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MATLIS REFLEXIVE AND GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ an ideal of R and N and L two Matlis reflexive R-modules with $\operatorname{Supp}(L) \subseteq V(\mathfrak{a})$. We prove that if M is a finitely generated R-module, then $\operatorname{Ext}^i_R(L, H^j_{\mathfrak{a}}(M, N))$ is Matlis reflexive for all i and j in the following cases:
(a) $\dim R/\mathfrak{a} = 1$;
(b) $\operatorname{cd}(\mathfrak{a}) = 1$; where cd is the cohomological dimension of $\mathfrak{a}$ in R;
(c) $\dim R \leqslant 2$.
In these cases we also prove that the Bass numbers of $H^j_{\mathfrak{a}}(M, N)$ are finite.

Keywords: Bass numbers, generalized local cohomology modules, Matlis reflexive

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1. Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and $\mathfrak{a}$ an ideal of R. For an integer $j \geqslant 0$, the jth generalized local cohomology module $H^j_{\mathfrak{a}}(M, N)$ of two R-modules M and N with respect to an ideal $\mathfrak{a}$ was defined by Herzog [10] as follows:

$$H^j_{\mathfrak{a}}(M, N) = \varinjlim_n \operatorname{Ext}^j_R(M/\mathfrak{a}^n M, N).$$

It is clear that $H^j_{\mathfrak{a}}(R, N)$ is just the ordinary local cohomology module $H^j_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$ (cf. [1]).

Hartshorne [9] defined a module T to be $\mathfrak{a}$-cofinite if $\operatorname{Supp}(T) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, T)$ is finitely generated for all i. He proved that if R is a complete regular ring and $\mathfrak{a}$ is either a principal ideal or a prime ideal with $\dim R/\mathfrak{a} = 1$, then the

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local cohomology modules $H^j_{\mathfrak{a}}(N)$ are $\mathfrak{a}$-cofinite for all finitely generated modules N. Kawasaki [13] showed that, in general, the local cohomology modules $H^j_{\mathfrak{a}}(N)$ are $\mathfrak{a}$-cofinite for all finitely generated modules N, where the ideal $\mathfrak{a}$ is principal. Delfino and Marley [6] and Yoshida [22], in general, proved that if $\mathfrak{a}$ is an ideal of R with $\dim R/\mathfrak{a} = 1$ and L is finitely generated with $\operatorname{Supp}(L) \subseteq V(\mathfrak{a})$, then $\operatorname{Ext}^i_R(L, H^j_{\mathfrak{a}}(N))$ is finitely generated for all i and j and for all finitely generated modules N (see also [5] and [11]). Melkersson [17] proved that if R is a local ring with $\dim R \leqslant 2$, then $H^j_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all j and all finitely generated modules N (see also [18]). Belshoff, Slattery and Wickham [3] and Belshoff and Slattery [2] extended the results of Hartshorne to larger class of modules. In fact, they showed that if R is a complete Gorenstein domain, $\mathfrak{a}$ is either a principal ideal or an ideal with $\dim R/\mathfrak{a} = 1$, and M and N are Matlis reflexive modules with $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$, then $\operatorname{Ext}^i_R(M, H^j_{\mathfrak{a}}(N))$ is Matlis reflexive for all i and j. Recall that an R-module N is Matlis reflexive if $D(D(N)) = N$, where $D(-) = \operatorname{Hom}_R(-, E(R/\mathfrak{m}))$ is the Matlis duality functor. Khashyarmanesh and Khosh-Ahang [14] proved that if R is a complete ring and M and N are Matlis reflexive modules with $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$, then $\operatorname{Ext}^i_R(M, H^j_{\mathfrak{a}}(N))$ is Matlis reflexive for all i and j in the following cases:

(a) $\dim R/\mathfrak{a} = 1$;
(b) $\mathfrak{a}$ is a principal ideal;
(c) $\dim R \leqslant 2$.

The goal of the present paper is to extend the main results of Khashyarmanesh and Khosh-Ahang [14] to generalized local cohomology modules.

2. Preliminaries

Throughout this paper we assume that R is a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, M a finitely generated R-module and N a Matlis reflexive R-module. We now briefly recall some known facts on generalized local cohomology modules.

