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WEIGHTED SHARING AND UNIQUENESS OF ENTIRE FUNCTIONS

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Abstract. In this paper we study the uniqueness for meromorphic functions sharing one value, and obtain some results which improve and generalize the related results due to M. L. Fang, X. Y. Zhang, W. C. Lin, T. D. Zhang, W. R. Lü and others.

Keywords: entire function, weighted sharing, differential polynomial, uniqueness

MSC 2010: 30D35

1. Introduction and Results

In this paper, the term "meromorphic" will always mean meromorphic in the complex plane $\mathbb{C}$. Let f and g be two nonconstant meromorphic functions, and let a be a complex number. We say that f and g share a IM (ignoring multiplicity) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicity, we say that f and g share a CM (counting multiplicity). It is assumed that the reader is familiar with the standard notations of value distribution theory that can be found, for instance, in [3], [7], [8]. We denote by $S(r,f)$ any function satisfying

$$S(r,f)=o(T(r,f))$$

as $r\to\infty$, possibly outside a set of finite measure.

In addition, we shall also use the following notation.

For a positive integer k, we denote by $N_k(r,1/(f-a))$ the counting function for zeros of $f-a$ with multiplicities at least k, and by $\overline{N}_k(r,1/(f-a))$ the corresponding

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one for which multiplicity is not counted. Set

$$N_k\Big(r, \frac{1}{f-a}\Big) = \overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}_2\Big(r, \frac{1}{f-a}\Big) + \ldots + \overline{N}_k\Big(r, \frac{1}{f-a}\Big).$$

Let f and g share a IM. We denote by $N_{11}(r, 1/(f-a))$ the counting function for the common simple zeros of both $f-a$ and $g-a$, by $\overline{N}_L(r, 1/(f-a))$ the counting function for the zeros of both $f-a$ and $g-a$ about which $f-a$ has larger multiplicity than $g-a$, with multiplicity being not counted.

In 2002, Fang [1] proved the following uniqueness theorems.

Theorem A. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = t\,g(z)$ for a constant t such that $t^n = 1$ or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$.*

Theorem B. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n \geqslant 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Recently, Zhang and Lin [10] proved the following results, which generalize and improve Theorem A and B.

Theorem C. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k be three positive integers with $n > 2k + m^* + 4$, and let λ, μ be constants such that $|\lambda| + |\mu| \neq 0$. If $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 CM, then*

(i) *when $\lambda\mu \neq 0$, then $f(z) \equiv g(z)$;*

(ii) *when $\lambda\mu = 0$, then either $f(z) = t\,g(z)$ for a constant t such that $t^{n+m^*} = 1$ or*

$$f(z) = c_1 e^{cz} \quad \text{and} \quad g(z) = c_2 e^{-cz}$$

for three constants c, c_1 and c_2 satisfying

$$(-1)^k \lambda^2 (c_1c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1,$$

or

$$(-1)^k \mu^2 (c_1c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1,$$

where $m^* = 0$ if $\mu = 0$, and $m^* = m$ if $\mu \neq 0$.

Theorem D. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k be three positive integers with $n > 2k + m + 4$. If $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 CM, then either $f(z) \equiv g(z)$, or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^n(\omega_1-1)^m - \omega_2^n(\omega_2-1)^m$.*

Remark 1. The conclusion (i) in Theorem C is incomplete. In fact, if $\lambda\mu \neq 0$ and both m, n are even integers then for $f(z) \equiv -g(z)$ the hypotheses of Theorem C are still satisfied.

We rewrite Theorem C as follows.

Theorem C′. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k be three positive integers with $n > 2k + m^* + 4$, and let λ, μ be such constants that $|\lambda| + |\mu| \neq 0$. If $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 CM, then*

(i) *when $\lambda\mu \neq 0$, then $f(z) \equiv h\,g(z)$ for a constant h such that $h^n = 1$ and $h^{n+m} = 1$;*

(ii) *when $\lambda\mu = 0$, then either $f(z) = t\,g(z)$ for a constant t such that $t^{n+m^*} = 1$ or*

$$f(z) = c_1 e^{cz} \quad \text{and} \quad g(z) = c_2 e^{-cz},$$

for three constants c, c_1 and c_2 satisfying

$$(-1)^k \lambda^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1$$

or

$$(-1)^k \mu^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1,$$

where $m^* = 0$ if $\mu = 0$, and $m^* = m$ if $\mu \neq 0$.

