac{|V(G) - (S \cup A)|}{2} \leqslant \frac{|V(G) - S|}{2} = \frac{n(G) - \gamma_k(G)}{2}.$$

Since $\delta(G) \geqslant k+1$, every vertex of A has at least $k+1$ neighbors in S, and therefore $D \cup S$ is a $(k+1)$-dominating set of G. Thus we obtain

$$\gamma_{k+1}(G) \leqslant |S \cup D| \leqslant \gamma_k(G) + \frac{n(G) - \gamma_k(G)}{2} = \frac{n(G) + \gamma_k(G)}{2},$$

and the desired inequality is proved. □

Corollary 2 (Blidia, Chellali, Volkmann [1] 2006)**.** *If G is a graph of minimum degree $\delta(G) \geqslant 2$, then*

$$\gamma_2(G) \leqslant \frac{n(G) + \gamma(G)}{2}.$$

The following family of graphs demonstrates that the bound in Theorem 1 is the best possible.

Example 3. Let H be a connected graph, and let $k \geqslant 1$ be an integer. If $G = H \circ K_{k+1}$, then $n(G) = (k+2)n(H)$, and it is easy to verify that

$$\gamma_{k+1}(G) = n(H)(k+1) = \frac{n(G) + \gamma_k(G)}{2}.$$

The graphs G of even order n and without isolated vertices with $\gamma(G) = n/2$ have been characterized independently by Payan and Xuong [11] and Fink, Jacobson, Kinch and Roberts [8].

Theorem 4 (Payan, Xuong [11] 1982 and Fink, Jacobson, Kinch, Roberts [8] 1985)**.** *Let G be a graph of even order n without isolated vertices. Then $\gamma(G) = n/2$ if and only if each component of G is either a cycle C_4 of length four or the corona $F \circ K_1$ of some connected graph F.*

A graph is P_4-free if and only if it contains no induced subgraph isomorphic to the path P_4 of order four. A graph is $(K_4 - e)$-free if and only if it contains no induced subgraph isomorphic to the graph $K_4 - e$, where e is an arbitrary edge of the complete graph K_4. The graph $\overline{G}$ denotes the complement of the graph G. Next we present a characterization of some special graphs attaining equality in Theorem 1.

Theorem 5. *Let G be a connected P_4-free graph such that $\overline{G}$ is $(K_4 - e)$-free. If k is an integer such that $1 \leqslant k \leqslant \delta(G) - 1$, then*

$$\gamma_{k+1}(G) = \frac{n(G) + \gamma_k(G)}{2}$$

if and only if

1. *$G = K_{k+2}$, or*
2. *$\overline{G} = H \cup 2K_{1,1}$ such that $n(H) = k$ and all components of H are isomorphic to $K_{1,1}$ or to $K_{2,2}$, or*
3. *$G = (Q_1 \cup Q_2) + F$, where Q_1, Q_2 and F are three pairwise disjoint graphs with $1 \leqslant |V(F)| \leqslant k$, $\alpha(F) \leqslant 2$, and Q_1 and Q_2 are cliques with $|V(Q_1)| = |V(Q_2)| = k + 2 - |V(F)|$ such that $|V(F)| \leqslant 2$ or $\alpha(F) = 2$ and $F = K_k - M$, where M is a perfect matching of F or $\alpha(F) = 2$ and $|V(F)| = k - t$ for $0 \leqslant t \leqslant k - 3$ with $k \geqslant 3t + 4$ and all components of $\overline{F}$ are isomorphic to $K_{t+2,t+2}$.*

Proof. Assume that $\gamma_{k+1}(G) = (n(G) + \gamma_k(G))/2$. Following the notation used in the proof of Theorem 1 we obtain $|D| = \frac{1}{2}|V(G) - S|$, and we observe that $S \cup D$ is a $\gamma_{k+1}(G)$-set. It follows that $G[V(G) - S]$ has no isolated vertices and so by Theorem 4, each component of $G[V(G) - S]$ is either a cycle C_4 or the corona of some connected graph. Using the hypothesis that G is P_4-free, we deduce that each component of $G[V(G) - S]$ is isomorphic to C_4 or to K_2. Since $\overline{G}$ is $(K_4 - e)$-free, there remain exactly the three cases that $G[V(G) - S]$ is isomorphic to K_2, to C_4 or to $2K_2$.

