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ON THE UNIQUENESS OF AN ENTIRE FUNCTION SHARING A SMALL ENTIRE FUNCTION WITH SOME LINEAR DIFFERENTIAL POLYNOMIAL

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Abstract. We prove a theorem on the growth of nonconstant solutions of a linear differential equation. From this we obtain some uniqueness theorems concerning that a nonconstant entire function and its linear differential polynomial share a small entire function. The results in this paper improve many known results. Some examples are provided to show that the results in this paper are the best possible.

Keywords: entire functions, order of growth, shared values, uniqueness theorems

MSC 2010: 30D30, 30D35

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notation in the Nevanlinna theory of meromorphic functions as explained in [6], [9] and [14]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h)=o(T(r,h))$ $(r\to\infty, r\notin E)$. Let f and g be two nonconstant meromorphic functions and let a be a complex number. We say that f and g share a CM provided $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a IM provided $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM if $1/f$ and $1/g$

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share 0 CM, and we say that f and g share ∞ IM if $1/f$ and $1/g$ share 0 IM (see [15]). A nonconstant meromorphic function b is called a small function related to f if $T(r,b) = S(r,f)$. If $f-b$ and $g-b$ share 0 CM, we say that f and g share b CM, and we say that f and g share b IM, if $f-b$ and $g-b$ share 0 IM. In this paper, we also need the following definition.

Definition 1.1. For a nonconstant entire function f, the order $\sigma(f)$, lower order $\mu(f)$, hyper order $\sigma_2(f)$ and lower hyper order $\mu_2(f)$ are defined by

$$\begin{aligned}
\sigma(f) &= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},\\
\mu(f) &= \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r},\\
\sigma_2(f) &= \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}
\end{aligned}$$

and

$$\mu_2(f) = \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

respectively, where and in what follows, $M(r,f) = \max\limits_{|z|=r} |f(z)|$.

In 1977, L. A. Rubel and C. C. Yang [11] proved that if an entire function f shares two distinct complex numbers CM with its derivative f', then $f = f'$. What is the relation between f and f', if an entire function f shares one complex number a CM with its derivative f'? In 1996, R. Brück [1] made a conjecture that if f is a nonconstant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f)$ is not a positive integer, and if f and f' share one complex number a CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$. For the case $a = 0$, the above conjecture was proved by R. Brück [1]. In the same paper, R. Brück proved the above conjecture is true provided $a \neq 0$ and $N(r, 1/f') = S(r,f)$. In 1998, G. G. Gundersen and L. Z. Yang proved that the conjecture is true for $a \neq 0$ provided f satisfies the additional assumption $\sigma(f) < \infty$ (see [5]). In 1999, L. Z. Yang proved that if a nonconstant entire function f and one of its derivatives $f^{(k)}$ $(k \geqslant 1)$ share one complex number $a(\neq 0)$ CM, where f satisfies $\sigma(f) < \infty$, then $f - a = c(f^{(k)} - a)$ for some complex number $c \neq 0$ (see [16]). In 2004, J. P. Wang proved the following theorem.

Theorem A (see [13, Theorem 1]). *Let f be a nonconstant entire function of finite order, let P be a polynomial with degree $p \geqslant 1$, and let k be a positive integer. If $f - P$ and $f^{(k)} - P$ share 0 CM, then $f^{(k)} - P = c(f - P)$ for some complex number $c \neq 0$.*

Consider the following linear differential polynomial related to f:

$$L[f] = f^{(k)} + b_{k-1}f^{(k-1)} + \ldots + b_1 f' + b_0 f, \tag{1.1}$$

where k is a positive integer and $b_0, b_1, \ldots, b_{k-1}$ are complex numbers.

Regarding Theorem A, it is natural to ask the following question.

Question 1.1. *What can be said if a nonconstant entire function f and $L[f]$ share a small entire function a related to f ?*

In this paper, we will prove the following result, which improves Theorem A and deals with Question 1.1.

Theorem 1.1. *If f is a transcendental entire solution of the differential equation*

$$L[f] - a_1(z) = (f - a_2(z)) \cdot \mathrm{e}^{Q(z)}, \tag{1.2}$$

where $L[f]$ is defined by (1.1), a_1 and a_2 are two entire functions such that $\sigma(a_j) < 1$ ($j = 1, 2$), and Q is a nonconstant polynomial, then one of the following two cases occurs.

(i) *If $\mu(f) > 1$, then $\mu(f) = \infty$ and $\mu_2(f) = \sigma_2(f) = \gamma_Q$, where and in what follows, γ_Q is the degree of Q;*

(ii) *If $\mu(f) \leqslant 1$, then $\mu(f) = 1$ and $Q(z) = p_1 z + p_0$, where p_1 ($\neq 0$) and p_0 are two complex numbers and $b_0, b_1, \ldots, b_{k-1}$ are not all equal to zero.*

From Theorem 1.1 we get the following three corollaries, of which Corollary 1.1 improves Theorem 1 in [16]and Corollaries 1.2–1.3 improve Theorem 2 in [16].

Corollary 1.1. *Let f be a nonconstant solution of the differential equation* (1.2), *where $L[f]$ is defined by (1.1), a_1 and a_2 are entire functions such that $\sigma(a_j) < 1$ ($j = 1, 2$), and Q is a nonconstant polynomial. If $\mu(f) \neq 1$, then $\mu_2(f) = \sigma_2(f) = \gamma_Q \geqslant 1$.*

Corollary 1.2. *Let f be a nonconstant solution of the differential equation*

$$L[f] - a = (f - a) \cdot \mathrm{e}^{Q(z)}, \tag{1.3}$$

where $L[f]$ is defined by (1.1), Q is a polynomial, and a ($\not\equiv 0$) is an entire function such that $\sigma(a) < 1$. If $\mu_2(f)$ is not a positive integer and $\mu(f) \neq 1$, then $L[f] - a = c(f - a)$ for some complex number $c \neq 0$.

