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COVER MATRICES OF POSETS AND THEIR SPECTRA

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Abstract. We analyze the spectra of the cover matrix of a given poset. Some consequences on the multiplicities are provided.

Keywords: poset, Boolean algebra, cover matrix, spectra, multiplicities

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1. Introduction

For any positive integers m and n, let $\mathcal{P}_n^{(m)}$ be the set of all functions with domain $[n]=\{1,2,\ldots,n\}$ and range $\mathbb{Z}_m=\{0,1,\ldots,m-1\}$, i.e., $\mathcal{P}_n^{(m)}=\mathbb{Z}_m^{[n]}$. The set $\mathcal{P}_n^{(m)}$ equipped with the partial order

$$f \leqslant g \quad \text{if and only if} \quad f(x) \leqslant g(x) \text{ for all } x \in [n]$$

is a partially ordered set or, simply, a poset, denoted by $(\mathcal{P}_n^{(m)}, \leqslant)$. The so-called cover graph of this poset, denoted by $\mathcal{G}_n^{(m)}$, is an undirected graph whose vertices are labeled by the elements of $\mathcal{P}_n^{(m)}$ and there is an edge connecting f and g if and only if one vertex covers the other one, i.e., if $f \sim g$, then f and g are comparable and there is no $h \in \mathcal{P}_n^{(m)} \setminus \{f,g\}$ such that $f \leqslant h \leqslant g$ or $g \leqslant h \leqslant f$. Analogously, we may say that f is adjacent to g if and only if they differ at only one point and the difference is one unit. The adjacency matrix of $\mathcal{G}_n^{(m)}$, also known as the cover matrix, will be denoted by $A_n^{(m)}$, and the spectrum of $\mathcal{P}_n^{(m)}$, denoted by $\sigma(\mathcal{P}_n^{(m)})$, is the set of eigenvalues of $A_n^{(m)}$.

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Spectral technique for general cover graphs have not been well developed due to the computational cost for calculating the spectrum, although this concept has a vast variety of different applications, playing an important role in enumerative combinatorics, computer and information sciences, quantum mechanics, theoretical chemistry, statistical physics, among others (cf. [2], [10], [11], [12], [13] and references therein).

There are some interesting recent extensions to double cover graphs, where the study of the characteristic polynomial and the spectra is essential to obtain expanders from magnifiers for studying networks (cf. [1], [9]).

There are other ways of portraying partially ordered sets, as the Hasse diagrams and zeta matrices. Recently, in [2], Ballantine et al. discussed the properties of the determinant of the matrix $\mathfrak{Z}_P = Z_P + Z_P^t$, where $Z_P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the zeta matrix of the Boolean algebra P of rank n, providing a combinatorial interpretation of $\det(\mathfrak{Z}_P)$ in terms of the adjacency matrices of comparability graphs, and showing that if n is even, then $\det(\mathfrak{Z}_n) = 2^{\alpha_n}$, where $\alpha_n = 4\alpha_{n-2} - 2$ for $n \geqslant 4$, with the initial condition $\alpha_2 = 2$. The main drawback of Hasse diagrams is the possibility of unduly largeness of the poset. On the other hand, zeta matrices provide a better analytic approach, since many operations in posets can be expressed by matrix multiplication. Cover matrices can be computed from the zeta matrices and the converse is true as well. So, as a variation of zeta matrices, we shall focus our attention on cover matrices, since these are more tractable.

In this note we examine the spectra of the cover matrix of the poset $(\mathcal{P}_n^{(m)}, \leqslant)$ and construct a modified Pascal triangle for the multiplicities of the eigenvalues in some particular cases. Our purpose is to find the adjacency matrix of $\mathcal{G}_n^{(m)}$, and using Chebyshev polynomials of the second kind, calculate the characteristic polynomial. We will be able to generalize some recent spectral results on Boolean algebras (cf. [2], [4]).

2. Cover matrices

Let us fix a positive integer m and recall that

$$\mathcal{P}_n^{(m)} = \{f \mid f\colon \{1, \ldots, n\} \longrightarrow \mathbb{Z}_m\}.$$

The cardinality of $\mathcal{P}_n^{(m)}$ is m^n.