Case 1: First assume that $G[V(G) - S] = K_2$. Suppose that G has an independent set Q of size at least two. Then the hypothesis $\delta(G) \geqslant k + 1$ implies that

$V(G) - Q$ is a $(k+1)$-dominating set of G of size $n - |Q| < |S \cup D| = n - 1$, a contradiction. Therefore $\alpha(G) = 1$ and thus G is isomorphic to the complete graph K_{k+2}.

Case 2: Secondly, assume that $G[V(G) - S]$ is a cycle $C_4 = x_0x_1x_2x_3x_0$. In the following the indices of the vertices x_i are taken modulo 4. Recall that $S \cup D$ is a $\gamma_{k+1}(G)$-set, and D contains two vertices of the cycle C_4. Clearly, since S is a $\gamma_k(G)$-set, every vertex of the cycle C_4 has degree at least $k+2$. Suppose that $d_G(x_i) \geqslant k+3$ for an $i \in \{0,1,2,3\}$. Then $\{x_{i+2}\} \cup S$ is a $(k+1)$-dominating set of G of size $|S| + 1 < |S \cup D| = |S| + 2$, a contradiction. Thus $d_G(x_i) = k+2$ for every $i \in \{0,1,2,3\}$. If Q is an independent set of G, then $|Q| \leqslant 2$, for otherwise the hypothesis $\delta(G) \geqslant k+1$ implies that $V(G) - Q$ is a $(k+1)$-dominating set of G of size $|V(G) - Q| < |S \cup D| = n(G) - 2$, a contradiction too. Since there are two non-adjacent vertices on the cycle C_4 and G is P_4-free, it follows that every vertex of S has at least three neighbors on the cycle C_4.

Subcase 2.1: Assume that $\alpha(G[S]) = 1$. Then the subgraph induced by S is complete and $|S| \geqslant k$. If $|S| = k$, then we observe that every vertex of S has exactly four neighbors on the cycle C_4. Thus, in each case, we deduce that $d_G(y) \geqslant k+3$ for every $y \in S$. But then for any subset W of S of size three, the set $V(G) - W$ is a $(k+1)$-dominating set of G of size less than $|S \cup D|$, a contradiction.

Subcase 2.2: Assume that $\alpha(G[S]) = 2$. Suppose that there exists a vertex $w \in S$ with at least $k-1$ neighbors in S. Then, since $|N(w) \cap V(C_4)| \geqslant 3$, say $\{x_0, x_1, x_2\} \subseteq N(w) \cap V(C_4)$, we observe that $(S - \{w\}) \cup \{x_0, x_2\}$ is a $(k+1)$-dominating set of G of size $|S| + 1 < |S \cup D|$, a contradiction. Thus every vertex of S has at most $k-2$ neighbors in S.

Let $S = X \cup Y$ be such that every vertex of X has exactly three and every vertex of Y exactly 4 neighbors on C_4. We shall show that $X = \emptyset$. If $X \neq \emptyset$, then let $S_{x_i} \subseteq X$ be the set of vertices such that no vertex of S_{x_i} is adjacent to x_{i+2} for $i \in \{0,1,2,3\}$. Because of $\alpha(G) = 2$, we observe that the set $S_{x_i} \cup \{x_i\}$ induces a complete graph for each $i \in \{0,1,2,3\}$. In additon, since G is P_4-free, it is straightforward to verify that all vertices of $X \cup V(C_4)$ are adjacent to all vertices of Y and that $S_{x_i} \cup S_{x_{i+1}} \cup \{x_i, x_{i+1}\}$ induces a complete graph for each $i \in \{0,1,2,3\}$. Now assume, without loss of generality, that $S_{x_0} \neq \emptyset$, and let $w \in S_{x_0}$. Using the fact that every vertex of S has at most $k-2$ neighbors in S, we conclude that $d_G(w) \leqslant k+1$. Furthermore, we observe that $d_G(w) = d_G(x_0)$. But since we have seen above that $d_G(x_0) = k+2$, we arrive at a contradiction.