Corollary 1.3. *Let f be a nonconstant entire function such that $\mu(f) < \infty$, let $L[f]$ be defined by* (1.1)*, and let a ($\not\equiv 0$) be an entire function such that $\sigma(a) < \mu(f)$. If $f - a$ and $L[f] - a$ share* 0 *CM, then one of the following two cases occurs.*

(i) *$L[f] - a = c(f - a)$, where $c(\neq 0)$ is a complex number;*

(ii) *f is a solution of* (1.3) *such that $\mu(f) = 1$ and $Q(z) = p_1 z + p_0$, where p_1 ($\neq 0$) and p_0 are two complex numbers and $b_0, b_1, \ldots, b_{k-1}$ are not all equal to zero.*

Proof. From the assumptions of Corollary 1.3 we have (1.3), where Q is an entire function. From Definition 1.1 and the condition $\sigma(a) < \mu(f)$ we get $\mu(f) > 0$, and so f is a transcendental entire function. Combining (1.3), Definition 1.1 and Lemma 2.5 in Section 2 of this paper, we get

$$T(r, \mathrm{e}^Q) \leqslant 3T(r, f) + O(\log T(r, f) + \log r) \ (r \notin E,\ r \to \infty). \tag{1.4}$$

From (1.4) and Lemma 2.6 in Section 2 of this paper we see that there exists a sufficiently large positive number r_0 such that

$$T(r, \mathrm{e}^Q) \leqslant 3T(2r, f) + O(\log T(2r, f) + \log r + \log 2) \ (r \geqslant r_0),$$

which together with $\mu(f) < \infty$, implies

$$\sigma(\mathrm{e}^Q) = \mu(\mathrm{e}^Q) \leqslant \mu(f) < \infty. \tag{1.5}$$

From (1.5) we see that Q is a polynomial. If Q is a constant, from (1.3) we get (i) of Corollary 1.3. Next we suppose that Q is a nonconstant polynomial. By Theorem 1.1, we discuss the following two cases.

Case 1. Suppose that $\mu(f) > 1$. Then from (i) of Theorem 1.1 we get $\mu(f) = \infty$, which is impossible.

Case 2. Suppose that $\mu(f) \leqslant 1$. Then from (ii) of Theorem 1.1 we get (ii) of Corollary 1.3.

From Corollary 1.3 we get the following result improving Theorem 2 in [16].

Corollary 1.4. *Let f be a nonconstant entire function such that $\mu(f) < \infty$ and $\mu(f) \neq 1$, let $L[f]$ be defined by* (1.1)*, and let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \mu(f)$. If $f - a$ and $L[f] - a$ share* 0 *CM, then $L[f] - a = c(f - a)$ for some complex number $c \neq 0$.*

Now we give the following examples.

Example 1.1 (see [4]). Let $f(z) = \mathrm{e}^{\mathrm{e}^z} + \mathrm{e}^z$. Then it is easy to see that $f'(z) - \mathrm{e}^z = \mathrm{e}^z(f(z) - \mathrm{e}^z)$ and $\mu_2(f) = \sigma_2(f) = 1$. This example shows that the conclusion (i) of Theorem 1.1 can occur.

Example 1.2. Let $f(z) = (e^z - 1)^2$ and $L[f](z) = f^{(3)}(z) - 3f''(z) + \frac{5}{2}f'(z) - f(z)$. Then it is easy to see that $L[f](z) - 1 = (f(z) - 1) \cdot e^{-z}$ and $\mu(f) = 1$. This example shows that the conclusion (ii) of Theorem 1.1 can occur.

In 1995, H. X. Yi and C. C. Yang posed the following question.

Question 1.2 (see [15, pp. 398]). Let f be a nonconstant meromorphic function, and let a be a nonzero complex number. If f, $f^{(n)}$ and $f^{(m)}$ share the value a CM, where n and m $(n < m)$ are distinct positive integers not both even or odd, can we get the result $f = f^{(n)}$?

Regarding Question 1.2, G. G. Gundersen and L. Z. Yang proved the following result in 1998.

Theorem B (see [5, Theorem 2]). *Let f be a nonconstant entire function of finite order, let $a (\neq 0)$ be a complex number, and let n be a positive integer. If the value a is shared by f, $f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f = f'$.*

In this paper, we will prove the following theorem corresponding to Theorem B.

Theorem 1.2. *Let f be a nonconstant solution of the differential equation*

$$f^{(n+1)} + a_n f^{(n)} - z = (f^{(n)} - z) \cdot e^{Q(z)}, \tag{1.6}$$

such that $\sigma_2(f)$ is not a positive integer, where n $(\geqslant 1)$ is a positive integer, a_n is a complex number, and Q is a polynomial. If $f - z$ and $f^{(n)} - z$ share 0 CM, then e^Q is a constant, and one of the following two cases occurs.

(i) *If f is a transcendental entire function, then*

$$f(z) = \frac{c}{(1 - a_n)^n} \cdot e^{(1-a_n)z} + \frac{(1 - a_n)^n - 1}{(1 - a_n)^n} \cdot z,$$

where c $(\neq 0)$ is a complex number and $a_n \neq 1$.

(ii) *If f is a polynomial, then $f(z) = cz + c(1 - c)$, where $c \neq 0, 1$ is a complex number.*

From Theorem 1.2 we get the following corollary.