Proof. We only need to prove the conclusion (i). As the proof of Theorem C in [10], we have $f^n(\mu f^m + \lambda) = g^n(\mu g^m + \lambda)$ (see (3.29), p. 947, [10]). For the case $\lambda\mu \neq 0$, set $h = f/g$. It follows that

$$(h^{n+m} - 1)g^m + \frac{\lambda}{\mu}(h^n - 1) = 0. \tag{$*$}$$

Suppose that h is nonconstant. Then

$$g^m = -\frac{\lambda}{\mu}\frac{h^n - 1}{h^{n+m} - 1}. \tag{$**$}$$

Since g is entire, we see from $(**)$ that each zero of $h^{n+m} - 1$ must be a zero of $h^n - 1$, and hence of $h^m - 1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n+m}$ be distinct roots of $z^{n+m} = 1$, and $\beta_1, \beta_2, \ldots, \beta_m$ be distinct roots of $z^m = 1$. Thus

$$\sum_{i=1}^{n+m} \overline{N}\Big(r, \frac{1}{h - \alpha_i}\Big) \leqslant \sum_{i=1}^{m} N\Big(r, \frac{1}{h - \beta_i}\Big).$$

By Nevanlinna first and second fundamental theorems, we have

$$
\begin{aligned}
(n+m-2)T(r,h) &\leqslant \sum_{i=1}^{n+m} \overline{N}\Big(r, \frac{1}{h-\alpha_i}\Big) + S(r,h) \\
&\leqslant \sum_{i=1}^{m} N\Big(r, \frac{1}{h-\beta_i}\Big) + S(r,h) \leqslant mT(r,h) + S(r,h),
\end{aligned}
$$

that is,

$$(n-2)T(r,h) \leqslant S(r,h),$$

which is impossible since $n > 2k + m + 4$. Hence h is a constant. The conclusion (i) follows from $(*)$ and the fact that g is a nonconstant entire function. □

Next we explain the notion of weighted sharing of a value.

Definition 1. Let k be a nonnegative integer or infinity. For a complex number a, we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m-times if $m \leqslant k$ and $(k+1)$-times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f and g share the value a with weight k.

We write f and g share (a, k) meaning that f and g share the value a with weight k. Obviously, f and g share (a, k) means that z_0 is a zero of $f - a$ with multiplicity m $(\leqslant k)$ if and only if it is a zero of $g - a$ with multiplicity m $(\leqslant k)$ and z_0 is a zero of $f - a$ with multiplicity m $(> k)$ if and only if it is a zero of $g - a$ with multiplicity n $(> k)$ where n is not necessarily equal to m.

Clearly, if f and g share (a, k), then f and g share (a, p) for any integer $0 \leqslant p \leqslant k$. We also note that f and g share $(a, 0)$ or (a, ∞) if and only if f and g share a IM or CM, respectively. So, the weighted sharing is indeed a scaling between IM and CM.

Remark 2. Fujimoto [2] used an idea similar to the above under the name of "truncated multiplicity" in connection with meromorphic maps of $\mathbb{C}^n$ into $P^N(C)$. Lahiri [4], [5] was the first to give the above simplified definition and successfully apply the idea to the uniqueness problems of meromorphic functions under the name "weighted sharing".

In this paper, we shall use the idea of weighted sharing of values and prove the following results, which improve and extend Theorems A–D.

Theorem 1. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k, l be four positive integers and λ, μ constants such that $|\lambda| + |\mu| \neq 0$. Suppose that $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share $(1, l)$. If $l = 2$ and $n > 2k + m^* + 4$ or if $l = 1$ and $n > 3k + 2m^* + 6$ or if $l = 0$ and $n > 5k + 4m^* + 7$, where $m^* = 0$ if $\mu = 0$ and $m^* = m$ if $\mu \neq 0$, then the conclusion of Theorem C′ holds.*

Theorem 2. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k be three positive integers. Suppose that $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share $(1,l)$. If $l=2$ and $n>2k+m+4$ or if $l=1$ and $n>3k+2m+6$ or if $l=0$ and $n>5k+4m+7$, then the conclusion of Theorem* D *holds.*

From Theorem 1, we obtain the following corollary, which is a result of Zhang and Lü [9].