Hence we have shown that $X = \emptyset$. Since $d_G(x_i) = k+2$ for every $i \in \{0,1,2,3\}$, it follows that $|S| = k$. If we define $H = \overline{G[S]}$, then we deduce that $\omega(H) = 2$ and $\delta(H) \geqslant 1$. In addition, the hypotheses $\delta(G) \geqslant k+1$ and $n(G) = k+4$ lead to $\Delta(H) \leqslant 2$. Since H is also P_4-free, H contains no induced cycle of odd length. Using

$\omega(H) = 2$, we deduce that H is a bipartite graph. Now let H_i be a component of H. If H_i is not complete, then H_i contains a P_4, a contradiction. Thus the components of H consist of $K_{1,1}$, $K_{1,2}$ or $K_{2,2}$.

If $K_{1,2}$ is a component of H, then $V(G) - V(K_{1,2})$ is a $(k+1)$-dominating set of G of size $n-3$, a contradiction.

Case 3: Thirdly assume that $G[V(G) - S] = 2K_2$. Let $2K_2 = J_1 \cup J_2 = J$ be such that $V(J_1) = \{u_1, u_2\}$ and $V(J_2) = \{u_3, u_4\}$. If $\alpha(G) \geqslant 3$, then we obtain the contradiction $\gamma_{k+1}(G) \leqslant n-3$. Thus $\alpha(G) = 2$. Since S is a $\gamma_k(G)$-set, every vertex of J has degree at least $k+1$. Suppose that $d_G(u_1) \geqslant k+2$ and $d_G(u_2) \geqslant k+2$. Then $\{u_3\} \cup S$ is a $(k+1)$-dominating set of G of size $|S|+1 < |S \cup D| = |S|+2$, a contradiction. Thus J_1 contains at least one vertex of degree $k+1$, and for reason of symmetry, also J_2 contains a vertex of degree $k+1$. Since $\alpha(G) = 2$, every vertex of S has at least two neighbors in J_1 or in J_2. Now let $x \in S$. If x has two neighbors in J_i and one neighbor in J_{3-i} for $i = 1, 2$, then the hypothesis that G is P_4-free implies that x is adjacent to each vertex of J. Consequently, S can be partitioned into three subsets S_1, S_2 and A such that all vertices of S_1 are adjacent to all vertices of J_1 and there is no edge between S_1 and J_2, all vertices of S_2 are adjacent to all vertices of J_2 and there is no edge between S_2 and J_1, all vertices of A are adjacent to all vertices of J. Since G is P_4-free, it follows that there is no edge between S_1 and S_2, and that all vertices of S_i are adjacent to all vertices of A for $i = 1, 2$. Furthermore, $\alpha(G) = 2$ shows that $G[S_1]$ and $G[S_2]$ are cliques. Altogether we see that $d_G(u_i) = k+1$ for each $i \in \{1, 2, 3, 4\}$ and therefore $|S_1| + |A| = |S_2| + |A| = k$. It follows that $|S_1| = |S_2|$ and $|S| + |A| = 2k$. Since G is connected, we deduce that $|A| \geqslant 1$ and so $1 \leqslant |A| \leqslant k$. If we define $F = G[A]$ and $Q_i = G[S_i \cup V(J_i)]$ for $i = 1, 2$, then we derive the desired structure, since $\alpha(G[A]) \leqslant 2$.

Assume that $|V(F)| \geqslant 3$ and $\alpha(F) = 1$. If x_1, x_2, x_3 are three arbitrary vertices in F, then let $S_0 = V(G) - \{x_1, x_2, x_3\}$. If $d_G(x_i) \geqslant k+3$ for each $i = 1, 2, 3$, then S_0 is a $(k+1)$-dominating set of G, a contradiction. Otherwise, we have $n-1 = d_G(x_i) \leqslant k+2$ for at least one $i \in \{1, 2, 3\}$ and so $n \leqslant k+3$, a contradiction to $n \geqslant k+4$.

Assume next that $|V(F)| \geqslant 3$ and $\alpha(F) = 2$. As we have seen in Case 2, all components of $\overline{F}$ are complete bipartite graphs.