Corollary 1.5. *Let f be a nonconstant entire function such that $\mu(f) < \infty$, and let n be a positive integer. If $f - z$, $f^{(n)} - z$ and $f^{(n+1)} + a_n f^{(n)} - z$ share 0 CM, where a_n is a complex number, then* (i) *or* (ii) *of Theorem* 1.2 *holds.*

2. Some lemmas

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function. We define by $\mu(r) = \max\{|a_n| r^n \colon n = 0, 1, 2, \ldots\}$ the maximum term of f, and by $\nu(r, f) = \max\{m \colon \mu(r) = |a_m| r^m\}$ the central index of f (see [7, Definition 1.5] or [9, pp. 50]).

Lemma 2.1 (see [9, Proposition 8.1]). *Suppose that all the coefficients a_0 ($\not\equiv 0$), $a_1, a_2, \ldots, a_{n-1}$ and g ($\not\equiv 0$) of the non-homogeneous linear differential equation*

$$f^{(n)} + a_{n-1}(z) f^{(n-1)} + \ldots + a_1(z) f' + a_0(z) f = g(z) \tag{2.1}$$

are entire functions. Then all solutions of (2.1) *are entire functions.*

Lemma 2.2. *Let f be an entire function of infinite order, with the lower order $\mu(f)$ and the lower hyper order $\mu_2(f)$. Then*

(i) $\mu(f) = \liminf\limits_{r\to\infty} \log \nu(r, f) / \log r$;

(ii) $\mu_2(f) = \liminf\limits_{r\to\infty} \log\log \nu(r, f) / \log r$.

Proof. Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. Without loss of generality, we may assume $|a_0| \neq 0$. By [7, Theorem 1.9], the maximum term $\mu(r)$ of f satisfies

$$\log \mu(2r) = \log |a_0| + \int_0^{2r} \frac{\nu(t, f)}{t}\, \mathrm{d}t \geqslant \log |a_0| + \nu(r, f) \log 2. \tag{2.2}$$

On the other hand, by Cauchy's inequality, we get $|a_n| r^n \leqslant M(r, f)$ ($r > 0$, $n = 0, 1, 2, 3, \ldots$). This together with the definition of the maximum term of f implies

$$\mu(2r) \leqslant M(2r, f). \tag{2.3}$$

Therefore, from (2.2) and (2.3) we get

$$\nu(r, f) \log 2 \leqslant \log M(2r, f) + C, \tag{2.4}$$

where C (> 0) is a suitable constant. By the definition of $\mu_2(f)$, we have

$$\mu_2(f) = \liminf_{r\to\infty} \frac{\log\log\log M(r, f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log T(r, f)}{\log r}. \tag{2.5}$$

From (2.4) and (2.5) we get

$$\liminf_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}\leqslant\liminf_{r\to\infty}\frac{\log\log\log M(r,f)}{\log r}=\mu_2(f). \tag{2.6}$$

On the other hand, by [7, Theorem 1.10] we have

$$M(r,f)<\mu(r)\{\nu(2r,f)+2\}=|a_{\nu(r,f)}|r^{\nu(r,f)}\cdot\{\nu(2r,f)+2\}. \tag{2.7}$$

Since $|a_n|<M_1$ for all nonnegative integers n and some constant $M_1>0$, we get from (2.7) that

$$\begin{aligned}\log\log M(r,f)&\leqslant\log\nu(r,f)+\log\log\nu(2r,f)+\log\log r+C_1\\&\leqslant\log\nu(2r,f)\cdot\Big(1+\frac{\log\log\nu(2r,f)}{\log\nu(2r,f)}\Big)+\log\log r+C_2,\end{aligned} \tag{2.8}$$

where C_j (>0) $(j=1,2)$ are suitable constants. By (2.5) and (2.8) we get

$$\begin{aligned}\mu_2(f)&=\liminf_{r\to\infty}\frac{\log\log\log M(r,f)}{\log r}\leqslant\liminf_{r\to\infty}\frac{\log\log\nu(2r,f)}{\log 2r}\\&=\liminf_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}.\end{aligned} \tag{2.9}$$

By (2.6) and (2.9), we get (ii). Proceeding as above we get (i).

Lemma 2.2 is thus completely proved.

Lemma 2.3 (see [9, Lemma 1.1.2]). *Let $g,h\colon(0,+\infty)\to\mathbb{R}$ be monotonically increasing functions such that $g(r)\leqslant h(r)$ outside of an exceptional set F of finite logarithmic measure. Then, for any $\alpha>1$, there exists $r_0>0$ such that $g(r)\leqslant h(r^\alpha)$ for all $r>r_0$.*

Lemma 2.4 (see [2, Lemma 2] or [3, Lemma 4]). *If f is a transcendental entire function of hyper order $\sigma_2(f)$, then $\sigma_2(f)=\limsup\limits_{r\to\infty}(\log\log\nu(r,f))/\log r$.*

Lemma 2.5 (see [9, Corollary 2.3.4]). *Let f be a transcendental meromorphic function and $k\geqslant 1$ an integer. Then $m(r,f^{(k)}/f)=O(\log(rT(r,f)))$, possibly outside an exceptional set E of finite linear measure, and if f is of finite order of growth, then $m(r,f^{(k)}/f)=O(\log r)$.*

Lemma 2.6 (see [9, Lemma 1.1.1]). *Let $g, h\colon (0, +\infty) \to \mathbb{R}$ be monotonically increasing functions such that $g(r) \leqslant h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leqslant h(\alpha r)$ for all $r > r_0$.*

Lemma 2.7 (see [15, Corollary of Theorem 1.20] or [17]). *Suppose that f is a meromorphic function. Then $T(r, f) \leqslant O(T(2r, f') + \log r)$ as $r \to \infty$.*