Lemma 2.1 (see [15]). *Let X be an $\mathfrak{a}$-torsion R-module; that is, $\Gamma_{\mathfrak{a}}(X) = X$. Then $H^j_{\mathfrak{a}}(M, X) \cong \operatorname{Ext}^j_R(M, X)$ for all $j \geqslant 0$.*

Lemma 2.2 (see [8]). *The following assertions hold:*

(i) *If $0 \longrightarrow N \longrightarrow E^\bullet$ is an injective resolution of N, then $H^j_{\mathfrak{a}}(M, N) \cong H^j(\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M, E^\bullet))) \cong H^j(\operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^\bullet)))$ for all $j \geqslant 0$. In particular, $H^j_{\mathfrak{a}}(M, N) \cong H^j_{\sqrt{\mathfrak{a}}}(M, N)$ for all $j \geqslant 0$.*

(ii) *If $f\colon R \longrightarrow S$ is a flat ring homomorphism, then $H^j_{\mathfrak{a}}(M, N) \otimes_R S \cong H^j_{\mathfrak{a}S}(M \otimes_R S, N \otimes_R S)$ for all $j \geqslant 0$.*

Lemma 2.3 (see [16]). *Let X be a finitely generated R-module. Then $H^j_{\mathfrak{a}}(M, X)$ is $\mathfrak{a}$-cofinite for all $j \geqslant 0$, whenever one of the following conditions holds:*

(i) $\dim R \leqslant 2$;

(ii) $\mathrm{cd}(\mathfrak{a}) = 1$.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{p}$ a prime ideal of R with $\dim R/\mathfrak{p} = 1$, and X a finitely generated module. Then $H^j_{\mathfrak{p}}(M, X)$ is $\mathfrak{p}$-cofinite for all $j \geqslant 0$.*

Proof. By [6, Proposition 2] and Lemma 2.2, we can assume that R is a complete regular local ring. Hence the result follows by [7, Theorem 2.9]. □

Lemma 2.5 (see [6]). *Let X be an R-module. Then the following assertions are equivalent:*

(i) $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ *is finitely generated for all* $i \geqslant 0$;

(ii) $\mathrm{Ext}^i_R(L, X)$ *is finitely generated for all finitely generated modules* L *with* $\mathrm{Supp}(L) \subseteq V(\mathfrak{a})$ *and all* $i \geqslant 0$.

3. The results

We begin this section by recalling some general facts about Matlis reflexive modules. First, any module of finite length and any Artinian module over a local ring are Matlis reflexive (see, for example, [19] and [20]).

Lemma 3.1 (see [4]). *Let $(R, \mathfrak{m})$ be a local ring. If N is a Matlis reflexive R-module, then there is a short exact sequence*

$$0 \longrightarrow S \longrightarrow N \longrightarrow A \longrightarrow 0$$

with S finitely generated and complete, and A Artinian.

Lemma 3.2 (see [20]). *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules and R-homomorphisms. Then B is Matlis reflexive if and only if A and C are Matlis reflexive.*

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a complete local ring and let L be a finitely generated R-module with* $\operatorname{Supp}(L) \subseteq V(\mathfrak{a})$. *Suppose that one of the following cases hold:*

(α) $\operatorname{cd}(\mathfrak{a}) = 1$;

(β) $\dim R \leqslant 2$.

Then $\operatorname{Ext}^i_R(L, H^j_{\mathfrak{a}}(M, N))$ *is Matlis reflexive for all i and j.*

Proof. Since N is Matlis reflexive, by Lemma 3.1 there is a short exact sequence

$$0 \longrightarrow S \longrightarrow N \longrightarrow A \longrightarrow 0$$

with S finitely generated and A Artinian. Hence we obtain the long exact sequence of generalized local cohomology

$$\ldots \longrightarrow H^j_{\mathfrak{a}}(M, S) \xrightarrow{h^j} H^j_{\mathfrak{a}}(M, N) \xrightarrow{f^j} \operatorname{Ext}^j_R(M, A) \xrightarrow{g^j} H^{j+1}_{\mathfrak{a}}(M, S) \longrightarrow \ldots.$$

