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COMPOSITION-DIAMOND LEMMA FOR MODULES

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Abstract. We investigate the relationship between the Gröbner-Shirshov bases in free associative algebras, free left modules and "double-free" left modules (that is, free modules over a free algebra). We first give Chibrikov's Composition-Diamond lemma for modules and then we show that Kang-Lee's Composition-Diamond lemma follows from it. We give the Gröbner-Shirshov bases for the following modules: the highest weight module over a Lie algebra sl_2, the Verma module over a Kac-Moody algebra, the Verma module over the Lie algebra of coefficients of a free conformal algebra, and a universal enveloping module for a Sabinin algebra. As applications, we also obtain linear bases for the above modules.

Keywords: Gröbner-Shirshov basis, module, Lie algebra, Kac-Moody algebra, conformal algebra, Sabinin algebra

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1. Introduction

In literature, the Composition-Diamond lemma for modules was first proved by S.-J. Kang and K.-H. Lee in [16], [17]. According to their approach, a Gröbner-Shirshov basis of a cyclic module M over an algebra A is a pair (S,T), where S is the defining relations of $A=k\langle X|S\rangle$ and T is the defining relations of the A-module ${}_AM=\mathrm{mod}_A\langle e|T\rangle$. Then Kang-Lee's lemma says that (S,T) is a Gröbner-Shirshov pair for the A-module ${}_AM=\mathrm{mod}_A\langle e|T\rangle$ if S is a Gröbner-Shirshov basis of A and T is closed under the right-justified composition with respect to S and T, and for $f\in S$, $g\in T$ such that $(f,g)_w$ is defined, $(f,g)_w\equiv 0 \bmod(S,T;w)$. They also gave some applications of this lemma for irreducible modules over $sl_n(k)$ in [17], the Specht modules over the Hecke algebras and the Ariki-Koike algebras in [18], [19]. Some years later, E. S. Chibrikov [11] proposed a new Composition-Diamond lemma

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for modules that treats any module as a factor module of a "double-free" module $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ over a free algebra $k\langle X\rangle$. When using this approach, any A-module ${}_AM$ is presented in the form

$$ {}_AM = \mathrm{mod}_{k\langle X\rangle}\langle Y|SX^*Y, T\rangle, $$

where $A = k\langle X|S\rangle$, ${}_AM = \mathrm{mod}_A\langle Y|T\rangle$, X^* is the free monoid generated by X.

The aim of this paper is to describe a relationship between the Gröbner-Shirshov bases in free associative algebras, free left modules and "double-free" left modules, respectively. We also give some applications of the Composition-Diamond lemma to "double-free" modules. The paper is organized as follows. In Section 2, we deal with the Gröbner-Shirshov bases and the Composition-Diamond lemma for left ideals of a free algebra. Actually, this is a special case of cyclic "double-free" modules. By using this lemma, we can easily get the well-known Cohn's theorem (see [12] p. 333). In Section 3, we give a relationship between the Gröbner-Shirshov bases in free associative algebras, free left modules and "double-free" modules, respectively. In particular, we give a proof of Chibrikov's Composition-Diamond lemma and formulate Kang-Lee's Composition-Diamond lemma. Then we show that the latter follows from the former. In Sections 4, 5, 6 and 7, we give Gröbner-Shirshov bases for the highest weight module over the Lie algebra sl_2, the Verma module over a Kac-Moody algebra, the Verma module over the Lie algebra of coefficients of a free conformal algebra, and a universal enveloping module for a Sabinin algebra, respectively. As applications, in particular, we also obtain linear bases for the above modules. For the universal enveloping module for a Sabinin algebra it was done before by Perez-Izquierdo [21] using another method.

Let k be a field and X a set. Let X^* be the free monoid generated by X and $k\langle X\rangle$ the free associative algebra over X and k. For a word $w \in X^*$, we denote the length of w by $\deg(w)$. Suppose that $<$ is a well ordering on X^*. For any polynomial f, let $\overline{f}$ be the leading term of f. If the coefficient of $\overline{f}$ is 1, then this polynomial is said to be monic. The following lemma will be used in Sections 4, 5 and 6.

Lemma 1.1 ([9], [10], [3]). *Let* $\mathrm{Lie}(X)$ *be a free Lie algebra over a set* X *and a field* k. *Let* $S \subset \mathrm{Lie}(X)$ *be a nonempty set of monic Lie polynomials. Then, with a deg-lex ordering on* X^*, S *is a Gröbner-Shirshov basis in* $\mathrm{Lie}(X)$ *if and only if* $S^{(-)}$ *is a Gröbner-Shirshov basis in* $k\langle X\rangle$ *where* $S^{(-)}$ *is just* S *but all* $[xy]$ *substituted by* $xy - yx$.

2. Composition-Diamond lemma for left ideals of a free algebra

Let X be a set and $<$ a well ordering on X^*. Let $S \subset k\langle X\rangle$ in which every $s \in S$ is monic. Then $k\langle X\rangle S$ is the left ideal of $k\langle X\rangle$ generated by S. For the left ideal $k\langle X\rangle S$, we define the compositions in S as follows.

Definition 2.1. For any $f, g \in S$, if $\overline{f} = a\overline{g}$ for some $a \in X^*$, then the composition of f and g is defined to be $(f,g)_{\overline{f}} = f - ag$. The transformation $f \to f - ag$ is called the elimination of the leading word (ELW) of g in f. If $(f,g)_{\overline{f}} = \sum \alpha_i a_i s_i$, where $\alpha_i \in k$, $a_i \in X^*$, $s_i \in S$ and $a_i\bar{s}_i < \overline{f}$, then the composition $(f,g)_{\overline{f}}$ is trivial modulo $(S, \overline{f})$, denoted by $(f,g)_{\overline{f}} \equiv 0 \mod(S, \overline{f})$.

Definition 2.2. Let $S \subset k\langle X\rangle$ with each $s \in S$ monic. Then S is called a Gröbner-Shirshov basis of the left ideal $k\langle X\rangle S$ if all compositions are trivial modulo S. The set S is now called the minimal Gröbner-Shirshov basis of $k\langle X\rangle S$ if there exists no composition of polynomials in S, i.e., $\overline{f} \neq a\overline{g}$ for any $a \in X^*$, f, $g \in S$, $f \neq g$.

A well ordering $<$ on X^* is left compatible if for any $u, v \in X^*$,

$$u > v \Rightarrow wu > wv \text{ for all } w \in X^*.$$

That $<$ is right compatible can be similarly defined. Moreover, $<$ is monomial if it is both left and right compatible.

Now we formulate the Composition-Diamond lemma for left ideals of a free associative algebra.