To construct the cover matrix $A_n^{(m)}$ of $\mathcal{G}_n^{(m)}$, we order the elements of $\mathcal{P}_n^{(m)}$ in a reverse lexicographic manner. For example, when $m = 3$, the 9 maps of $\mathcal{P}_2^{(3)}$ are ordered as follows:

$$f_1 := f_{00} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \ f_2 := f_{10} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \ f_3 := f_{20} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix},$$
$$f_4 := f_{01} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ f_5 := f_{11} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \ f_6 := f_{21} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$
$$f_7 := f_{02} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \ f_8 := f_{12} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \ f_9 := f_{22} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

Two vertices, f and g, of the cover graph of $\mathcal{P}_n^{(m)}$ are adjacent provided they are comparable and there exists $x \in \{1, \dots, n\}$ such that $f(x) = g(x) + 1$ or $f(x) = g(x) - 1$. Defining $f(i) = a_i$ and $g(i) = b_i$ for $a_i, b_i \in \mathbb{Z}_m$ and $i = 1, \dots, n$, an induction argument leads to the concise formula for the distance between f and g:

$$d(f, g) = \sum_{i=1}^{n} |a_i - b_i| .$$

Still, about the structure of the cover graph $\mathcal{G}_n^{(m)}$, we can also say that the number of shortest paths between f and g is given by

$$\frac{\left(\displaystyle\sum_{i=1}^{n} |a_i - b_i|\right)!}{\displaystyle\prod_{i=1}^{n} (|a_i - b_i|!)}.$$

If one considers the maximum of the distance between any two vertices, one gets the following proposition:

Proposition 2.1. *The diameter of* $\mathcal{G}_n^{(m)}$ *is* $(m-1)n$.

If $m = 2$, i.e., in the Boolean case, the diameter of $\mathcal{G}_n^{(2)}$ is n.

Taking into account that the adjacency matrix of the cover graph of $\mathcal{P}_1^{(m)}$ is the tridiagonal Toeplitz matrix

$$A_1^{(m)} = \begin{pmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix}_{m \times m},$$

and since we have the decomposition

$$\mathcal{P}_n^{(m)} = G_0^{(m)} \oplus G_1^{(m)} \oplus \ldots \oplus G_{m-1}^{(m)},$$

where

$$G_i^{(m)} = \{f \in \mathcal{P}_n^{(m)} | f(n) = i\},$$

we may conclude that the adjacency matrix of $\mathcal{G}_n^{(m)}$ is

$$A_n^{(m)} = \begin{pmatrix} A_{n-1}^{(m)} & I_{m^{n-1}} & & \\ I_{m^{n-1}} & \ddots & \ddots & \\ & \ddots & \ddots & I_{m^{n-1}} \\ & & I_{m^{n-1}} & A_{n-1}^{(m)} \end{pmatrix}_{m^n \times m^n} \tag{2.1}$$

for any integer $n \geqslant 2$. Of course, we can use the convention that $A_0^{(m)} = (0)$.

3. Chebyshev polynomials of second kind

We proceed with one of the most important families of orthogonal polynomials: the Chebyshev polynomials of the second kind, denoted by $\{U_m(x)\}_{m \geqslant 0}$. These polynomials satisfy the three-term recurrence relation

$$U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x) \quad \text{for all } m = 1, 2, \ldots, \tag{3.1}$$

with the initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. Since each $U_m(x)$ verifies

$$U_m(x) = \frac{\sin(m+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi)$$

for all $m = 0, 1, 2 \ldots$, we can deduce the orthogonality relations

$$\int_{-1}^{1} U_i(x)U_j(x)\sqrt{1 - x^2}\, \mathrm{d}x = \tfrac{1}{2}\pi\delta_{i,j}$$

(cf. [3], e.g.). It is also well known that the explicit formula for Chebyshev polynomials of the second kind is

$$U_m(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-)^k \frac{(m-k)!}{k!(m-2k)!} (2x)^{m-2k}.$$