Corollary 1. *Let $f(z)$ and $g(z)$ be two nonconstant transcendental entire functions, and let n, k, l be three positive integers. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share $(1,l)$. If $l=2$ and $n>2k+4$ or if $l=1$ and $n>3k+6$ or if $l=0$ and $n>5k+7$, then the conclusion of Theorem* A *holds.*

The next result follows from Theorem 2 and the fact that for two polynomials f, g, $f^n(f-1)\equiv g^n(g-1)$ implies $f\equiv g$ (for details, see [1] or [10]), or it follows from Theorem 1 immediately.

Corollary 2. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers. Suppose $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share $(1,l)$. If $l=2$ and $n>2k+5$ or if $l=1$ and $n>3k+8$ or if $l=0$ and $n>5k+11$, then $f(z)\equiv g(z)$.*

2. Some lemmas

For proofs of our results, we need the following lemmas.

Lemma 1 (see [6]). *Let f be a nonconstant meromorphic function, and let $a_0, a_1, \ldots, a_n$ be finite complex numbers such that $a_n\neq 0$. Then*

$$T(r, a_nf^n+\ldots+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2 (see [3], [7], [8]). *Let f be a nonconstant meromorphic function, and let k be a positive integer. Then*

$$N\Big(r,\frac{1}{f^{(k)}}\Big)\leqslant N\Big(r,\frac{1}{f}\Big)+k\overline{N}(r,f)+S(r,f).$$

Lemma 3. *Let F, G be two nonconstant entire functions and let k be a positive integer. If $F^{(k)}$ and $G^{(k)}$ share 1 IM, then*

$$\overline{N}_L\Big(r,\frac{1}{F^{(k)}-1}\Big)\leqslant N_{k+1}\Big(r,\frac{1}{F}\Big)+S(r,F).$$

Proof. Since $F^{(k)}$ and $G^{(k)}$ share 1 IM, we have

$$\overline{N}_L\Big(r,\frac{1}{F^{(k)}-1}\Big)\leqslant \overline{N}_2\Big(r,\frac{1}{F^{(k)}-1}\Big)\leqslant N\Big(r,\frac{1}{F^{(k)}-1}\Big)-\overline{N}\Big(r,\frac{1}{F^{(k)}-1}\Big). \tag{1}$$

By Lemma 2, we get

$$\begin{aligned}N\Big(r,\frac{1}{F^{(k)}-1}\Big)-\overline{N}\Big(r,\frac{1}{F^{(k)}-1}\Big)+N\Big(r,\frac{1}{F^{(k)}}\Big)-\overline{N}\Big(r,\frac{1}{F^{(k)}}\Big)\\ \leqslant N\Big(r,\frac{1}{F^{(k+1)}}\Big)\leqslant N\Big(r,\frac{1}{F^{(k)}}\Big)+S(r,F),\end{aligned}$$

that is,

$$N\Big(r,\frac{1}{F^{(k)}-1}\Big)-\overline{N}\Big(r,\frac{1}{F^{(k)}-1}\Big)\leqslant \overline{N}\Big(r,\frac{1}{F^{(k)}}\Big)+S(r,F). \tag{2}$$

It is easy to see that

$$\overline{N}\Big(r,\frac{1}{F^{(k)}}\Big)\leqslant N\Big(r,\frac{1}{F^{(k)}}\Big)-\Big[N_{k+1}\Big(r,\frac{1}{F}\Big)-(k+1)\overline{N}_{k+1}\Big(r,\frac{1}{F}\Big)\Big].$$

It follows from Lemma 2 that

$$\overline{N}\Big(r,\frac{1}{F^{(k)}}\Big)\leqslant \Big[N\Big(r,\frac{1}{F}\Big)-N_{k+1}\Big(r,\frac{1}{F}\Big)+(k+1)\overline{N}_{k+1}\Big(r,\frac{1}{F}\Big)\Big]+S(r,F).$$