Lemma 2.8 (see [12]). *Let f be a meromorphic function and k a positive integer. If f is a solution of the differential equation $a_0 f^{(k)} + a_1 f^{(k-1)} + \ldots + a_k f = 0$, where $a_0, a_1, \ldots, a_k$ are complex numbers with $a_0 \neq 0$, then $T(r, f) = O(r)$. Moreover, if f is transcendental, then $r = O(T(r, f))$.*

Lemma 2.9 (see [10]). *Let f be a nonconstant meromorphic function such that $\sigma(f) = \sigma < \infty$. Then*

$$\limsup_{r\to\infty} \frac{m(r, f'/f)}{\log r} \leqslant \max\{0, \sigma - 1\}.$$

Lemma 2.10 (see [15, Theorem 1.57]). *Suppose that f_1, f_2, f_3 are meromorphic functions satisfying $f_1 + f_2 + f_3 = 1$. If f_1 is not a constant and*

$$\sum_{i=1}^{3} N\Big(r, \frac{1}{f_i}\Big) + 2\sum_{i=1}^{3} \overline{N}(r, f_i) < \lambda T(r, f_1) + S(r, f_1),$$

where $\lambda < 1$, then $f_2 = 1$ or $f_3 = 1$.

Lemma 2.11. *Suppose that α and β are nonconstant entire functions, and that a_1, a_2, b_1 and b_2 are meromorphic functions satisfying $T(r, a_1) + T(r, a_2) = S(r, \mathrm{e}^{\alpha})$, $T(r, b_1) + T(r, b_2) = S(r, \mathrm{e}^{\beta})$ and $a_1 a_2 b_1 b_2 \not\equiv 0$. If $a_1 \mathrm{e}^{\alpha} - a_2$ and $b_1 \mathrm{e}^{\beta} - b_2$ share 0 IM, then one of the following relations holds:*
(i) $a_1 b_2 \mathrm{e}^{\alpha} = a_2 b_1 \mathrm{e}^{\beta}$;
(ii) $a_1 b_1 \mathrm{e}^{\alpha+\beta} = a_2 b_2$.

P r o o f. By the second fundamental theorem, we have

$$T(r, \mathrm{e}^{\alpha}) = \overline{N}\Big(r, \frac{1}{a_1 \mathrm{e}^{\alpha} - a_2}\Big) + S(r, \mathrm{e}^{\alpha}) = N_{1)}\Big(r, \frac{1}{a_1 \mathrm{e}^{\alpha} - a_2}\Big) + S(r, \mathrm{e}^{\alpha}) \tag{2.10}$$

and

$$T(r, \mathrm{e}^{\beta}) = \overline{N}\Big(r, \frac{1}{b_1 \mathrm{e}^{\beta} - b_2}\Big) + S(r, \mathrm{e}^{\beta}) = N_{1)}\Big(r, \frac{1}{b_1 \mathrm{e}^{\beta} - b_2}\Big) + S(r, \mathrm{e}^{\beta}), \tag{2.11}$$

where $N_{1)}(r,1/f)$ denotes the counting function of simple zeros of a meromorphic function f in $|z|<r$. Let

$$H=\frac{a_1\mathrm{e}^{\alpha}-a_2}{b_1\mathrm{e}^{\beta}-b_2}. \tag{2.12}$$

Noting that $a_1\mathrm{e}^{\alpha}-a_2$ and $b_1\mathrm{e}^{\beta}-b_2$ share 0 IM, from (2.10)–(2.12) we obtain

$$N(r,H)=S(r,\mathrm{e}^{\alpha}) \quad \text{and} \quad N\Big(r,\frac{1}{H}\Big)=S(r,\mathrm{e}^{\alpha}). \tag{2.13}$$

Since (2.12) can be rewritten as

$$\frac{a_1}{a_2}\mathrm{e}^{\alpha}-\frac{b_1}{a_2}H\mathrm{e}^{\beta}+\frac{b_2}{a_2}H=1, \tag{2.14}$$

we obtain from (2.13), (2.14) and Lemma 2.10 that $b_2a_2^{-1}H=1$ or $-b_2a_2^{-1}H\mathrm{e}^{\beta}=1$. If $b_2a_2^{-1}H=1$, from (2.14) we have $a_1a_2^{-1}\mathrm{e}^{\alpha}=b_1a_2^{-1}H\mathrm{e}^{\beta}$. Hence we have the relation (i) of Lemma 2.11. If $-b_1a_2^{-1}H\mathrm{e}^{\beta}=1$, from (2.14) we have $a_1a_2^{-1}\mathrm{e}^{\alpha}=-b_2a_2^{-1}H$. Hence we have the relation (ii) of Lemma 2.11.

3. Proof of theorems

Proof of Theorem 1.1. By (1.1) we see that (1.2) can be rewritten as

$$f^{(k)}+b_{k-1}f^{(k-1)}+\ldots+b_1f'+(b_0-\mathrm{e}^{Q(z)})\cdot f=a_1(z)-a_2(z)\mathrm{e}^{Q(z)}. \tag{3.1}$$

From (3.1) and Lemma 2.1 we see that all solutions of (3.1) are entire functions. We discuss the following two cases.