If we consider the $m \times m$ tridiagonal Toeplitz matrix $A_1^{(m)}$, the recurrence relation (3.1) can be rewritten in the matrix form:

$$2x \begin{pmatrix} U_0(x) \\ U_1(x) \\ \vdots \\ U_{m-2}(x) \\ U_{m-1}(x) \end{pmatrix} = A_1^{(m)} \begin{pmatrix} U_0(x) \\ U_1(x) \\ \vdots \\ U_{m-2}(x) \\ U_{m-1}(x) \end{pmatrix} + U_m(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Hence, the characteristic polynomial of $A_1^{(m)}$ is $p_m(x) = U_m(\frac{1}{2}x)$ and it follows immediately (see [5]) that the eigenvalues are

$$\lambda_l = 2\cos\Big(\frac{l\pi}{m+1}\Big) \quad \text{for } l = 1, 2 \dots, m. \tag{3.2}$$

Therefore, if X is any square matrix, then

$$p_m(X) = \prod_{l=1}^{m} (X - \lambda_l I)\,. \tag{3.3}$$

4. The spectra

Finding the characteristic polynomials of the adjacency matrices of comparability graphs is a hard task. The same difficulties can be met with the determinant of special structured integral matrices (cf. e.g. [7], [8]). In this section, we focus our attention on the characteristic polynomial of $A_n^{(m)}$.

Let us define

$$A_n^{(m)}(t) = tI_{m^n} - A_n^{(m)}. \tag{4.1}$$

Thus, the characteristic polynomial of $A_n^{(m)}$ is $\det A_n^{(m)}(t)$.

Theorem 4.1.

$$\det A_{n+1}^{(m)}(t) = \det p_m(A_n^{(m)}(t)). \tag{4.2}$$

Proof. From (2.1) and (4.1) we have

$$\det A_{n+1}^{(m)}(t) = \det \begin{pmatrix} A_n^{(m)}(t) & I_{m^n} & & \\ I_{m^n} & \ddots & \ddots & \\ & \ddots & \ddots & I_{m^n} \\ & & I_{m^n} & A_n^{(m)}(t) \end{pmatrix}_{m^{n+1}\times m^{n+1}}.$$

To simplify the notation, from now on we write $A_n(t) = A_n^{(m)} - tI_{m^n}$ and $I = I_{m^n}$. To establish the relation (4.2), we perform block-wise row and column operations.

First we switch the positions of the first and the second rows to obtain

$$\det\left(\begin{array}{c|cccccc} I & A_n(t) & I & & & & \\ \hline A_n(t) & I & 0 & 0 & & & \\ & I & A_n(t) & I & 0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & I \\ & & & & & I & A_n(t) \end{array}\right)_{m^{n+1}\times m^{n+1}}.$$

Applying the Schur complement to the determinant (cf., e.g., [6]), the previous determinant is equal to

$$\det\begin{pmatrix} A_n^2(t)-I & A_n(t) & & & \\ I & A_n(t) & I & & \\ & I & \ddots & \ddots & \\ & & \ddots & \ddots & I \\ & & & I & A_n(t) \end{pmatrix}_{(m-1)m^n\times(m-1)m^n}.$$

We can apply the same procedure $(m-2)$ times more. In the k-th step we get

$$\det\begin{pmatrix} p_{k+1}(A_n(t)) & p_k(A_n(t)) & & & \\ I & A_n(t) & I & & \\ & I & \ddots & \ddots & \\ & & \ddots & \ddots & I \\ & & & I & A_n(t) \end{pmatrix}_{(m-k)m^n\times(m-k)m^n},$$

where $p_k(x) = U_k(\frac{x}{2})$, $U_k(x)$ being the Chebyshev polynomial of the second kind defined e.g. by the three-term recurrence relation (3.1). Therefore, in the final $(m-1)$-st step we obtain (4.2) as desired. $\square$

Theorem 4.2. *The eigenvalues of $A_{n+1}^{(m)}$ are of the form*

$$\lambda + \lambda_l,$$

where λ is an eigenvalue of $A_n^{(m)}$ and λ_l is an eigenvalue of $A_1^{(m)}$ defined as in (3.2).