Lemma 2.3 (Composition-Diamond lemma for left ideals of $k\langle X\rangle$)**.** *Let $S \subset k\langle X\rangle$ in which every $s \in S$ is monic and let $<$ be a left compatible well ordering on X^*. Then the following statements are equivalent:*

(1) *S is a Gröbner-Shirshov basis of the left ideal $k\langle X\rangle S$.*

(2) *If $0 \neq f \in k\langle X\rangle S$, then $\overline{f} = a\bar{s}$ for some $a \in X^*$, $s \in S$.*

(2′) *If $0 \neq f \in k\langle X\rangle S$, then $f = \sum \alpha_i a_i s_i$ with $a_1\bar{s}_1 > a_2\bar{s}_2 > \ldots$, where each $\alpha_i \in k$, $a_i \in X^*$, $s_i \in S$.*

(3) $\mathrm{Irr}(S) = \{w \in X^* \mid w \neq a\bar{s},\ a \in X^*,\ s \in S\}$ *is a k-linear basis for the factor $k\langle X\rangle$-module ${}_{k\langle X\rangle}k\langle X\rangle / k\langle X\rangle S$.*

Lemma 2.3 is a special case of Lemma 3.2 (see the next section).

Assume that S is a Gröbner-Shirshov basis for the left ideal $k\langle X\rangle S$ of $k\langle X\rangle$. We may assume that the leading terms of the elements in S are different. Then

$$S_1 = \{s \in S \mid \bar{s} \neq a\bar{t},\ a \in X^*,\ t \in S \setminus \{s\}\}$$

is clearly a minimal Gröbner-Shirshov basis for the left ideal $k\langle X\rangle S$. Then $k\langle X\rangle S$ is a free $k\langle X\rangle$-module with the basis S_1 by Lemma 2.3. Thus, we get the following well-known result.

Corollary 2.4 (Cohn [12]). *Any left (right) ideal of a free algebra $k\langle X\rangle$ is a free left (right) $k\langle X\rangle$-module.*

Now, we quote below Kang-Lee's Composition-Diamond lemma. Let $S, T \subset k\langle X\rangle$, $A = k\langle X|S\rangle$, let ${}_AM = {}_A A/A(T + \mathrm{Id}(S))$ be a left A-module and $f, g \in k\langle X\rangle$. In Kang-Lee's paper [16], the composition of f and g is defined as follows.

Definition 2.5 ([16], [17]). Let $<$ be a monomial ordering on X^*.

(a) If there exist $a, b \in X^*$ such that $w = \overline{f}a = b\overline{g}$ with $\deg(\overline{f}) > \deg(b)$, then the composition of intersection is defined to be $(f, g)_w = fa - bg$.

(b) If there exist $a, b \in X^*$ such that $w = a\overline{f}b = \overline{g}$, then the composition of inclusion is defined to be $(f, g)_w = afb - g$.

(c) A composition $(f, g)_w$ is said to be right-justified if $w = \overline{f} = a\overline{g}$ for some $a \in X^*$.

If $f - g = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j$, where α_i, $\beta_j \in k$, a_i, b_i, $c_j \in X^*$, $s_i \in S$, $t_j \in T$ with $a_i\overline{s}_i b_i < w$ and $c_j\overline{t}_j < w$ for each i and j, then $f - g$ is called trivial with respect to S and T and denoted by $f \equiv g \mod(S, T; w)$. When $T = \emptyset$, we simply write $f \equiv g \mod(S, w)$. If for any $f, g \in S$, $(f, g)_w$ is defined and $f \equiv g \bmod(S, w)$, then we say S is closed under composition. Note that if this is the case, S is called a Gröbner-Shirshov basis in $k\langle X\rangle$ which was first introduced by Shirshov [26] (see also [1], [2]).

Remark. A Gröbner-Shirshov basis S in $k\langle X\rangle$ is called minimal if there is no inclusion composition in S. If a subset S of $k\langle X\rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner-Shirshov basis S^c in $k\langle X\rangle$. Such a process is called the Shirshov algorithm. If we delete from S^c all polynomials with the leading term containing the leading term of other polynomials in S^c as subwords, then we will get a minimal Gröbner-Shirshov basis equivalent to S^c.

Definition 2.6 ([16], [17]). Let S, T be monic subsets of $k\langle X\rangle$. We call (S, T) a Gröbner-Shirshov pair for the A-module ${}_AM = {}_A A/A(T + \mathrm{Id}(S))$, where $A = k\langle X|S\rangle$, if S is closed under composition, T is closed under the right-justified composition with respect to S and T, and for any $f \in S$, $g \in T$ and $w \in X^*$ such that if $(f, g)_w$ is defined (it means that $a\overline{f}b = c\overline{g}$, where a, b, $c \in X^*$, $f \in S$, $g \in T$ and $\deg(\overline{f}) > \deg(c)$), we have $(f, g)_w \equiv 0 \mod(S, T; w)$.

The following is Kang-Lee's Composition-Diamond lemma for a left module.

Theorem 2.7 ([16], [17]). *Let (S,T) be a pair of subsets of monic elements in $k\langle X\rangle$ and $A=k\langle X|S\rangle$ the associative algebra defined by S. Let ${}_AM={}_AA/A(T+\mathrm{Id}(S))$ be a left A-module defined by (S,T). If (S,T) is a Gröbner-Shirshov pair for the A-module ${}_AM$ and $p\in k\langle X\rangle T+\mathrm{Id}(S)$, then $\bar p=a\bar s b$ or $\bar p=c\bar t$, where $a,b,c\in X^*$, $s\in S$, $t\in T$.*

Lemma 2.3 is a special case of Theorem 2.7 when $S=\emptyset$.

3. Composition-Diamond lemma for "double-free" modules

Let X, Y be sets and $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ a free left $k\langle X\rangle$-module with the basis Y. Then $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle=\bigoplus_{y\in Y}k\langle X\rangle y$ is called a "double-free" module. We now define the Gröbner-Shirshov basis in $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$. Suppose that $<$ is a monomial ordering on X^*, $<$ a well ordering on Y and $X^*Y=\{uy\mid u\in X^*,y\in Y\}$. We define an ordering $\prec$ on X^*Y as follows: for any $w_1=u_1y_1,w_2=u_2y_2\in X^*Y$,

$$(*)\qquad w_1\prec w_2\Leftrightarrow u_1<u_2\quad\text{or } u_1=u_2,\ y_1<y_2.$$

It is clear that the ordering $\prec$ is left compatible in the sense

$$w\prec w'\Rightarrow aw\prec aw'\ \text{ for any } a\in X^*.$$

Let $S\subset\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ with each $s\in S$ monic. Then we define the composition in S only the inclusion composition which means $\overline{f}=a\overline{g}$ for some $a\in X^*$, where $f,g\in S$. If $(f,g)_{\overline{f}}=f-ag=\sum\alpha_ia_is_i$, where $\alpha_i\in k$, $a_i\in X^*$, $s_i\in S$ and $a_i\bar s_i\prec\overline{f}$, then this composition is called trivial modulo $(S,\overline{f})$ and is denoted by

$$(f,g)_{\overline{f}}\equiv 0\ \ \mathrm{mod}(S,\overline{f}).$$

Definition 3.1 ([11]). Let $S\subset\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ be a non-empty set with each $s\in S$ monic. Let the ordering $\prec$ be defined as before. Then we call S a Gröbner-Shirshov basis in the module $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ if all compositions in S are trivial modulo S.