Case 1. Suppose that

$$\liminf_{r\to\infty}\frac{\log\nu(r,f)}{\log r}>1. \tag{3.2}$$

Then (3.2) and (i) of Lemma 2.2 yield

$$\mu(f)=\liminf_{r\to\infty}\frac{\log\nu(r,f)}{\log r}>1. \tag{3.3}$$

From the condition that f is a nonconstant entire function we have

$$M(r,f)\to\infty \tag{3.4}$$

as $r \to \infty$. Let

$$M(r, f) = |f(z_r)|, \tag{3.5}$$

where $z_r = r\mathrm{e}^{\mathrm{i}\theta(r)}$ and $\theta(r) \in [0, 2\pi)$. From (3.5) and the Wiman-Valiron theory (see [9, Theorem 3.2]) we see that there exist subsets $F_j \subset (1, \infty)$ $(1 \leqslant j \leqslant n)$ with finite logarithmic measure, i.e., $\int_{F_j} \mathrm{d}t/t < \infty$, such that for some point $z_r = r\mathrm{e}^{\mathrm{i}\theta(r)}$ $(\theta(r) \in [0, 2\pi))$ satisfying $|z_r| = r \notin F_j$ and $M(r, f) = |f(z_r)|$ we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \Big(\frac{\nu(r, f)}{z_r}\Big)^j (1 + o(1)) \quad (1 \leqslant j \leqslant n,\ r \notin F_j,\ r \to \infty). \tag{3.6}$$

Since $\sigma(a_j) < 1$ $(j = 1, 2)$, from (3.3)–(3.5) and Definition 1.1 we get

$$a_j(z_r)/f(z_r) \to 0, \tag{3.7}$$

as $r \to \infty$. Since

$$\frac{L[f](z) - a_1(z)}{f(z) - a_2(z)} = \frac{L[f](z)/f(z) - a_1(z)/f(z)}{1 - a_2(z)/f(z)}, \tag{3.8}$$

from (1.1), (3.2) and (3.6)–(3.8) we get

$$\frac{L[f](z_r) - a_1(z_r)}{f(z_r) - a_2(z_r)} = \Big(\frac{\nu(r, f)}{z_r}\Big)^k (1 + o(1)) \ \Big(r \notin \bigcup_{j=1}^{n} F_j,\ r \to \infty\Big). \tag{3.9}$$

From (3.2) and (3.9) we have

$$\log \Big|\frac{L[f](z_r) - a_1(z_r)}{f(z_r) - a_2(z_r)}\Big| = k(\log \nu(r, f) - \log r) + o(1) \ \Big(r \notin \bigcup_{j=1}^{n} F_j,\ r \to \infty\Big). \tag{3.10}$$

Let

$$Q(z) = p_n z^n + p_{n-1} z^{n-1} + \ldots + p_1 z + p_0, \tag{3.11}$$

where $p_0, p_1, \ldots, p_{n-1}, p_n$ are complex numbers with $p_n \neq 0$. It follows from (3.11) that $\lim\limits_{|z| \to \infty} |Q(z)|/|p_n z^n| = 1$ and $|Q(z)|/|p_n z^n| > 1/\mathrm{e}$ $(|z| > r_0)$. Using this and (1.2) we get

$$\begin{aligned} n \log |z| + \log |p_n| - 1 \leqslant \log |Q(z)| = \log|\log \mathrm{e}^{Q(z)}| &\leqslant |\log \log \mathrm{e}^{Q(z)}| \\ &= \Big|\log \log \frac{L[f] - a_1}{f - a_2}\Big| \ (|z| > r_0). \end{aligned} \tag{3.12}$$

From (3.2), (3.10) and (3.12) we get

$$
\begin{aligned}
(3.13)\quad & n\log|z_r|+\log|p_n|-1\\
&\leqslant \Bigl|\log\log\frac{L[f](z_r)-a_1(z_r)}{f(z_r)-a_2(z_r)}\Bigr|\\
&= \Bigl|\log\Bigl|\log\frac{L[f](z_r)-a_1(z_r)}{f(z_r)-a_2(z_r)}\Bigr|+\mathrm{i}\arg\Bigl(\log\frac{L[f](z_r)-a_1(z_r)}{f(z_r)-a_2(z_r)}\Bigr)\Bigr|\\
&\leqslant \Bigl|\log\Bigl|\log\frac{L[f](z_r)-a_1(z_r)}{f(z_r)-a_2(z_r)}\Bigr|\Bigr|+2\pi\\
&\leqslant \log\log\nu(r,f)+\log\log r+O(1)\ \Bigl(r\notin\bigcup_{j=1}^{n}F_j,\ r\to\infty\Bigr).
\end{aligned}
$$

From (3.13), Lemma 2.3 and $|z_r| = r$ we see that for any $\beta > 1$, there exists a sufficiently large positive number r_0 such that

$$(3.14)\qquad n\log r+\log|p_n|-1\leqslant \log\log\nu(r^{\beta},f)+\log\log r^{\beta}+O(1)\ (r>r_0).$$

From (3.14) and Lemma 2.4 we deduce

$$(3.15)\qquad n/\beta\leqslant \limsup_{r^{\beta}\to\infty}\frac{\log\log\nu(r^{\beta},f)}{\log r^{\beta}}=\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}=\sigma_2(f).$$

By letting $\beta\to 1^+$, we have

$$(3.16)\qquad n\leqslant\sigma_2(f).$$

In the same manner as above and by (ii) of Lemma 2.2 we get

$$(3.17)\qquad n\leqslant\mu_2(f).$$

From (3.11) we obtain

$$(3.18)\qquad \sigma(\mathrm{e}^{Q})=\gamma_Q=n.$$

From (3.16) and (3.18) we get

$$(3.19)\qquad \sigma(\mathrm{e}^{Q})\leqslant\sigma_2(f).$$

Again from (1.2) and (3.9) we have

$$(3.20)\qquad \Bigl(\frac{\nu(r,f)}{z_r}\Bigr)^{k}(1+o(1))=\mathrm{e}^{Q(z_r)}\ \Bigl(r\notin\bigcup_{j=1}^{n}F_j,\ r\to\infty\Bigr).$$

From (3.20) we get

$$(3.21)\qquad (\nu(r,f))^k \leqslant 2r^k M(r,\mathrm{e}^Q)\ \Bigl(r \notin \bigcup_{j=1}^{n} F_j,\ r\to\infty\Bigr),$$