Proof. From (3.3) we have

$$p_m(A_n^{(m)}(t)) = \prod_{l=1}^{m}(A_n^{(m)}(t) - \lambda_l I_{m^n}).$$

Applying now (4.2), we obtain

$$\det A_{n+1}^{(m)}(t) = \prod_{l=1}^{m} \det(A_n^{(m)}(t) - \lambda_l I_{m^n}).$$

This means that all eigenvalues of $A_{n+1}^{(m)}$ have the form $\lambda + \lambda_l$ for an eigenvalue λ of $A_n^{(m)}$. □

Corollary 4.3. *If $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $A_1^{(m)}$, as in (3.2), then the spectrum of $A_n^{(m)}$ consists of the real numbers*

$$\lambda_{i_1} + \ldots + \lambda_{i_n} \quad \textit{for } i_k \in \{1, \ldots, m\}.$$

Corollary 4.4. *The largest eigenvalue of $A_n^{(m)}$ is*

$$2n \cos\left(\frac{\pi}{m+1}\right).$$

Obviously, the multiplicity of the largest (and the lowest) eigenvalue of $A_n^{(m)}$ is 1. In fact, the value symmetric to any eigenvalue of $A_n^{(m)}$ is also an eigenvalue with the same multiplicity. In the next section, we will give a more detail account on the spectra and on the eigenvalue multiplicity of $A_n^{(m)}$ for $m = 2, 3$.

5. Examples

We start this final section considering $m = 2$. In this case the poset $\mathcal{P}_n^{(2)}$ is commonly known as the Boolean algebra of rank n.

By Corollary 4.3, $A_n^{(2)}$ has $n+1$ distinct eigenvalues, and they are of the form

$$\underbrace{\pm 1 \pm \ldots \pm 1}_{n},$$

i.e.,

$$n - 2k, \quad \text{for } k = 0, 1, \ldots, n.$$

We can display them in the triangle, for each n,

$$\begin{array}{lccccccc}
n=0 & & & & 0 & & & \\
n=1 & & & -1 & & 1 & & \\
n=2 & & -2 & & 0 & & 2 & \\
n=3 & -3 & & -1 & & 1 & & 3 \\
\multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

The multiplicities are, respectively,

$$\begin{array}{lccccccc}
n=0 & & & & 1 & & & \\
n=1 & & & 1 & & 1 & & \\
n=2 & & 1 & & 2 & & 1 & \\
n=3 & 1 & & 3 & & 3 & & 1 \\
\multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

which corresponds to the Pascal triangle, i.e., the multiplicity of the eigenvalue $n-2k$ is $\binom{n}{k}$ for $k = 0, 1, \dots, n$. These results were also obtained in [4] by a different approach.

Let us now consider $\mathcal{P}_n^{(3)}$. From Theorem 4.2 we get

$$\begin{array}{lccccccc}
n=0 & & & & 0 & & & \\
n=1 & & & -\sqrt{2} & 0 & \sqrt{2} & & \\
n=2 & & -2\sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} & 2\sqrt{2} & \\
n=3 & -3\sqrt{2} & -2\sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} & 2\sqrt{2} & 3\sqrt{2} \\
\multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

In fact, the eigenvalues of $A_n^{(3)}$ are of the form $\pm k\sqrt{2}$, for $k = 0, \dots, n$, and, in this case, the multiplicities are, respectively,

$$\begin{array}{lccccccc}
n=0 & & & & 1 & & & \\
n=1 & & & 1 & 1 & 1 & & \\
n=2 & & 1 & 2 & 3 & 2 & 1 & \\
n=3 & 1 & 3 & 6 & 7 & 6 & 3 & 1 \\
\multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

In general, we may state the following proposition:

Proposition 5.1. *For $k = 0, \dots, n$, the multiplicity of the eigenvalue $\pm k\sqrt{2}$ of $A_n^{(3)}$ is*

$$\sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n}{k+l} \binom{n-k-l}{l}.$$

P r o o f. The multiplicity of $k\sqrt{2}$ coincides with the number of all possible ways in which the nonnegative integer k can be decomposed as a sum of n summands 1, 0 or -1, bearing in mind that here we do not apply commutative rule. In general, we consider all sums of the form

$$\underbrace{1 + \ldots + 1}_{k+l} + \underbrace{0 + \ldots + 0}_{n-k-2l} + \underbrace{(-1) + \ldots + (-1)}_{l},$$

and the number of them is
$$\binom{n}{k+l}\binom{n-k-l}{l}.$$
The greatest integer l is determined by the condition: $k+2l \leqslant n$, i.e., when $l = \lfloor \frac{1}{2}(n-k) \rfloor$. Summing up all possibilities we reach the desired conclusion. $\square$

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