Set $X^j = \operatorname{Im} f^j$, $Y^j = \operatorname{Im} h^j$ and $Z^j = \operatorname{Im} g^j$. Now, consider the exact sequences

$$(\dagger) \qquad 0 \longrightarrow Z^{j-1} \longrightarrow H^j_{\mathfrak{a}}(M, S) \longrightarrow Y^j \longrightarrow 0,$$

$$(\ddagger) \qquad 0 \longrightarrow Y^j \longrightarrow H^j_{\mathfrak{a}}(M, N) \longrightarrow X^j \longrightarrow 0.$$

Hence we note that X^j and Z^j are Artinian for all $j \geqslant 0$, since $\operatorname{Ext}^j_R(M, A)$ is Artinian for all $j \geqslant 0$. Let $j \geqslant 0$ be fixed arbitrary. By the exact sequence ($\dagger$), we have an exact sequence:

$$\begin{aligned} \ldots \longrightarrow \operatorname{Ext}^i_R(L, Z^{j-1}) \longrightarrow \operatorname{Ext}^i_R(L, H^j_{\mathfrak{a}}(M, S)) \longrightarrow \operatorname{Ext}^i_R(L, Y^j) \\ \longrightarrow \operatorname{Ext}^{i+1}_R(L, Z^{j-1}) \longrightarrow \ldots. \end{aligned}$$

Now, $\operatorname{Ext}^i_R(L, Z^{j-1})$ is Matlis reflexive for all $i \geqslant 0$ and by Lemmas 2.3, 2.5 $\operatorname{Ext}^i_R(L, H^j_{\mathfrak{a}}(M, S))$ is Matlis reflexive for all $i \geqslant 0$. Hence $\operatorname{Ext}^i_R(L, Y^j)$ is Matlis reflexive for all $i \geqslant 0$. Furthermore, we obtain an exact sequence by ($\ddagger$):

$$\ldots \longrightarrow \operatorname{Ext}^i_R(L, Y^j) \longrightarrow \operatorname{Ext}^i_R(L, H^j_{\mathfrak{a}}(M, N)) \longrightarrow \operatorname{Ext}^i_R(L, X^j) \longrightarrow \ldots.$$

Since $\operatorname{Ext}^i_R(L, Y^j)$ and $\operatorname{Ext}^i_R(L, X^j)$ are Matlis reflexive for all $i \geqslant 0$, $\operatorname{Ext}^i_R(L, H^j_{\mathfrak{a}}(M, N))$ is Matlis reflexive for all $i \geqslant 0$. The proof is complete. $\square$

Lemma 3.4. *Let $(R,\mathfrak{m})$ be a local ring. Then $H^j_{\mathfrak{m}}(M,N)$ is Artinian for all $j \geqslant 0$.*

Proof. By Lemma 3.1, there is an exact sequence

$$0 \longrightarrow S \longrightarrow N \longrightarrow A \longrightarrow 0$$

with S finitely generated and A Artinian. This induces the long exact sequence

$$\ldots \longrightarrow H^j_{\mathfrak{m}}(M,S) \longrightarrow H^j_{\mathfrak{m}}(M,N) \longrightarrow \operatorname{Ext}^j_R(M,A) \longrightarrow \ldots.$$

Since $H^j_{\mathfrak{m}}(M,S)$ by [8, Theorem 2.2] and $\operatorname{Ext}^j_R(M,A)$ are Artinian for all $i \geqslant 0$, we have that $H^j_{\mathfrak{m}}(M,N)$ is Artinian for all $i \geqslant 0$. □

Theorem 3.5. *Let $(R,\mathfrak{m})$ be a complete local ring and $\mathfrak{a}$ an ideal of R with $\dim R/\mathfrak{a} = 1$. Then for any finitely generated R-module L with $\operatorname{Supp}(L) \subseteq V(\mathfrak{a})$, $\operatorname{Ext}^i_R(L, H^j_{\mathfrak{a}}(M,N))$ is Matlis reflexive for all i and j.*