From the definition of $N_{k+1}(r,\frac{1}{F})$ we see that

$$N\Big(r,\frac{1}{F}\Big)-N_{k+1}\Big(r,\frac{1}{F}\Big)+(k+1)\overline{N}_{k+1}\Big(r,\frac{1}{F}\Big)\leqslant N_{k+1}\Big(r,\frac{1}{F}\Big).$$

The above two inequalities give

$$\overline{N}\Big(r,\frac{1}{F^{(k)}}\Big)\leqslant N_{k+1}\Big(r,\frac{1}{F}\Big)+S(r,F). \tag{3}$$

Combining (1)–(3), we obtain

$$\overline{N}_L\Big(r,\frac{1}{F^{(k)}-1}\Big)\leqslant \overline{N}\Big(r,\frac{1}{F^{(k)}}\Big)+S(r,F)\leqslant N_{k+1}\Big(r,\frac{1}{F}\Big)+S(r,F).$$

Lemma 3 is proved. □

Lemma 4. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k be three positive integers with $n > k+2$ and λ, μ constants such that $|\lambda|+|\mu| \neq 0$. Set*

$$F = f^n(\mu f^m + \lambda), \quad G = g^n(\mu g^m + \lambda), \tag{4}$$

$$H = \Big(\frac{F^{(k+2)}}{F^{(k+1)}} - 2\frac{F^{(k+1)}}{F^{(k)}-1}\Big) - \Big(\frac{G^{(k+2)}}{G^{(k+1)}} - 2\frac{G^{(k+1)}}{G^{(k)}-1}\Big). \tag{5}$$

Suppose that $F^{(k)}$ and $G^{(k)}$ share $(1, l)$. If $H \not\equiv 0$, then

(i) *when $l = 2$, then*

$$m\Big(r, \frac{1}{F^{(k)}}\Big) \leqslant N\Big(r, \frac{1}{G^{(k)}}\Big) - (n-k-2)N\Big(r, \frac{1}{f}\Big) - (n-k-2)N\Big(r, \frac{1}{g}\Big) + S(r,f) + S(r,g); \tag{6}$$

(ii) *when $l = 1$, then*

$$m\Big(r, \frac{1}{F^{(k)}}\Big) \leqslant N\Big(r, \frac{1}{G^{(k)}}\Big) - (n-k-2)N\Big(r, \frac{1}{f}\Big) - (n-k-2)N\Big(r, \frac{1}{g}\Big) + N_{k+2}\Big(r, \frac{1}{F}\Big) + S(r,f) + S(r,g); \tag{7}$$

(iii) *when $l = 0$, then*

$$m\Big(r, \frac{1}{F^{(k)}}\Big) \leqslant N\Big(r, \frac{1}{G^{(k)}}\Big) - (n-k-2)N\Big(r, \frac{1}{f}\Big) - (n-k-2)N\Big(r, \frac{1}{g}\Big) + 2N_{k+1}\Big(r, \frac{1}{F}\Big) + N_{k+1}\Big(r, \frac{1}{G}\Big) + S(r,f) + S(r,g). \tag{8}$$

Proof. Since $F^{(k)}$ and $G^{(k)}$ share $(1, l)$, using local expansion we see from (5) that, if z_0 is a common simple 1-point of $F^{(k)}$ and $G^{(k)}$, then $H(z_0) = 0$. Thus

$$N_{11}\Big(r, \frac{1}{F^{(k)}-1}\Big) = N_{11}\Big(r, \frac{1}{G^{(k)}-1}\Big) \leqslant N\Big(r, \frac{1}{H}\Big) \leqslant T(r,H) + O(1) \leqslant N(r,H) + S(r,f) + S(r,g). \tag{9}$$

By the Second Fundamental Theorem we have

$$T(r, F^{(k)}) \leqslant \overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{F^{(k)}-1}\Big) - N_0\Big(r, \frac{1}{F^{(k+1)}}\Big) + S(r,f),$$
$$T(r, G^{(k)}) \leqslant \overline{N}\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}-1}\Big) - N_0\Big(r, \frac{1}{G^{(k+1)}}\Big) + S(r,g),$$

where $N_0(r, 1/F^{(k+1)})$ denotes the counting function which only counts points such that $F^{(k+1)} = 0$ but $F^{(k)}(F^{(k)} - 1) \neq 0$ and $N_0(r, 1/G^{(k+1)})$ is defined similarly.