The proof of the following lemma is basically taken from [11]. For the sake of convenience, we sketch the proof.

Lemma 3.2 ([11], Composition-Diamond lemma for "double-free" modules)**.** *Let $S \subset \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ be a non-empty set with each $s \in S$ monic and $\prec$ the ordering on X^*Y as before. Then the following statements are equivalent:*

(1) *S is a Gröbner-Shirshov basis in $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$.*

(2) *If $0 \neq f \in k\langle X\rangle S$, then $\overline{f} = a\bar{s}$ for some $a \in X^*$, $s \in S$.*

(2′) *If $0 \neq f \in k\langle X\rangle S$, then $f = \sum \alpha_i a_i s_i$ with $a_1\bar{s}_1 \succ a_2\bar{s}_2 \succ \ldots$, where each $\alpha_i \in k$, $a_i \in X^*$, $s_i \in S$.*

(3) *$\mathrm{Irr}(S) = \{w \in X^*Y \mid w \neq a\bar{s},\ a \in X^*,\ s \in S\}$ is a k-linear basis for the factor $\mathrm{mod}_{k\langle X\rangle}\langle Y|S\rangle = \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle / k\langle X\rangle S$.*

P r o o f. (1) $\Rightarrow$ (2). Suppose that $0 \neq f \in k\langle X\rangle S$. Then $f = \sum \alpha_i a_i s_i$ for some $\alpha_i \in k$, $a_i \in X^*$, $s_i \in S$. Let $w_i = a_i\bar{s}_i$ and $w_1 = w_2 = \ldots = w_l \succ w_{l+1} \succeq \ldots$. We now prove that $\overline{f} = a\bar{s}$ for some $a \in X^*$, $s \in S$, by using induction on l and w_1. If $l = 1$, then the result is clear. If $l > 1$, then $a_1\bar{s}_1 = a_2\bar{s}_2$. Thus, we may assume that $a_1 = a_2 a$, $\bar{s}_2 = a\bar{s}_1$ for some $a \in X^*$. Now, by (1),

$$a_1 s_1 - a_2 s_2 = a_2 a s_1 - a_2 s_2 = a_2(a s_1 - s_2) = a_2 \sum \beta_j b_j u_j = \sum \beta_j a_2 b_j u_j,$$

where $\beta_j \in k$, $b_j \in X^*$, $u_j \in S$ and $b_j\bar{u}_j \prec \bar{s}_2$. Therefore, $a_2 b_j \bar{u}_j \prec w_1$. By using induction on l and w_1, we obtain the result.

It is clear that (2) is equivalent to (2′).

(2) $\Rightarrow$ (3). For any $0 \neq f \in \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$, if $\overline{f} = u_1 \in \mathrm{Irr}(S)$, then $f = \beta_1 u_1 + \ldots$. If $\overline{f} \notin \mathrm{Irr}(S)$, then $f = \alpha_1 a_1 s_1 + \ldots$. Consequently, f can be expressed by

$$f = \sum \alpha_i a_i s_i + \sum \beta_j u_j,$$

where $\alpha_i, \beta_j \in k$, $a_i \in X^*$, $s_i \in S$ and $u_j \in \mathrm{Irr}(S)$. Then $\mathrm{Irr}(S)$ generates the factor module. Moreover, if $0 \neq \sum \alpha_i a_i s_i = \sum \beta_j u_j$, where $a_i \in X^*$, $s_i \in S$, $u_j \in \mathrm{Irr}(S)$, $a_1\bar{s}_1 \succ a_2\bar{s}_2 \succ \ldots$ and $u_1 \succ u_2 \succ \ldots$, then $u_1 = a_1\bar{s}_1$, which is clearly a contradiction. Hence, $\mathrm{Irr}(S)$ is a k-linear basis of the factor module.

(3) $\Rightarrow$ (1). For any $f, g \in S$, suppose that $\overline{f} = a\overline{g}$. Since $(f,g)_{\overline{f}} \in k\langle X\rangle S$, by (3) we have $(f,g)_{\overline{f}} = f - ag = \sum \alpha_i a_i s_i$, where $s_i \in S$, $a_i \in X^*$ and $a_i\bar{s}_i \preceq \overline{(f,g)_{\overline{f}}} \prec \overline{f}$. Now, it is clear that S is a Gröbner-Shirshov basis in $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$. $\square$

Remark. We view $k\langle X\rangle$ as a free left $k\langle X\rangle$-module with one generator e. Then $\mathrm{mod}_{k\langle X\rangle}\langle e\rangle = k\langle X\rangle e = {}_{k\langle X\rangle}k\langle X\rangle$ is a cyclic $k\langle X\rangle$-module. If $S \subset k\langle X\rangle$, then $k\langle X\rangle S$ is a left ideal of $k\langle X\rangle$ which is also a left $k\langle X\rangle$-submodule of $k\langle X\rangle e$. This implies that Lemma 2.3 is a special case of Lemma 3.2.

Let $S \subset k\langle X\rangle$ and let $A = k\langle X|S\rangle$ be an associative algebra. Then, for any left A-module ${}_AM$, we can regard ${}_AM$ as a $k\langle X\rangle$-module in a natural way: for any

$f \in k\langle X\rangle$, $m \in M$,

$$fm = (f + \mathrm{Id}(S))m.$$

We note that ${}_AM$ is an epimorphic image of some free A-module. Now, we assume that ${}_AM = \mathrm{mod}_A\langle Y|T\rangle = \mathrm{mod}_A\langle Y\rangle/AT$, where $T \subset \mathrm{mod}_A\langle Y\rangle$ and $\mathrm{mod}_A\langle Y\rangle$ is a free left A-module with the basis Y. Let $T_1 = \{\sum f_i y_i \in \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle | \sum(f_i + \mathrm{Id}(S))y_i \in T\}$ and $R = SX^*Y \cup T_1$. Then, by the following Lemma 3.3, we have, as $k\langle X\rangle$-modules, ${}_AM \cong \mathrm{mod}_{k\langle X\rangle}\langle Y|R\rangle$.

Lemma 3.3 ([11]). *Let the notation be the same as above. Then, as $k\langle X\rangle$-modules,*

$$\sigma\colon\ {}_AM \to \mathrm{mod}_{k\langle X\rangle}\langle Y|R\rangle,\ \ \sum(f_i + \mathrm{Id}(S))(y_i + AT) \mapsto \sum f_i y_i + k\langle X\rangle R$$

is an isomorphism, where each $f_i \in k\langle X\rangle$.