Proof. We may assume that $\mathfrak{a} = \sqrt{\mathfrak{a}}$ by Lemma 2.2. Let $\mathfrak{a} = \bigcap\limits_{i=1}^{n} \mathfrak{p}$ be the irredundant primary decomposition. Now, we proceed by induction on the number n. Let $n = 1$. We consider the exact sequence

$$0 \longrightarrow S \longrightarrow N \longrightarrow A \longrightarrow 0$$

with S finitely generated and A Artinian. This induces a long exact sequence

$$\ldots \longrightarrow H^j_{\mathfrak{a}}(M,S) \longrightarrow H^j_{\mathfrak{a}}(M,N) \longrightarrow \operatorname{Ext}^j_R(M,A) \longrightarrow \ldots.$$

By Lemmas 2.4, 2.5 and using the same arguments as in the proof of Theorem 3.3 the result follows in this case. Now suppose that $n \geqslant 2$, and that the assertion holds for $n-1$. Put $\mathfrak{a}_1 = \mathfrak{p}_1$ and $\mathfrak{a}_2 = \bigcap\limits_{i=2}^{n} \mathfrak{p}_i$. One can easily see that $V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) = V(\mathfrak{a})$ and $V(\mathfrak{a}_1) \cap V(\mathfrak{a}_2) = V(\mathfrak{m})$, since $\mathfrak{a}$ is an ideal of dimension one. We have a Mayer-Vietoris exact sequence (cf. [21, Corollary 2.14])

$$\ldots \longrightarrow H^j_{\mathfrak{m}}(M,N) \xrightarrow{f^j} H^j_{\mathfrak{a}_1}(M,N) \oplus H^j_{\mathfrak{a}_2}(M,N) \xrightarrow{h^j} H^j_{\mathfrak{a}}(M,N) \xrightarrow{g^j} \ldots$$

Set $X^j = \operatorname{Im} f^j$, $Y^j = \operatorname{Im} h^j$ and $Z^j = \operatorname{Im} g^j$. Hence there are exact sequences

$$(\dagger) \qquad 0 \longrightarrow Y^j \longrightarrow H^j_{\mathfrak{a}}(M,N) \longrightarrow Z^j \longrightarrow 0,$$

$$(\ddagger) \qquad 0 \longrightarrow X^j \longrightarrow H^j_{\mathfrak{a}_1}(M,N) \oplus H^j_{\mathfrak{a}_2}(M,N) \longrightarrow Y^j \longrightarrow 0.$$

Here we note that X^j and Z^j are Artinian for all $j \geqslant 0$, since by Lemma 3.4 $H^j_{\mathfrak{m}}(M,N)$ is Artinian for all $j \geqslant 0$. Let $j \geqslant 0$ be fixed arbitrary. By the exact sequence (‡), we have an exact sequence

$$\ldots \longrightarrow \operatorname{Ext}^i_R(L,X^j) \longrightarrow \operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}_1}(M,N)) \oplus \operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}_2}(M,N)) \longrightarrow \\ \operatorname{Ext}^i_R(L,Y^j) \longrightarrow \operatorname{Ext}^{i+1}_R(L,X^j) \longrightarrow \ldots.$$

Now $\operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}_1}(M,N))$ and $\operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}_2}(M,N))$ are Matlis reflexive for all $i \geqslant 0$, by induction hypothesis. Since $\operatorname{Ext}^i_R(L,X^j)$ is Matlis reflexive, $\operatorname{Ext}^i_R(L,Y^j)$ is Matlis reflexive for all $i \geqslant 0$. Moreover, we obtain an exact sequence by (†):

$$\ldots \longrightarrow \operatorname{Ext}^i_R(L,Y^j) \longrightarrow \operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}}(M,N)) \longrightarrow \operatorname{Ext}^i_R(L,Z^j) \longrightarrow \ldots.$$

Since Z^j is also Artinian, $\operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}}(M,N))$ is Matlis reflexive for all $i \geqslant 0$ by the above exact sequence, as required. □

The following result extends [12, Theorem 1].

Corollary 3.6. *Let $(R,\mathfrak{m})$ be a complete local ring and suppose that one of the following cases occurs*
(a) $\mathfrak{a}$ *is an ideal of R with* $\dim R/\mathfrak{a} = 1$;
(b) $\operatorname{cd}(\mathfrak{a}) = 1$;
(c) $\dim R \leqslant 2$.
Then the Bass numbers of generalized local cohomology modules $H^j_{\mathfrak{a}}(M,N)$ are finite for all $j \geqslant 0$.