By adding the above two inequalities and using (9), we get

$$
\begin{aligned}
(10)\quad & T(r, F^{(k)}) + T(r, G^{(k)}) \\
&\leqslant \overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{F^{(k)} - 1}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)} - 1}\Big) \\
&\quad - N_0\Big(r, \frac{1}{F^{(k+1)}}\Big) - N_0\Big(r, \frac{1}{G^{(k+1)}}\Big) + S(r, f) + S(r, g) \\
&= \overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{F^{(k)} - 1}\Big) \\
&\quad + \overline{N}\Big(r, \frac{1}{G^{(k)} - 1}\Big) + N(r, H) - N_{11}\Big(r, \frac{1}{F^{(k)} - 1}\Big) \\
&\quad - N_0\Big(r, \frac{1}{F^{(k+1)}}\Big) - N_0\Big(r, \frac{1}{G^{(k+1)}}\Big) + S(r, f) + S(r, g).
\end{aligned}
$$

For $l = 2$, $F^{(k)}$ and $G^{(k)}$ share 1 with weight 2. It follows from (5) that the poles of $H(z)$ possibly occur only at zeros of $F^{(k+1)}$ and $G^{(k+1)}$, and 1-points of $F^{(k)}$ (or $G^{(k)}$) with order at least 3. Then

$$
\begin{aligned}
(11)\qquad N(r, H) &\leqslant \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}_3\Big(r, \frac{1}{F^{(k)} - 1}\Big) \\
&\quad + N_0\Big(r, \frac{1}{F^{(k+1)}}\Big) + N_0\Big(r, \frac{1}{G^{(k+1)}}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
(12)\qquad & \overline{N}_3\Big(r, \frac{1}{F^{(k)} - 1}\Big) + \overline{N}\Big(r, \frac{1}{F^{(k)} - 1}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)} - 1}\Big) \\
&\leqslant N_{11}\Big(r, \frac{1}{F^{(k)} - 1}\Big) + N\Big(r, \frac{1}{G^{(k)} - 1}\Big) \\
&\leqslant N_{11}\Big(r, \frac{1}{F^{(k)} - 1}\Big) + T(r, G^{(k)}) + O(1).
\end{aligned}
$$

Combining (10)–(12), we obtain

$$
\begin{aligned}
(13)\quad T(r, F^{(k)}) &\leqslant \overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{G^{(k)}}\Big) \\
&\quad + S(r, f) + S(r, g).
\end{aligned}
$$

It is not difficult to see that

$$
\begin{aligned}
(14)\quad & \overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) \\
&= N\Big(r, \frac{1}{F^{(k)}}\Big) - \Big[N_2\Big(r, \frac{1}{F^{(k)}}\Big) - \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big)\Big] + \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) \\
&= N\Big(r, \frac{1}{F^{(k)}}\Big) - \Big[N_3\Big(r, \frac{1}{F^{(k)}}\Big) - 2\overline{N}_3\Big(r, \frac{1}{F^{(k)}}\Big)\Big].
\end{aligned}
$$

If z_0 is a zero of f with multiplicity l ($\geqslant 1$), then z_0 is a zero of $F^{(k)} = [f^n(\mu f^m + \lambda)]^{(k)}$ with multiplicity at least 3 since $nl - k > (k+2)l - k = (l-1)k + 2l \geqslant 2$, so we have

$$N_3\Big(r, \frac{1}{F^{(k)}}\Big) - 2\overline{N}_3\Big(r, \frac{1}{F^{(k)}}\Big) \geqslant (n-k-2)N\Big(r, \frac{1}{f}\Big). \tag{15}$$

Inequalities (14) and (15) yield that

$$\overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) \leqslant N\Big(r, \frac{1}{F^{(k)}}\Big) - (n-k-2)N\Big(r, \frac{1}{f}\Big). \tag{16}$$

Similarly, we have

$$\overline{N}\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{G^{(k)}}\Big) \leqslant N\Big(r, \frac{1}{G^{(k)}}\Big) - (n-k-2)N\Big(r, \frac{1}{g}\Big). \tag{17}$$

Substituting (16) and (17) in (13) and noting that

$$m\Big(r, \frac{1}{F^{(k)}}\Big) = T(r, F^{(k)}) - N\Big(r, \frac{1}{F^{(k)}}\Big) + O(1), \tag{18}$$

we get (6).