From (3.21) and Lemma 2.3 we see that for any $\beta > 1$ there exists a sufficiently large positive number r_0 such that

$$(3.22)\qquad (\nu(r,f))^k \leqslant 2r^{\beta k} M(r^\beta,\mathrm{e}^Q)\ (r > r_0).$$

From (3.22), Lemma 2.4 and Definition 1.1 we get

$$(3.23)\qquad \begin{aligned}\sigma_2(f) &= \limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r} = \limsup_{r\to\infty}\frac{\log\log(\nu(r,f))^k}{\log r}\\ &\leqslant \limsup_{r\to\infty}\frac{\log\log(2r^{\beta k}M(r^\beta,\mathrm{e}^Q))}{\log r} = \beta\limsup_{r\to\infty}\frac{\log\log M(r,\mathrm{e}^Q)}{\log r}\\ &= \beta\sigma(\mathrm{e}^Q).\end{aligned}$$

By letting $\beta\to 1^+$ on both sides of (3.23) we get

$$(3.24)\qquad \sigma_2(f) \leqslant \sigma(\mathrm{e}^Q).$$

From (3.18), (3.19) and (3.24) we deduce

$$(3.25)\qquad \sigma_2(f) = \sigma(\mathrm{e}^Q) = n.$$

From (3.17), (3.18), (3.25) and $\mu_2(f) \leqslant \sigma_2(f)$ we get

$$(3.26)\qquad \mu_2(f) = \sigma_2(f) = \gamma_Q.$$

If $\mu(f) < \infty$, then it follows from (3.26) that $\mu_2(f) = \gamma_Q = 0$, and so Q is a complex number, which is impossible. Thus $\mu(f) = \infty$. Using this and (3.26) we get (i) of Theorem 1.1.

Case 2. Suppose that

$$(3.27)\qquad \liminf_{r\to\infty}\frac{\log\nu(r,f)}{\log r} \leqslant 1.$$

From (3.27) and (i) of Lemma 2.2 we have

$$(3.28)\qquad \mu(f) = \liminf_{r\to\infty}\frac{\log\nu(r,f)}{\log r} \leqslant 1.$$

From (1.1), (1.2), Lemma 2.5 and $\sigma(a_j) < 1$ $(j = 1, 2)$ we get

$$T(r, \mathrm{e}^Q) \leqslant 2T(r, f) + O(\log T(r, f) + \log r) \ (r \notin E,\ r \to \infty). \tag{3.29}$$

From (3.29) and Lemma 2.6 we see that for sufficiently large positive number r_0 we have

$$T(r, \mathrm{e}^Q) \leqslant 2T(2r, f) + O(\log T(2r, f) + \log r) \ (r > r_0). \tag{3.30}$$

If Q is a nonconstant polynomial such that Q is given by (3.11), then (3.11) and (3.30) imply $1 \leqslant n = \gamma_Q = \sigma(\mathrm{e}^Q) = \mu(\mathrm{e}^Q) \leqslant \mu(f)$. From this and (3.28) we get $n = \mu(f) = 1$, and so $Q(z) = p_1 z + p_0$. If $b_j = 0$ $(0 \leqslant j \leqslant k-1)$, then (1.2) can be rewritten as

$$f^{(k)} - a_1(z) = (f - a_2(z)) \cdot \mathrm{e}^{Q(z)}. \tag{3.31}$$

Hence in the same manner as in Case 1 we get (3.17), and so $\mu_2(f) \geqslant 1$, which contradicts $\mu(f) = 1$. Thus $b_0, b_1, \ldots b_{k-2}$ and b_{k-1} are not all equal to zero. The conclusion (ii) of Theorem 1.1 is thus completely proved.

Theorem 1.1 is thus completely proved.

Proof of Theorem 1.2. First, by virtue of (1.6), Lemma 2.1 and the assumptions of Theorem 1.2 we see that f is a nonconstant entire function. Next we will prove

$$f^{(n+1)} + a_n f^{(n)} - z = c(f^{(n)} - z), \tag{3.32}$$

where and in what follows, c $(\neq 0)$ is a complex number. If f is a polynomial, or Q is a constant, then from (1.6) we get (3.32). Next we suppose that Q is a nonconstant polynomial, and suppose that f, and so $f^{(k)}$, is a transcendental entire function, where k is an arbitrary positive integer. We prove

$$\mu(f) = \mu(f^{(n)}), \quad \sigma(f) = \sigma(f^{(n)}), \quad \mu_2(f) = \mu_2(f^{(n)}), \quad \sigma_2(f) = \sigma_2(f^{(n)}). \tag{3.33}$$

In fact, from Lemma 2.7 we have

$$T(r, f) \leqslant O(T(2r, f') + \log r) \tag{3.34}$$

as $r \to \infty$. From (3.34) and Definition 1.1 we get

$$\mu_2(f) \leqslant \mu_2(f'). \tag{3.35}$$

Lemma 2.5 yields

$$T(r,f') \leqslant 2T(r,f) + O(\log T(r,f) + \log r)\ (r \notin E,\ r \to \infty). \tag{3.36}$$

From (3.36) and Lemma 2.6 we see that for a sufficiently large positive number r_0 we have

$$T(r,f') \leqslant 2T(2r,f) + O(\log T(2r,f) + \log r + \log 2)\ (r \geqslant r_0). \tag{3.37}$$

From (3.37) and Definition 1.1 we get

$$\mu_2(f') \leqslant \mu_2(f). \tag{3.38}$$

From (3.35) and (3.38) we get

$$\mu_2(f) = \mu_2(f'). \tag{3.39}$$

Similarly,

$$\mu_2(f^{(j)}) = \mu_2(f^{(j+1)})\ 1 \leqslant j \leqslant n-1). \tag{3.40}$$

From (3.39) and (3.40) we get $\mu_2(f) = \mu_2(f^{(n)})$ in (3.33). Proceeding as above we get $\mu(f) = \mu(f^{(n)})$, $\sigma(f) = \sigma(f^{(n)})$ and $\sigma_2(f) = \sigma_2(f^{(n)})$ in (3.33).