Proof. Let k be the residue field of R. Then $\operatorname{Ext}^i_R(k,H^j_{\mathfrak{a}}(M,N))$ is Matlis reflexive by Theorems 3.3, 3.5. Since $\operatorname{Ext}^i_R(k,H^j_{\mathfrak{a}}(M,N))$ is also a k vector space, it must be finitely generated. If $\mathfrak{p}$ is any non-maximal prime ideal, it follows from Lemma 3.1 that $N_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$. We have $(H^j_{\mathfrak{a}}(M,N))_{\mathfrak{p}} \cong H^j_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})$ if $\mathfrak{p} \supseteq \mathfrak{a}$ or $(H^j_{\mathfrak{a}}(M,N))_{\mathfrak{p}} = 0$ if $\mathfrak{p} \not\supseteq \mathfrak{a}$. In either case, it follows that $\operatorname{Ext}^i_R(R/\mathfrak{p},H^j_{\mathfrak{a}}(M,N))_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$. □

Corollary 3.7. *Let $(R,\mathfrak{m})$ be a complete local ring and suppose that one of the following cases occurs:*
(a) $\mathfrak{a}$ *is an ideal of R with* $\dim R/\mathfrak{a} = 1$;
(b) $\operatorname{cd}(\mathfrak{a}) = 1$;
(c) $\dim R \leqslant 2$.
If L is Artinian, then $\operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}}(M,N))$ is finitely generated (and thus Matlis reflexive) for all i and j.

Proof. By Corollary 3.6, $H^j_{\mathfrak{a}}(M,N)$ has finite Bass numbers. Fix j. Let $0 \longrightarrow H^j_{\mathfrak{a}}(M,N) \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots$ be a minimal injective resolution of $H^j_{\mathfrak{a}}(M,N)$. Hence for each t, $\operatorname{Hom}_R(L,E^t) = \oplus \operatorname{Hom}_R(L,E(R/\mathfrak{m}))$ where the direct sum is a finite direct sum, since $H^j_{\mathfrak{a}}(M,N)$ has finite Bass numbers. By Matlis duality, $\operatorname{Hom}_R(L,E(R/\mathfrak{m}))$ is finitely generated. Thus $\operatorname{Hom}_R(L,E^t)$, and hence $\operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}}(M,N))$, is finitely generated. □

The following corollary is a generalization of [14, Theorem 2.2].

Corollary 3.8. *Let $(R,\mathfrak{m})$ be a complete local ring and suppose that one of the following cases occurs*
(a) *$\mathfrak{a}$ is an ideal of R with $\dim R/\mathfrak{a} = 1$;*
(b) *$\operatorname{cd}(\mathfrak{a}) = 1$;*
(c) *$\dim R \leqslant 2$.*
If L is Matlis reflexive with $\operatorname{Supp}(L) \subseteq V(\mathfrak{a})$, then $\operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}}(M,N))$ is Matlis reflexive for all i and j.

Proof. Since L is Matlis reflexive, there is a short exact sequence

$$0 \longrightarrow S \longrightarrow L \longrightarrow A \longrightarrow 0$$

with S finitely generated and A Artinian. This induces a long exact sequence

$$\ldots \longrightarrow \operatorname{Ext}^i_R(A,H^j_{\mathfrak{a}}(M,N)) \longrightarrow \operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}}(M,N)) \longrightarrow \operatorname{Ext}^i_R(S,H^j_{\mathfrak{a}}(M,N)) \longrightarrow \ldots.$$

By Theorems 3.3, 3.5 $\operatorname{Ext}^i_R(S,H^j_{\mathfrak{a}}(M,N))$ is Matlis reflexive. By Corollary 3.7, $\operatorname{Ext}^i_R(A,H^j_{\mathfrak{a}}(M,N))$ is Matlis reflexive. Thus $\operatorname{Ext}^i_R(L,H^j_{\mathfrak{a}}(M,N))$ is Matlis reflexive for all i and j. □

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