For $l = 1$, $F^{(k)}$ and $G^{(k)}$ share (1,1). From (5), we see that the poles of H possibly occur only at zeros of $F^{(k+1)}$ and $G^{(k+1)}$, and 1-points of $F^{(k)}$ and $G^{(k)}$ are of order at least 2. Then we have

$$\begin{aligned} N(r, H) \leqslant \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{F^{(k)}-1}\Big) \\ + N_0\Big(r, \frac{1}{F^{(k+1)}}\Big) + N_0\Big(r, \frac{1}{G^{(k+1)}}\Big) \end{aligned} \tag{19}$$

and

$$\begin{aligned} \overline{N}\Big(r, \frac{1}{F^{(k)}-1}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}-1}\Big) &\leqslant N_{11}\Big(r, \frac{1}{F^{(k)}-1}\Big) + N\Big(r, \frac{1}{G^{(k)}-1}\Big) \\ &\leqslant N_{11}\Big(r, \frac{1}{F^{(k)}-1}\Big) + T(r, G^{(k)}) + O(1). \end{aligned} \tag{20}$$

Combining (10), (19) and (20), we get

$$\begin{aligned} T\Big(r, F^{(k)}\Big) \leqslant \overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{G^{(k)}}\Big) \\ + \overline{N}_2\Big(r, \frac{1}{F^{(k)}-1}\Big) + S(r, f) + S(r, g). \end{aligned} \tag{21}$$

It follows from (3) that

$$\overline{N}_2\Big(r, \frac{1}{F^{(k)}-1}\Big) \leqslant \overline{N}\Big(r, \frac{1}{F^{(k+1)}}\Big) \leqslant N_{k+2}\Big(r, \frac{1}{F}\Big). \tag{22}$$

Then, from (16)–(18) and (21)–(22) we obtain (7).

For $l = 0$, $F^{(k)}$ and $G^{(k)}$ share 1 IM. We see from (5) that H has poles possibly only at zeros of $F^{(k+1)}$ and $G^{(k+1)}$, and 1-points of $F^{(k)}$ and $G^{(k)}$ with different order. Then

$$\begin{aligned} N(r,H) \leqslant{} & \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}_L\Big(r, \frac{1}{F^{(k)}-1}\Big) \\ & + \overline{N}_L\Big(r, \frac{1}{G^{(k)}-1}\Big) + N_0\Big(r, \frac{1}{F^{(k+1)}}\Big) + N_0\Big(r, \frac{1}{G^{(k+1)}}\Big) \end{aligned} \tag{23}$$

and

$$\begin{aligned} & \overline{N}\Big(r, \frac{1}{F^{(k)}-1}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}-1}\Big) \\ & \leqslant N_{11}\Big(r, \frac{1}{F^{(k)}-1}\Big) + \overline{N}_L\Big(r, \frac{1}{F^{(k)}-1}\Big) + N\Big(r, \frac{1}{G^{(k)}-1}\Big) \\ & \leqslant N_{11}\Big(r, \frac{1}{F^{(k)}-1}\Big) + \overline{N}_L\Big(r, \frac{1}{F^{(k)}-1}\Big) + T(r, G^{(k)}) + O(1). \end{aligned} \tag{24}$$

Combining (10), (23) and (24), we have

$$\begin{aligned} T(r, F^{(k)}) \leqslant{} & \overline{N}\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}\Big(r, \frac{1}{G^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{F^{(k)}}\Big) + \overline{N}_2\Big(r, \frac{1}{G^{(k)}}\Big) \\ & + 2\overline{N}_L\Big(r, \frac{1}{F^{(k)}-1}\Big) + \overline{N}_L\Big(r, \frac{1}{G^{(k)}-1}\Big) + S(r,f) + S(r,g). \end{aligned} \tag{25}$$

Lemma 3 implies that

$$\begin{aligned} & 2\overline{N}_L\Big(r, \frac{1}{F^{(k)}-1}\Big) + \overline{N}_L\Big(r, \frac{1}{G^{(k)}-1}\Big) \\ & \leqslant 2N_{k+1}\Big(r, \frac{1}{F}\Big) + N_{k+1}\Big(r, \frac{1}{G}\Big) + S(r,f) + S(r,g). \end{aligned} \tag{26}$$

Combining (16)–(18) and (25)–(26), we have (8). Lemma 4 is proved. □

Remark 3. Clearly, Lemma 4 is still valid if $F = f^n(\mu f^m + \lambda)$ and $G = g^n(\mu g^m + \lambda)$ are replaced by $F = f^n(f-1)^m$ and $G = g^n(g-1)^m$.