P r o o f. For any $\sum(f_i + \mathrm{Id}(S))(y_i + AT), \sum(g_i + \mathrm{Id}(S))(y_i + AT) \in_A M$ we have

$$\begin{aligned}\sum(f_i + \mathrm{Id}(S))(y_i + AT) = {} & \sum(g_i + \mathrm{Id}(S))(y_i + AT) \quad \text{in } {}_AM\\ & \Leftrightarrow \sum(f_i - g_i)y_i \in AT \quad \text{in } {}_AM\\ & \Leftrightarrow \sum(f_i - g_i)y_i \in k\langle X\rangle R\\ & \Leftrightarrow \sum f_i y_i + k\langle X\rangle R = \sum g_i y_i + k\langle X\rangle R.\end{aligned}$$

Hence, σ is injective. It is easy to see that σ is also surjective and consequently, it is a $k\langle X\rangle$-module isomorphism. $\square$

By Lemma 3.2 and Lemma 3.3, we know that if we want to find a k-linear basis for the module ${}_AM = \mathrm{mod}_A\langle Y|T\rangle$, where $A = k\langle X|S\rangle$, we only need to find a Gröbner-Shirshov basis for the module $\mathrm{mod}_{k\langle X\rangle}\langle Y|SX^*Y \cup T_1\rangle$, where $T_1 = \{\sum f_i y_i \in \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle \mid \sum(f_i + \mathrm{Id}(S))y_i \in T\}$.

The next theorem gives a relationship between the Gröbner-Shirshov bases (pairs) in free associative algebras and in "double-free" modules.

Theorem 3.4. *Let X, Y be well ordered sets, $<$ a monomial ordering on X^* and $\prec$ the ordering on X^*Y as in $(*)$. Let S, $T \subset k\langle X\rangle$ be monic sets. Then the following statements hold:*

(1) *$S \subset k\langle X\rangle$ is a Gröbner-Shirshov basis in $k\langle X\rangle$ with respect to the ordering $<$ if and only if $SX^*Y \subset \mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ is a Gröbner-Shirshov basis in $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$ with respect to the ordering $\prec$.*

(2) *We consider $k\langle X\rangle$ as a free $k\langle X\rangle$-module having one generator e. Then (S,T) is a Gröbner-Shirshov pair for the A-module $M = A/A(T + \mathrm{Id}(S))$, where $A = k\langle X|S\rangle$ if and only if S is a Gröbner-Shirshov basis in the algebra $k\langle X\rangle$ with respect to the ordering $<$ and $(SX^* \cup T)e$ is a Gröbner-Shirshov basis in the free module $\mathrm{mod}_{k\langle X\rangle}\langle e\rangle$ with respect to the ordering $\prec$.*

Proof. (1) Suppose that S is a Gröbner-Shirshov basis in $k\langle X\rangle$. We shall prove that all compositions in SX^*Y are trivial modulo SX^*Y. For any $f,\ g \in SX^*Y$, let $f = s_1a_1y,\ g = s_2a_2y,\ s_1, s_2 \in S,\ a_1, a_2 \in X^*,\ y \in Y$ and $w = \overline{f} = a\overline{g}$. Then $\bar{s}_1a_1 = a\bar{s}_2a_2$. Since S is a Gröbner-Shirshov basis in $k\langle X\rangle$, we have

$$(f,g)_w = f - ag = s_1a_1y - as_2a_2y = (s_1a_1 - as_2a_2)y = \sum(\alpha_iu_ir_iv_i)y,$$

where $u_i, v_i \in X^*$, $r_i \in S$ and $u_i\bar{r}_iv_iy \prec w$. Thus, every composition is trivial modulo SX^*Y and hence, SX^*Y is a Gröbner-Shirshov basis in $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$. Conversely, assume that SX^*Y is a Gröbner-Shirshov basis in the module $\mathrm{mod}_{k\langle X\rangle}\langle Y\rangle$. For any $f,\ g \in S$ and $w = \overline{f}a = b\overline{g}$, we have $w_1 = \overline{fay} = b\overline{gy}$ and

$$(fay, bgy)_{w_1} = (fa - bg)y = \sum \alpha_i(a_ir_i)y,$$

where $\alpha_i \in k,\ r_i = s_ib_i,\ a_i, b_i \in X^*,\ s_i \in S$ and $a_i\bar{r}_iy \prec w_1$. Then

$$(f,g)_w = fa - bg = \sum \alpha_ia_is_ib_i$$

with $a_i\bar{s}_ib_i < w$. This shows that each composition of intersection in S is trivial modulo S. Similarly, every composition of inclusion in S is trivial modulo S. Therefore, S is indeed a Gröbner-Shirshov basis in $k\langle X\rangle$.

(2) The results follow directly from Definitions 2.6 and 3.1. □

Remark. By Theorem 3.4 it is clear that Theorem 2.7 follows from Lemma 3.2.

4. Highest weight modules over sl_2

In this section we give a Gröbner-Shirshov basis for the highest weight module over sl_2. By using this result and Lemma 3.2, we re-prove that the highest weight module over sl_2 is irreducible (see [13]) and show that any finite dimensional irreducible sl_2-module has the presentation $(**)$ given below.

Let $X = \{x, y, h\}$ and let $sl_2 = \mathrm{Lie}(X|S)$ be a Lie algebra over a field k with $chk = 0$, where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } S = \{[hx] - 2x, [hy] + 2y, [xy] - h\}.$$

Then the universal enveloping algebra of sl_2 is $\mathcal{U}(sl_2) = k\langle X|S^{(-)}\rangle$. Define the deg-lex ordering on X^* with $x > h > y$. Then S is a Gröbner-Shirshov basis in the free Lie algebra Lie(X) since $S^{(-)}$ is a Gröbner-Shirshov basis in $k\langle X\rangle$ (see Lemma 1.1). Let

$$_{sl_2}V(\lambda) = \mathrm{mod}_{sl_2}\langle v_0 \mid xv_0 = 0,\ hv_0 = \lambda v_0,\ y^{m+1}v_0 = 0\rangle$$

be a highest weight module generated by v_0 with the highest weight λ. We can rewrite it as

$$\begin{aligned} _{sl_2}V(\lambda) &= \mathrm{mod}_{\mathcal{U}(sl_2)}\langle v_0 \mid xv_0 = 0,\ hv_0 = \lambda v_0,\ y^{m+1}v_0 = 0\rangle \\ &= \mathrm{mod}_{k\langle X\rangle}\langle v_0 \mid xv_0 = 0,\ hv_0 = \lambda v_0,\ y^{m+1}v_0 = 0,\ S^{(-)}X^*v_0 = 0\rangle. \end{aligned}$$