If $\mu(f^{(n)}) > 1$, then (3.33) and (i) of Theorem 1.1 imply $\sigma_2(f) = \sigma_2(f^{(n)}) = \gamma_Q \geqslant 1$, which contradicts the condition that $\sigma_2(f)$ is not a positive integer. If $\mu(f^{(n)}) \leqslant 1$, then from (1.6) and (ii) of Theorem 1.1 we have $\mu(f^{(n)}) = 1$ and $Q(z) = p_1 z + p_0$, where p_0, p_1 are two complex numbers with $p_1 \neq 0$. Thus (1.6) can be rewritten as

$$f^{(n+1)} + a_n f^{(n)} - z = (f^{(n)} - z)\cdot \mathrm{e}^{p_1 z + p_0}. \tag{3.41}$$

The condition that $f - z$ and $f^{(n)} - z$ share 0 CM implies

$$f^{(n)} - z = \mathrm{e}^{Q_0(z)}(f - z), \tag{3.42}$$

where Q_0 is an entire function. From (3.42) and Lemma 2.5 we get

$$T(r,\mathrm{e}^{Q_0}) \leqslant 2T(r,f) + O(\log T(r,f) + \log r)\ (r \notin E,\ r \to \infty). \tag{3.43}$$

From (3.43) and Lemma 2.6 we see that for a sufficiently large positive number r_0 we have

$$T(r,\mathrm{e}^{Q_0}) \leqslant 2T(2r,f) + O(\log T(2r,f) + \log r + \log 2)\ (r \geqslant r_0). \tag{3.44}$$

From (3.33), (3.44), Definition 1.1 and $\mu(f^{(n)}) = 1$, we get

$$\sigma(\mathrm{e}^{Q_0}) = \mu(\mathrm{e}^{Q_0}) \leqslant \mu(f) = \mu(f^{(n)}) = 1. \tag{3.45}$$

From (3.45) we see that Q_0 is a polynomial. If Q_0 is a nonconstant polynomial, then (3.45) yields

$$Q_0(z) = q_1 z + q_0, \tag{3.46}$$

where $q_1 (\neq 0)$ and q_0 are two complex numbers. Proceeding as in the proof of (3.26), we get $\mu_2(f) = \sigma_2(f) = \gamma_{Q_0} = 1$, which contradicts the condition that $\sigma_2(f)$ is not a positive integer. Thus Q_0 is a constant, and so (3.42) can be rewritten as

$$f^{(n)} - z = c_0(f - z), \tag{3.47}$$

where c_0 is a nonzero complex number. From (3.47) we get $f^{(n+2)} - c_0 f'' = 0$. Combining Lemma 2.8 and the above supposition that f, and so $f^{(k)}$, is a transcendental entire function, where k $(\geqslant 1)$ is an arbitrary positive integer, we get $\sigma(f'') = 1$. Proceeding as in the proof of (3.33) we get $\sigma(f) = \sigma(f'')$, and so

$$\sigma(f - z) = \sigma(f^{(k)}) = 1. \tag{3.48}$$

From (3.41), (3.42) and (3.47) we have

$$\frac{f^{(n+1)} + (a_n - 1)f^{(n)}}{f - z} = c_0(\mathrm{e}^{p_1 z + p_0} - 1). \tag{3.49}$$

If $n \geqslant 2$, then (3.48) and Lemma 2.9 imply

$$\begin{aligned} m\Big(r, \frac{f^{(n+1)}}{f - z}\Big) & \\ &\leqslant m\Big(r, \frac{f^{(n+1)}}{f^{(n)}}\Big) + m\Big(r, \frac{f^{(n)}}{f^{(n-1)}}\Big) + \ldots + m\Big(r, \frac{f''}{f' - 1}\Big) + m\Big(r, \frac{f' - 1}{f - z}\Big) \\ &= o(\log r) \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} m\Big(r, \frac{f^{(n)}}{f - z}\Big) & \\ &\leqslant m\Big(r, \frac{f^{(n)}}{f^{(n-1)}}\Big) + m\Big(r, \frac{f^{(n-1)}}{f^{(n-2)}}\Big) + \ldots + m\Big(r, \frac{f''}{f' - 1}\Big) + m\Big(r, \frac{f' - 1}{f - z}\Big) \\ &= o(\log r). \end{aligned} \tag{3.51}$$

From (3.49)–(3.51) we get

$$m(r,\mathrm{e}^{p_1z+p_0}) = m(r,c_0(\mathrm{e}^{p_1z+p_0}-1))+O(1) \leqslant m\Big(r,\frac{f^{(n+1)}}{f-z}\Big)+m\Big(r,\frac{f^n}{f-z}\Big)+O(1)$$
$$= o(\log r),$$

which is impossible.