3. Proof of Theorem 1

Proof of Theorem 1. Let F, G be defined by (4). Then $F^{(k)}$ and $G^{(k)}$ share $(1,l)$. By Lemma 1 and Nevanlinna first fundamental theorem, we have

$$(27)\qquad \begin{aligned}(n+m^*)T(r,f) &= T(r,F)+S(r,f)\\ &\leqslant m\Big(r,\frac{1}{F^{(k)}}\Big)+m\Big(r,\frac{F^{(k)}}{F}\Big)+N(r,\frac{1}{F})+S(r,f)\\ &= m\Big(r,\frac{1}{F^{(k)}}\Big)+N\Big(r,\frac{1}{F}\Big)+S(r,f).\end{aligned}$$

Suppose that $H\not\equiv 0$, where H is defined by (5).

If $l=2$, we have (6). Substituting (6) in (27) and using Lemma 2, we have

$$(28)\qquad \begin{aligned}&(n+m^*)T(r,f)\\ &\leqslant N\Big(r,\frac{1}{F}\Big)+N\Big(r,\frac{1}{G^{(k)}}\Big)-(n-k-2)\Big[N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big]\\ &\quad+S(r,f)+S(r,g)\\ &\leqslant N\Big(r,\frac{1}{F}\Big)+N\Big(r,\frac{1}{G}\Big)-(n-k-2)\Big[N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big]\\ &\quad+S(r,f)+S(r,g)\\ &= N\Big(r,\frac{1}{\mu f^m+\lambda}\Big)+N\Big(r,\frac{1}{\mu g^m+\lambda}\Big)+(k+2)\Big[N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big]\\ &\quad+S(r,f)+S(r,g).\end{aligned}$$

Similarly, we have

$$(29)\qquad \begin{aligned}(n+m^*)T(r,g) &\leqslant N\Big(r,\frac{1}{\mu f^m+\lambda}\Big)+N\Big(r,\frac{1}{\mu g^m+\lambda}\Big)\\ &\quad+(k+2)\Big[N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big]+S(r,f)+S(r,g).\end{aligned}$$

By adding the above two inequalities, we obtain

$$\begin{aligned}&(n+m^*)[T(r,f)+T(r,g)]\\ &\leqslant 2\Big[N\Big(r,\frac{1}{\mu f^m+\lambda}\Big)+N\Big(r,\frac{1}{\mu g^m+\lambda}\Big)\Big]+(2k+4)\Big[N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big]\\ &\quad+S(r,f)+S(r,g)\\ &\leqslant (2k+2m^*+4)[T(r,f)+T(r,g)]+S(r,f)+S(r,g),\end{aligned}$$

that is,

$$(n-2k-m^*-4)[T(r,f)+T(r,g)]\leqslant S(r,f)+S(r,g),$$

which is impossible since $n>2k+m^*+4$.

If $l=1$, then substituting (7) in (27), and using Lemma 2, we get

$$(30)\ (n+m^*)T(r,f) \leqslant N\Big(r,\frac{1}{F}\Big) + N\Big(r,\frac{1}{G}\Big) + N_{k+2}\Big(r,\frac{1}{F}\Big) - (n-k-2)\Big[N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\Big] + S(r,f) + S(r,g).$$

Noting that $n > 3k+2m^*+6$, we have

$$N_{k+2}\Big(r,\frac{1}{F}\Big) = (k+2)\overline{N}\Big(r,\frac{1}{f}\Big) + N_{k+2}\Big(r,\frac{1}{\mu f^m+\lambda}\Big).$$