Subcase 3.1: Assume that $K_{1,1}$ is the greatest component of $\overline{F}$. Let u and v be the two vertices of the complete bipartite graph $K_{1,1}$. If $n \geqslant k+5$, then let w be a further vertex in F. It is easy to verify that $V(G) - \{u, v, w\}$ is a $(k+1)$-dominating set of G of size $n-3$, a contradiction. If $n = k+4$ and there exists a vertex w in F of degree $k+3$, then $V(G) - \{u, v, w\}$ is a $(k+1)$-dominating of G of size $n-3$, a contradiction. Thus $F = K_k - M$, where M is a perfect matching of F.

S u b c a s e 3.2: Assume that $|V(F)| = k - t$ for $0 \leqslant t \leqslant k-3$ and $\overline{F}$ contains a component $K_{p,q}$ with $1 \leqslant p \leqslant q$ and $p+q \geqslant 3$. Let $\{v_1, v_2, \ldots, v_q\}$ and $\{u_1, u_2, \ldots, u_p\}$ be a partition of $K_{p,q}$.

If $K_{1,s} \subseteq \overline{F}$ with $s \geqslant t+3$, then $\delta(G) \leqslant k$, a contradiction to $\delta(G) \geqslant k+1$. Thus $q \leqslant t+2$.

If $q \leqslant t+1$ or $q = t+2$ and $p \leqslant t+1$, then it is easy to see that $V(G) - \{u_1, v_1, v_2\}$ is a $(k+1)$-dominating set of G of size $n-3$, a contradiction. So all components of $\overline{F}$ are isomorphic to $K_{t+2,t+2}$ and $k \geqslant 3t+4$.

Conversely, if $G = K_{k+2}$, then obviously $\gamma_k(G) = k$, $\gamma_{k+1}(G) = k+1$ and so $\gamma_{k+1}(G) = (\gamma_k(G) + n(G))/2$.

Now let $\overline{G} = H \cup 2K_{1,1}$ be such that $n(H) = k$ and the components of H are complete bipartite graphs $K_{1,1}$ or $K_{2,2}$. This yields $k+1 \leqslant d_G(u) \leqslant k+2$ for every $u \in V(G)$, and G contains a cycle C on four vertices, where each vertex of C has degree $k+2$ in G.

Clearly, $V(H)$ is a k-dominating set of G and so $\gamma_k(G) \leqslant n(G) - 4$. Since $n(G) = k+4$, we observe that $\gamma_k(G) \geqslant k = n(G) - 4$ and thus $\gamma_k(G) = n(G) - 4$.

Now let us prove that $\gamma_{k+1}(G) = n(G) - 2$. Since $n(G) = k+4$, it follows that $\gamma_{k+1}(G) \geqslant k+1 = n(G) - 3$. Suppose that D is a $\gamma_{k+1}(G)$-set such that $|D| = n - 3 = k+1$. Then every vertex of $V(G) - D$ is adjacent to every vertex in D. Since every vertex has degree at most $k+2$ in G, no vertex of $V(G) - D$ has two neighbors in $V(G) - D$. Moreover, since $\alpha(G) = 2$, the subgraph $G[V(G) - D]$ is formed by two adjacent vertices x, y and an isolated vertex w. Hence the vertices x, y and w induce a $K_{1,2}$ in $\overline{G}$, a contradiction to the hypothesis. Thus $|D| \geqslant n(G) - 2$ and the equality follows from the fact that $V(G)$ minus any two non-adjacent vertices of C is a $(k+1)$-dominating set of G. Therefore $\gamma_{k+1}(G) = n(G) - 2 = (\gamma_k(G) + n(G))/2$.

Finally, let $G = (Q_1 \cup Q_2) + F$, where Q_1, Q_2 and F are three pairwise disjoint graphs with $1 \leqslant |V(F)| \leqslant k$, $\alpha(F) \leqslant 2$, and Q_1 and Q_2 are cliques with $|V(Q_1)| = |V(Q_2)| = k + 2 - |V(F)|$ such that $|V(F)| \leqslant 2$ or $\alpha(F) = 2$ and $F = K_k - M$, where M is a perfect matching of F or $\alpha(F) = 2$ and $|V(F)| = k - t$ for $0 \leqslant t \leqslant k-3$ with $k \geqslant 3t+4$ and all components of $\overline{F}$ are isomorphic to $K_{t+2,t+2}$.