Let $S_1 = \{xv_0, hv_0 - \lambda v_0, y^{m+1}v_0\} \cup S^{(-)}X^*v_0$. It is easy to see that all compositions in S_1 are trivial modulo S_1. Thus, S_1 is a Gröbner-Shirshov basis for this module with respect to the ordering $(*)$ as in Section 3, and by Lemma 3.2, $\mathrm{Irr}(S_1) = \{y^i v_0 \mid 0 \leqslant i \leqslant m\}$ is a k-linear basis for the module $_{sl_2}V(\lambda)$, and so $\dim(V(\lambda)) = m + 1$. Let $y^{(i)} = i!^{-1}y^i$, $v_i = i!^{-1}y^i v_0$ and $v_{-1} = 0$. Then v_i $(0 \leqslant i \leqslant m)$ is a linear basis of $V(\lambda)$. Now, by using ELW of the relations in S_1 on the left parts, we have the following equalities (see also [13], p. 32):

Lemma 4.1. $hv_i = (\lambda - 2i)v_i,$
$yv_i = (i+1)v_{i+1},$
$xv_i = (\lambda - i + 1)v_{i-1}$ $(0 \leqslant i)$.

Since $v_{m+1} = 0$ and $chk = 0$, we have $0 = xv_{m+1} = (\lambda - m)v_m$ and therefore, $\lambda = m$.

Lemma 4.2. *$V(\lambda)$ is irreducible.*

Proof. Let $0 \neq V_1 \leqslant V(\lambda)$ be a submodule. Since $V_1 \neq 0$, there exists $0 \neq a_i v_i + a_{i+1}v_{i+1} + \ldots + a_m v_m$, where i is the least number such that $a_i \neq 0$. Applying y to it $m-i$ times, we get $a_i(i+1)(i+2)\ldots m v_m \in V_1$ and hence, $v_m \in V_1$. Applying x to v_m, we get $v_i \in V_1$ $(0 \leqslant i < m)$ and hence $V_1 = V(\lambda)$. □

For any finite dimensional irreducible sl_2-module V, choose a maximal vector $v_0 \in V$ and $v_i = i!^{-1}y^i v_0$. Then we have the formulas as in Lemma 4.1. We can suppose that $\dim V = m$. Thus, $v_m \neq 0$, $v_{m+1} = 0$ and hence, V can be represented as

$$(**) \qquad _{sl_2}V = \mathrm{mod}_{sl_2}\langle v_0 | xv_0 = 0,\ hv_0 = \lambda v_0,\ y^{m+1}v_0 = 0\rangle.$$

This means that any finite dimensional irreducible sl_2-module has the above form.

Gröbner-Shirshov bases for Kac-Moody algebras of types $A_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $B_n^{(1)}$ are found by E. N. Poroshenko in [22], [23], [24].

In this section we give the definitions of Kac-Moody algebra $\mathcal{G}(A)$ and the Verma module over $\mathcal{G}(A)$. We find a Gröbner-Shirshov basis for this Verma module.

Let $A = (a_{ij})$ be an (integral) symmetrizable n-by-n Cartan matrix over $\mathbb{C}$, where $\mathbb{C}$ is the complex field. It means that $a_{ii} = 2$, $a_{ij} \leqslant 0$ $(i \neq j)$, and there exists a diagonal matrix D with nonzero integer diagonal entries d_i such that the product DA is symmetric. Let $\mathcal{G}(A) = \mathrm{Lie}(X|S)$ be a Lie algebra, where $X = \{x_i, y_i, h \mid 1 \leqslant i \leqslant n,\ h \in H\}$ and S consists of the following relations (see [14], p. 159):

$$[x_i, y_j] = \delta_{ij}\alpha_i^{\vee}\ (i, j = 1, \ldots, n), \tag{5.1}$$

$$[h, h'] = 0\ (h, h' \in H), \tag{5.2}$$

$$[h, x_i] = \langle\alpha_i, h\rangle x_i, \quad [h, y_i] = -\langle\alpha_i, h\rangle y_i,\ (i = 1, \ldots, n; h \in H), \tag{5.3}$$

$$(adx_i)^{1-a_{ij}} x_j = 0,\ (ady_i)^{1-a_{ij}} y_j = 0\ (i \neq j), \tag{5.4}$$

where ad is the derivation, H a complex vector space, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset H^\star$ (the dual space of H) and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset H$ indexed subsets in $H^\star$ and H, respectively, satisfying the following conditions (see [14], p. 1):

(a) both the sets Π and $\Pi^\vee$ are linearly independent,

(b) $\langle\alpha_i^\vee, \alpha_j\rangle = a_{ij}$ $(i, j = 1, \ldots, n)$,

(c) $n - l = \dim H - n$ $\mathrm{rank}(A) = l$.

Then we call this Lie algebra $\mathcal{G}(A)$ the Kac-Moody algebra. Let $\mathfrak{N}_+$ $(\mathfrak{N}_-)$ be the subalgebra of $\mathcal{G}(A)$ generated by x_i (y_i) $(0 \leqslant i \leqslant n)$. Then $\mathcal{G}(A) = \mathfrak{N}_- \oplus H \oplus \mathfrak{N}_+$ and $\mathcal{U}(\mathcal{G}(A)) = \mathcal{U}(\mathfrak{N}_+) \otimes k[H] \otimes \mathcal{U}(\mathfrak{N}_-)$ is the universal enveloping algebra of $\mathcal{G}(A)$, where $\mathcal{U}(\mathfrak{N}_+)$ $(\mathcal{U}(\mathfrak{N}_-))$ is the universal enveloping algebra of $\mathfrak{N}_+$ $(\mathfrak{N}_-)$. Let $\{h_j \mid 1 \leqslant j \leqslant 2n - l\}$ be a basis of H. We order the set $X = \{x_i,\ h_j,\ y_m \mid 1 \leqslant i, m \leqslant n,\ 1 \leqslant j \leqslant 2n - l\}$ by $x_i > x_j$, $h_i > h_j$, $y_i > y_j$, if $i > j$, and $x_i > h_j > y_m$ for all i, j, m. Then we define the deg-lex ordering on X^*. By [8], we can get a Gröbner-Shirshov basis T for $\mathcal{U}(\mathcal{G}(A))$, where T consists of the following relations:

$$h_i h_j - h_j h_i,\ x_j h_i - h_i x_j + d_i a_{ij} x_i,\ h_i y_j - y_j h_i + d_i a_{ij} y_j, \tag{5.5}$$

$$x_i y_j - y_j x_i - \delta_{ij} h_i, \tag{5.6}$$

$$\left\{ \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1 - a_{ij} \\ \nu \end{bmatrix} x_i^{1-a_{ij}-\nu} x_j x_i^\nu \right\}^c\ (i \neq j), \tag{5.7}$$

$$(5.8) \qquad \left\{ \sum_{\nu=0}^{1-a_{ij}} (-1)^{\nu} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix} y_i^{1-a_{ij}-\nu} y_j y_i^{\nu} \right\}^c \ (i \neq j),$$

where S^c is a Gröbner-Shirshov basis containing S.