Then we can deduce from (30) that

$$(31)\ (n+m^*)T(r,f) \leqslant 2N\Big(r,\frac{1}{\mu f^m+\lambda}\Big) + N\Big(r,\frac{1}{\mu g^m+\lambda}\Big) + (2k+4)N\Big(r,\frac{1}{f}\Big) + (k+2)N\Big(r,\frac{1}{g}\Big) + S(r,f) + S(r,g).$$

Similarly, we have

$$(32)\ (n+m^*)T(r,g) \leqslant 2N\Big(r,\frac{1}{\mu g^m+\lambda}\Big) + N\Big(r,\frac{1}{\mu f^m+\lambda}\Big) + (2k+4)N\Big(r,\frac{1}{g}\Big) + (k+2)N\Big(r,\frac{1}{f}\Big) + S(r,f) + S(r,g).$$

By adding (31)–(32) we obtain

$$\begin{aligned}(n+m^*)&[T(r,f)+T(r,g)]\\ &\leqslant 3\Big[N\Big(r,\frac{1}{\mu f^m+\lambda}\Big) + N\Big(r,\frac{1}{\mu g^m+\lambda}\Big)\Big] + (3k+6)\Big[N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\Big]\\ &\quad + S(r,f)+S(r,g)\\ &\leqslant (3k+3m^*+6)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),\end{aligned}$$

that is,

$$(n-3k-2m^*-6)[T(r,f)+T(r,g)] \leqslant S(r,f)+S(r,g),$$

which contradicts the assumption $n > 3k+2m^*+6$.

If $l=0$, we have (8). Using the same argument as above, we have

$$\begin{aligned}(n+m^*)&[T(r,f)+T(r,g)]\\ &\leqslant (5k+7)\Big[N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{g}\Big)\Big] + 5\Big[N\Big(r,\frac{1}{\mu f^m+\lambda}\Big) + N\Big(r,\frac{1}{\mu g^m+\lambda}\Big)\Big]\\ &\quad + S(r,f)+S(r,g)\\ &\leqslant (5k+5m^*+7)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),\end{aligned}$$

that is,

$$(n - 5k - 4m^* - 7)[T(r,f) + T(r,g)] \leqslant S(r,f) + S(r,g),$$

a contradiction, since $n > 5k + 4m^* + 7$.

Therefore $H \equiv 0$. Integrating $H \equiv 0$ yields

$$\frac{F^{(k+1)}}{(F^{(k)} - 1)^2} = A \frac{G^{(k+1)}}{(G^{(k)} - 1)^2},$$

where A is a nonzero constant. It follows that $F^{(k)}$ and $G^{(k)}$ share 1 CM. So by Theorem C′ we obtain the conclusion of Theorem 1. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Using almost the same argument as in the proof of Theorem 1, we can get the conclusion of Theorem 2. Here we omit the details. □

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References

[1] *M.-L. Fang*: Uniqueness and value-sharing of entire functions. Comput. Math. Appl. *44* (2002), 823–831.

[2] *H. Fujimoto*: Uniqueness problem with truncated multiplicities in value distribution theory. Nagoya Math. J. *152* (1998), 131–152.

[3] *W. K. Hayman*: Meromorphic Functions. Clarendon Press, Oxford, 1964.

[4] *I. Lahiri*: Weighted value sharing and uniqueness of meromorphic functions. Complex Variables, Theory Appl. *46* (2001), 241–253.

[5] *I. Lahiri*: Weighted sharing and uniqueness of meromorphic functions. Nagoya Math. J. *161* (2001), 193–206.

[6] *C.-C. Yang*: On deficiencies of differential polynomials. Math. Z. *152* (1972), 107–112.

[7] *L. Yang*: Value Distribution Theory. Springer-Science Press, Berlin-Beijing, 1993.

[8] *H.-X. Yi, C.-C. Yang*: Uniqueness Theory of Meromorphic Functions. Science Press, Beijing, 1995.

[9] *T. D. Zhang, W. R. Lü*: Uniqueness theorems on meromorphic functions sharing one value. Comput. Math. Appl. *55* (2008), 2981–2992.

[10] *X.-Y. Zhang, W.-C. Lin*: Uniqueness and value-sharing of entire functions. J. Math. Anal. Appl. *343* (2008), 938–950.

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