Let D be a $(k+1)$-dominating set of G. Since each vertex of Q_i has degree $k+1$, the set $V(G) - D$ contains at most one vertex of Q_i for every $i = 1, 2$. Moreover, if $(V(G) - D) \cap V(Q_i) \neq \emptyset$, then $V(F) \subseteq D$. Now suppose that $\gamma_{k+1}(G) \leqslant n - 3$ and assume, without loss of generality, that $V(G) - D = \{u, v, w\}$. Then as noted above $V(Q_1) \cup V(Q_2) \subseteq D$, and hence the vertices u, v, w belong to $V(F)$. It follows that $|V(F)| \geqslant 3$.

Assume next that $\alpha(F) = 2$. This implies that at least two vertices of $V(G) - D$ are adjacent in G.

First assume that $F = K_k - M$, where M is a perfect matching of F. Note that $n = k + 4$ and $|D| = k + 1$. It follows that $\{u, v, w\}$ induces either a path P_3 or a clique K_3 with the center vertex, say v, in G. But then v has at most k neighbors in D, a contradiction.

Assume now that $\alpha(F) = 2$ and $|V(F)| = k-t$ for $0 \leqslant t \leqslant k-2$ with $k \geqslant 3t+4$ and all components of $\overline{F}$ are isomorphic to $K_{t+2,t+2}$. Note that in this case $n = k+4+t$ and so $|D| = n - 3 = k + 1 + t$. Assume, without loss of generality, that u and v are adjacent in G. This leads to $|N_G(u) \cap D| \leqslant k$, a contradiction.

Altogether, we have shown that $\gamma_{k+1}(G) = n - 2$. Finally, it is a simple matter to obtain $\gamma_k(G) = n - 4$, and the proof of Theorem 5 is complete. □

References

[1] *M. Blidia, M. Chellali and L. Volkmann*: Bounds of the 2-domination number of graphs. Utilitas Math. *71* (2006), 209–216.
[2] *Y. Caro and Y. Roditty*: A note on the k-domination number of a graph. Int. J. Math. Sci. *13* (1990), 205–206.
[3] *B. Chen and S. Zhou*: Upper bounds for f-domination number of graphs. Discrete Math. *185* (1998), 239–243.
[4] *E. J. Cockayne, B. Gamble and B. Shepherd*: An upper bound for the k-domination number of a graph. J. Graph Theory *9* (1985), 533–534.
[5] *O. Favaron, A. Hansberg and L. Volkmann*: On k-domination and minimum degree in graphs. J. Graph Theory *57* (2008), 33–40.
[6] *J. F. Fink and M. S. Jacobson*: n-domination in graphs Graph Theory with Applications to Algorithms and Computer Science. John Wiley and Sons, New York, 1985, pp. 283–300.
[7] *J. F. Fink and M. S. Jacobson*: On n-domination, n-dependence and forbidden subgraphs. Graph Theory with Applications to Algorithms and Computer Science, John Wiley and Sons, New York, 1985, pp. 301–311.
[8] *J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts*: On graphs having domination number half their order. Period. Math. Hungar. *16* (1985), 287–293.
[9] *A. Hansberg and L. Volkmann*: Upper bounds on the k-domination number and the k-Roman domination number. Discrete Appl. Math. *157* (2009), 1634–1639.
[10] *O. Ore*: Theory of Graphs. vol. 38, Amer. Math. Soc. Colloq. Publ., 1962.
[11] *C. Payan and N. H. Xuong*: Domination-balanced graphs. J. Graph Theory *6* (1982), 23–32.
[12] *C. Stracke and L. Volkmann*: A new domination conception. J. Graph Theory *17* (1993), 315–323.
[13] *S. Zhou*: On f-domination number of a graph. Czech. Math. J. *46* (1996), 489–499.
[14] *S. Zhou*: Inequalities involving independence domination, f-domination, connected and total f-domination numbers. Czech. Math. J. *50* (2000), 321–330.

Author's address: Lutz Volkmann, Lehrstuhl II für Mathematik, RWTH Aachen University, Templergraben 55, D-52056 Aachen, Germany, e-mail: `volkm@math2.rwth-aachen.de`.