Definition 5.1 ([14])**.** A $\mathcal{G}(A)$-module V is called a highest weight module with highest weight $\Lambda \in H^\star$ if there exists a non-zero vector $v \in V$ such that

$$\mathfrak{N}_+(v) = 0, \quad h(v) = \Lambda(h)v, \quad h \in H$$

and $\mathcal{U}(\mathcal{G}(A))(v) = V$.

A Verma module $M(\Lambda)$ with highest weight Λ has the following presentation:

$$\begin{aligned} {}_{\mathcal{G}(A)}M(\Lambda) &= \mathrm{mod}_{\mathcal{U}(\mathcal{G}(A))}\langle v | \mathfrak{N}_+(v) = 0,\ h(v) = \Lambda(h)v,\ h \in H\rangle \\ &= \mathrm{mod}_{k\langle X\rangle}\langle v | TX^*(v) = 0,\ \mathfrak{N}_+(v) = 0,\ h(v) = \Lambda(h)v,\ h \in H\rangle. \end{aligned}$$

The proof of the following theorem is straightforward.

Theorem 5.2. *With the ordering $\prec$ on X^*v as $(*)$, $R = \{TX^*(v),\ \mathfrak{N}_+(v),\ h(v) - \Lambda(h)v\}$ is a Gröbner-Shirshov basis for the Verma module ${}_{\mathcal{G}(A)}M(\Lambda)$.*

Remark. In the book [13], the author considered only the semisimple Lie algebras and called this highest weight module the standard cyclic module.

6. Verma modules over the coefficient algebra of a free Lie conformal algebra

In this section we give a Gröbner-Shirshov basis for the Verma module over a Lie algebra having coefficients of some free conformal algebras. By using this result and Lemma 3.2, we find a linear basis for such a module.

Let $\mathcal{B}$ be a set of symbols. Let the locality function $N\colon \mathcal{B} \times \mathcal{B} \to \mathbb{Z}_+$ be a constant, i.e., $N(a,b) \equiv N$ for any $a, b \in \mathcal{B}$. Let $X = \{b(n) \mid\ b \in \mathcal{B},\ n \in \mathbb{Z}\}$ and let $L = \mathrm{Lie}(X|S)$ be a Lie algebra generated by X with the relation S, where

$$S = \left\{ \sum_s (-1)^s \binom{n}{s} [b(n-s)a(m+s)] = 0 \ \middle|\ a,\ b \in \mathcal{B},\ m, n \in \mathbb{Z} \right\}.$$

For any $b \in \mathcal{B}$, let $\tilde{b} = \sum_n b(n) z^{-n-1} \in L[[z, z^{-1}]]$. Then it is well-known that they generate a free Lie conformal algebra C with data $(\mathcal{B}, N)$ (see [25]). Moreover, the coefficient algebra of C is just L. Let $\mathcal{B}$ be a well ordered set. Define an ordering on X in the following way:

$$a(m) < b(n) \Leftrightarrow m < n \text{ or } (m = n \text{ and } a < b).$$

We use the deg-lex ordering on X^*. Then, it is clear that the leading term of each polynomial in S is $b(n)a(m)$ so that

$$n - m > N \text{ or } (n - m = N \text{ and } (b > a \text{ or } (b = a \text{ and } N \text{ is odd}))).$$

The following lemma is essentially taken from [25].

Lemma 6.1. *With the deg-lex ordering on X^*, S is a Gröbner-Shirshov basis in* $\mathrm{Lie}(X)$.

Corollary 6.2. *Let $\mathcal{U} = \mathcal{U}(L)$ be a universal enveloping algebra of L. Then a k-basis of $\mathcal{U}$ consists of monomials*

$$a_1(n_1)a_2(n_2)\dots a_k(n_k),\ a_i \in \mathcal{B},\ n_i \in \mathbb{Z}$$

such that for any $1 \leqslant i < k$,

$$(***) \qquad n_i - n_{i+1} \leqslant \begin{cases} N-1 & \text{if } a_i > a_{i+1} \text{ or } (a_i = a_{i+1} \text{ and } N \text{ is odd}),\\ N & \text{otherwise.}\end{cases}$$

P r o o f. We first regard $\mathcal{U}$ as a $k\langle X\rangle$-module. Then we have

$$_{\mathcal{U}}\mathcal{U} = \mathrm{mod}_{k\langle X\rangle}\langle e|\ S^{(-)}X^*e\rangle.$$

Since S is a Gröbner-Shirshov basis in $\mathrm{Lie}(X)$, $S^{(-)}$ is a Gröbner-Shirshov basis in $k\langle X\rangle$ by Lemma 1.1. Therefore, by Theorem 3.4, $S^{(-)}X^*e$ is a Gröbner-Shirshov basis in the free module $\mathrm{mod}_{k\langle X\rangle}\langle e\rangle$. Now, the result follows from Lemma 3.2. □

Definition 6.3 ([14], [15])**.**

(a) An L-module M is called restricted if for any $a \in C$, $v \in M$ there is an integer T such that for any $n \geqslant T$ one has $a(n)v = 0$.

(b) An L-module M is called a highest weight module if it is generated over L by a single element $m \in M$ such that $L_+m = 0$, where L_+ is the subspace of L generated by $\{a(n) \mid a \in \mathcal{B}, n \geqslant 0\}$. In this case, m is called the highest weight vector.

Now we construct a universal highest weight module V over L which is usually referred to as the Verma module. Let ke_v be a 1-dimensional trivial L_+-module generated by e_v, i.e., $a(n)e_v = 0$ for all $a \in \mathcal{B}$, $n \geqslant 0$. Clearly,

$$V = \mathrm{Ind}_{L_+}^{L} ke_v = \mathcal{U}(L) \otimes_{\mathcal{U}(L_+)} ke_v \cong \mathcal{U}(L)/\mathcal{U}(L)L_+.$$

Then V has a structure highest weight module over L with the action given by the multiplication on $\mathcal{U}(L)/\mathcal{U}(L)L_+$ and the highest weight vector $e \in \mathcal{U}(L)$. Also, $V = \mathcal{U}(L)/\mathcal{U}(L)L_+$ is the universal enveloping vertex algebra of C and the embedding $\varphi : C \to V$ is given by $a \mapsto a(-1)e$ (see also [25]).

Theorem 6.4. *Let the notions be defined as above. Then a k-basis of V consists of elements*

$$a_1(n_1)a_2(n_2)\dots a_k(n_k),\ a_i \in \mathcal{B},\ n_i \in \mathbb{Z}$$

such that the condition $(*{*}*)$ *holds and* $n_k < 0$.

P r o o f. Clearly, as the $k\langle X\rangle$-modules,

$${}_{\mathcal{U}}V =_{\mathcal{U}} (\mathcal{U}(L)/\mathcal{U}(L)L_+) = \mathrm{mod}_{k\langle X\rangle}\langle e|\ S^{(-)}X^*e,\ a(n)e,\ n \geqslant 0\rangle =_{k\langle X\rangle} \langle e|\ S'\rangle,$$

where $S' = \{S^{(-)}X^*e, a(n)e, n \geqslant 0\}$. In order to prove that S' is a Gröbner-Shirshov basis, we only need to check that $w = b(n)a(m)e$, where $m \geqslant 0$. Let

$$f = \sum_s (-1)^s \binom{n}{s} (b(n-s)a(m+s) - a(m+s)b(n-s))e \quad \text{and} \qquad g = a(m)e.$$

Then $(f,g)_w = f - b(n)a(m)e \equiv 0 \mod(S', w)$ since $n-m \geqslant N$, $m+s \geqslant 0$, $n-s \geqslant 0$, $0 \leqslant s \leqslant N$. It follows that S' is a Gröbner-Shirshov basis. Now, the result follows from Lemma 3.2. $\square$

7. Universal enveloping module for a Sabinin algebra

In this section we give a Gröbner-Shirshov basis for a universal enveloping module for a Sabinin algebra. By using this result and Lemma 3.2, we find a linear basis for such a module.

Definition 7.1 ([21])**.** A vector space V is called a Sabinin algebra if it is endowed with a multilinear operation $\langle ; \rangle$ such that for any $x_1, x_2, \dots, x_m, y, z \in V$ and any $m \geqslant 0$,

$$\langle x_1, x_2, \dots, x_m; y, z\rangle$$

satisfies the identities

$$
\begin{aligned}
&\langle x_1, x_2, \ldots, x_m; y, z\rangle = -\langle x_1, x_2, \ldots, x_m; z, y\rangle,\\
&\langle x_1, x_2, \ldots, x_r, a, b, x_{r+1}, \ldots, x_m; y, z\rangle - \langle x_1, x_2, \ldots, x_r, b, a, x_{r+1}, \ldots, x_m; y, z\rangle\\
&\quad + \sum_{k=0}^{r}\sum_{\alpha}\langle x_{\alpha_1}, \ldots, x_{\alpha_k}, \langle x_{\alpha_{k+1}}, \ldots, x_{\alpha_r}; a, b\rangle, \ldots, x_m; y, z\rangle = 0,\\
&\sigma_{x,y,z}(\langle x_1, x_2, \ldots, x_r, x; y, z\rangle + \sum_{k=0}^{r}\sum_{\alpha}\langle x_{\alpha_1}, \ldots, x_{\alpha_k}; \langle x_{\alpha_{k+1}}, \ldots, x_{\alpha_r}; y, z\rangle, x\rangle) = 0,
\end{aligned}
$$

where α runs over the set of all bijections of type $\alpha\colon \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}$, $i \mapsto \alpha_i$, $\alpha_1 < \alpha_2 < \ldots < \alpha_k$, $\alpha_{k+1} < \ldots < \alpha_r$, $r \geqslant 0$ and $\sigma_{x,y,z}$ denotes the cyclic sum by x, y, z.

Let $X = \{a_i \mid i \in \Lambda\}$ be a well ordered basis of V. We define the deg-lex ordering on X^*. Let $\Delta:\ V \to V \otimes V$ be a linear map which satisfies $\Delta(a_i) = 1 \otimes a_i + a_i \otimes 1$, $(\mathrm{Id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{Id})\Delta$ (coassociativity) and if $\tau\Delta = \Delta$ then $\tau(x \otimes y) = y \otimes x$ (cocommutativity). It is customary to write $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

Let $T(V)$ be the tensor algebra over V endowed with its natural structure of cocommutative Hopf algebra, that is, $V \subseteq \mathrm{Prim}\,(T(V))$ (the primitive element of $T(V)$). Let $\langle;\rangle\colon\ T(V) \otimes V \otimes V \to V$ be a map. Then we may shortly write the definition of a Sabinin algebra as

$$
\begin{aligned}
\langle x; a, b\rangle = -\langle x; b, a\rangle, \langle x[a, b]y; c, e\rangle + \sum \langle x_{(1)}\langle x_{(2)}; a, b\rangle y; c, e\rangle = 0,\\
\sigma_{a,b,c}(\langle xc; a, b\rangle + \sum \langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\rangle) = 0,
\end{aligned}
$$

where $[a, b] = ab - ba$.

Definition 7.2 ([21]). Let $(V, \langle;\rangle)$ be a Sabinin algebra. Then

$$
\tilde{S}(V) = T(V)/\mathrm{span}\Big\langle xaby - xbay + \sum x_{(1)}\langle x_{(2)}; a, b\rangle y | x, y \in T(V), a, b \in V\Big\rangle
$$

is called the universal enveloping module for V.

Since $T(V) \simeq k\langle X\rangle$ as k-algebras, we can view $\tilde{S}(V)$ as a right $k\langle X\rangle$-module:

$$
\tilde{S}(V) = \mathrm{mod}\langle X|I\rangle_{k\langle X\rangle},
$$

where $I = \{xab - xba + \sum x_{(1)}\langle x_{(2)}; a, b\rangle \mid x \in X^*, a > b,\ a, b \in X\}$.

For the right module, we have a right compatible well ordering $\prec$ on XX^* by a similar definition as in $(*)$. Then we have the following theorem.

Theorem 7.3. *Let I be as above. Then, with the ordering $\prec$ on XX^* as above, I is a Gröbner-Shirshov basis in* $\mathrm{mod}\langle X\rangle_{k\langle X\rangle}$.

P r o o f. There are two kinds of compositions: $w_1 = xabc$ ($a > b > c$) and $w_2 = ucdvab$ ($c > d$, $a > b$). Denote

$$
\begin{aligned}
f_1 &= xabc - xacb + \sum (xa)_{(1)}\langle (xa)_{(2)}; b, c\rangle,\\
f_2 &= xab - xba + \sum x_{(1)}\langle x_{(2)}; a, b\rangle,\\
f_3 &= ucdvab - ucdvba + \sum (ucdv)_{(1)}\langle (ucdv)_{(2)}; a, b\rangle,\\
f_4 &= ucd - udc + \sum u_{(1)}\langle u_{(2)}; c, d\rangle.
\end{aligned}
$$

Then, since $\sigma_{a,b,c}(\langle xc; a, b\rangle + \sum\langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\rangle) = 0$, we have

$$
\begin{aligned}
(f_1, f_2)_{w_1} = {} & xabc - xacb + \sum x_{(1)}a\langle x_{(2)}; b, c\rangle + \sum x_{(1)}\langle x_{(2)}a; b, c\rangle\\
& - xabc + xbac - \sum x_{(1)}\langle x_{(2)}; a, b\rangle c\\
\equiv {} & -xcab + \sum x_{(1)}\langle x_{(2)}; a, c\rangle b + \sum x_{(1)}a\langle x_{(2)}; b, c\rangle + \sum x_{(1)}\langle x_{(2)}a; b, c\rangle\\
& + xbca - \sum x_{(1)}b\langle x_{(2)}; a, c\rangle - \sum x_{(1)}\langle x_{(2)}b; a, c\rangle - \sum x_{(1)}\langle x_{(2)}; a, b\rangle c\\
\equiv {} & \sum x_{(1)}c\langle x_{(2)}; a, b\rangle + \sum x_{(1)}\langle x_{(2)}c; a, b\rangle + \sum x_{(1)}\langle x_{(2)}; a, c\rangle b\\
& + \sum x_{(1)}a\langle x_{(2)}; b, c\rangle + \sum x_{(1)}\langle x_{(2)}a; b, c\rangle - \sum x_{(1)}\langle x_{(2)}; b, c\rangle a\\
& - \sum x_{(1)}b\langle x_{(2)}; a, c\rangle - \sum x_{(1)}\langle x_{(2)}b; a, c\rangle - \sum x_{(1)}\langle x_{(2)}; a, b\rangle c\\
\equiv {} & \sum x_{(1)}\langle x_{(2)}a; b, c\rangle + \sum x_{(1)}\langle x_{(2)}; \langle x_{(3)}; b, c\rangle, a\rangle + \sum x_{(1)}\langle x_{(2)}b; c, a\rangle\\
& + \sum x_{(1)}\langle x_{(2)}c; a, b\rangle + \sum x_{(1)}\langle x_{(2)}; \langle x_{(3)}; c, a\rangle, b\rangle\\
& + \sum x_{(1)}\langle x_{(2)}; \langle x_{(3)}; a, b\rangle, c\rangle\\
\equiv {} & 0
\end{aligned}
$$

and since $\langle x[a, b]y; c, e\rangle + \sum\langle x_{(1)}\langle x_{(2)}; a, b\rangle y; c, e\rangle = 0$,

$$
\begin{aligned}
&(f_3, f_4)_{w_2}\\
&\quad = ucdvab - ucdvba + \sum u_{(1)}v_{(1)}\langle u_{(2)}cdv_{(2)}; a, b\rangle + \sum u_{(1)}cv_{(1)}\langle u_{(2)}dv_{(2)}; a, b\rangle\\
&\qquad + \sum u_{(1)}dv_{(1)}\langle u_{(2)}cv_{(2)}; a, b\rangle + \sum u_{(1)}cdv_{(1)}\langle u_{(2)}v_{(2)}; a, b\rangle\\
&\qquad - ucdvab + udcvab - \sum u_{(1)}\langle u_{(2)}; c, d\rangle vab
\end{aligned}
$$

$$
\begin{aligned}
&\equiv -udcvba + \sum u_{(1)}\langle u_{(2)}; c, d\rangle vba + \sum u_{(1)}v_{(1)}\langle u_{(2)}cdv_{(2)}; a, b\rangle \\
&\quad + \sum u_{(1)}cv_{(1)}\langle u_{(2)}dv_{(2)}; a, b\rangle + \sum u_{(1)}dv_{(1)}\langle u_{(2)}cv_{(2)}; a, b\rangle \\
&\quad + \sum u_{(1)}cdv_{(1)}\langle u_{(2)}v_{(2)}; a, b\rangle + udcvba - \sum u_{(1)}v_{(1)}\langle u_{(2)}dcv_{(2)}; a, b\rangle \\
&\quad - \sum u_{(1)}cv_{(1)}\langle u_{(2)}dv_{(2)}; a, b\rangle - \sum u_{(1)}dv_{(1)}\langle u_{(2)}cv_{(2)}; a, b\rangle \\
&\quad - \sum u_{(1)}dcv_{(1)}\langle u_{(2)}v_{(2)}; a, b\rangle + \sum u_{(1)}v_{(1)}\langle u_{(2)}\langle u_{(3)}; c, d\rangle v_{(2)}; a, b\rangle \\
&\quad + \sum u_{(1)}\langle u_{(2)}; c, d\rangle v_{(1)}\langle u_{(3)}v_{(2)}; a, b\rangle - \sum u_{(1)}\langle u_{(2)}; c, d\rangle vba \\
&\equiv \sum u_{(1)}v_{(1)}\langle u_{(2)}[c, d]v_{(2)}; a, b\rangle + \sum u_{(1)}[c, d]v_{(1)}\langle u_{(2)}v_{(2)}; a, b\rangle \\
&\quad + \sum u_{(1)}v_{(1)}\langle u_{(2)}\langle u_{(3)}; c, d\rangle v_{(2)}; a, b\rangle + \sum u_{(1)}\langle u_{(2)}; c, d\rangle v_{(1)}\langle u_{(3)}v_{(2)}; a, b\rangle \\
&\equiv \sum (u_{(1)}[c, d] + u_{(1)}\langle u_{(2)}; c, d\rangle)v_{(1)}\langle u_{(3)}v_{(2)}; a, b\rangle \\
&\quad + \sum u_{(1)}v_{(1)}\langle u_{(2)}[c, d]v_{(2)}; a, b\rangle + \sum u_{(1)}v_{(1)}\langle u_{(2)}\langle u_{(3)}; c, d\rangle v_{(2)}; a, b\rangle \\
&\equiv 0.
\end{aligned}
$$

Hence, I is a Gröbner-Shirshov basis in $\mathrm{mod}\langle X\rangle_{k\langle X\rangle}$. $\square$

Remark. From the above proof we can easily see that for $\tilde{S}(V) = \mathrm{mod}\langle X|I\rangle_{k\langle X\rangle}$, the minimal Gröbner-Shirshov basis is

$$
\begin{gathered}
G = \Big\{ xab - xba + \sum x_{(1)}\langle x_{(2)}; a, b\rangle \;\Big|\; x = a_{i_1} \dots a_{i_n} \\
(i_1 \leqslant \dots \leqslant i_n,\ n \geqslant 0),\ a > b,\ a, b \in X \Big\}.
\end{gathered}
$$

Now, by Lemma 3.2 and Theorem 7.3, we can easily get the following theorem.

Theorem 7.4 ([21], Poincaré-Birkhoff-Witt basis). *Let $\{a_i \mid i \in \Lambda\}$ be a well ordered basis of V. Then $\{a_{i_1} \dots a_{i_n} \mid i_1 \leqslant i_2 \leqslant \dots \leqslant i_n,\ n \geqslant 0\}$ is a basis of $\tilde{S}(V)$.*

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