Next we suppose that $n=1$. Then (3.41) and (3.47) can be rewritten as

$$f''+a_1f'-z=(f'-z)\cdot \mathrm{e}^{p_1z+p_0} \tag{3.52}$$

and

$$f'-z=c_0(f-z), \tag{3.53}$$

respectively. From (3.53) we get

$$f(z)=\lambda \mathrm{e}^{c_0z}+\frac{c_0(c_0-1)z+c_0-1}{c_0^2}, \tag{3.54}$$

where $\lambda(\neq 0)$ is a complex number. From (3.54) we get

$$f'(z)=\lambda c_0\mathrm{e}^{c_0z}+\frac{c_0-1}{c_0} \quad\text{and}\quad f''(z)=\lambda c_0^2\mathrm{e}^{c_0z}. \tag{3.55}$$

Substituting (3.55) into (3.52) we get

$$\frac{(\lambda c_0^2+\lambda a_1c_0)\cdot \mathrm{e}^{c_0z}+a_1(c_0-1)/c_0-z}{\lambda c_0\mathrm{e}^{c_0z}+(c_0-1)/c_0-z}=\mathrm{e}^{p_1z+p_0}. \tag{3.56}$$

From (3.56) we see that $\lambda c_0^2+a_1\lambda c_0\neq 0$. Combining (3.56) and Lemma 2.11 we get

$$\{\lambda c_0^2+\lambda a_1c_0\}\cdot\Big\{\frac{c_0-1}{c_0}-z\Big\}=\lambda c_0\Big\{\frac{a_1(c_0-1)}{c_0}-z\Big\}. \tag{3.57}$$

From (3.56) and (3.57) we get $\mathrm{e}^{p_1z+p_0}=c_0+a_1$, which is impossible. This gives (3.32).

Considering (3.32) we discuss the following two cases.

Case 1. Suppose that f is a transcendental entire function. If $c=1$, then (3.32) can be rewritten as

$$f^{(n+1)}=(1-a_n)f^{(n)}. \tag{3.58}$$

From (3.58) and the supposition that f, and so $f^{(n+1)}$ are transcendental entire functions, we get $a_n \neq 1$, and so it follows from (3.58) that

$$f^{(n)}(z) - z = A_0 e^{(1-a_n)z} - z, \tag{3.59}$$

where A_0 ($\neq 0$) is a complex number. From (3.59) we get

$$f(z) - z = \frac{A_0}{(1-a_n)^n} \cdot e^{(1-a_n)z} + P_{n-1}(z) - z, \tag{3.60}$$

where and in what follows, P_{n-1} denotes a polynomial with degree $\gamma_{P_{n-1}} \leqslant n-1$, not necessarily the same at each occurrence. From (3.59), (3.60), Lemma 2.11 and the condition that $f - z$ and $f^{(n)} - z$ share 0 CM, we get

$$P_{n-1}(z) = \frac{(1-a_n)^n z - z}{(1-a_n)^n}. \tag{3.61}$$

From (3.60) and (3.61) we get (i) of Theorem 1.2. If $c \neq 1$, then (3.32) can be rewritten as

$$f^{(n+1)} + (a_n - c) f^{(n)} = (1-c)z. \tag{3.62}$$

From (3.62) and the condition that f, and so $f^{(n)}$, is a transcendental entire function, we have $a_n \neq c$. This together with (3.62) yields

$$f^{(n)}(z) = A_1 \cdot e^{(c-a_n)z} + \frac{z - cz}{a_n - c} + \frac{c-1}{(a_n - c)^2}, \tag{3.63}$$

where A_1 ($\neq 0$) is a complex number. From (3.63) we get

(3.64)

$$f(z) = \frac{A_1}{(c-a_n)^n} \cdot e^{(c-a_n)z} + \frac{1}{(n+1)!} \cdot \frac{1-c}{a_n - c} \cdot z^{n+1} + \frac{1}{n!} \cdot \frac{c-1}{(a_n-c)^2} \cdot z^n + P_{n-1}(z).$$

From (3.63), (3.64), Lemma 2.11 and the condition that $f - z$ and $f^{(n)} - z$ share 0 CM we get

$$\begin{aligned} &\frac{1}{(n+1)!} \cdot \frac{1-c}{a_n - c} \cdot z^{n+1} + \frac{1}{n!} \cdot \frac{c-1}{(a_n-c)^2} \cdot z^n + P_{n-1}(z) - z \\ &\quad = \frac{1}{(c-a_n)^n} \cdot \left\{ \frac{z - cz}{a_n - c} + \frac{c-1}{(a_n-c)^2} - z \right\}. \end{aligned} \tag{3.65}$$

From (3.65) we have a contradiction.

Case 2. Suppose that f is a nonconstant polynomial. We discuss the following four subcases.

Subcase 2.1. Suppose that $a_n = c = 1$. Then from (3.32) we have $f^{(n+1)} \equiv 0$, which implies that f is a polynomial with degree $1 \leqslant \gamma_f \leqslant n$. If $n = 1$, then $\gamma_f = 1$, and so $f' = c_1$, where $c_1 \neq 0$ is a complex number. Let $f = c_1 z + c_0$, where c_0 is a complex number. Then from the condition that $f - z$ and $f' - z$ share 0 CM, we deduce $c_0 = c_1(1 - c_1)$ and $c_1 \neq 0, 1$. Hence we get (ii) of Theorem 1.2. If $2 \leqslant \gamma_f \leqslant n - 1$, then $f^{(n)} = 0$, and so $f^{(n)}(z) - z = -z$. Using this and the condition that $f - z$ and $f^{(n)} - z$ share 0 CM we get a contradiction. If $\gamma_f = n \geqslant 2$, then $f = d_n z^n + d_{n-1} z^{n-1} + \ldots + d_1 z + d_0$, where $d_0, d_1, \ldots, d_n$ are complex numbers with $d_n \neq 0$. Using this and the condition that $f - z$ and $f^{(n)} - z$ share 0 CM we get a contradiction.

Subcase 2.2. Suppose that $c = 1$ and $c \neq a_n$. Then (3.32) can be rewritten as $f^{(n+1)} + (a_n - c) f^{(n)} = 0$. From this we deduce that f is a nonconstant polynomial such that its degree γ_f satisfies $2 \leqslant \gamma_f \leqslant n - 1$. Hence we get $f^{(n)}(z) - z = -z$. Combining the condition that $f - z$ and $f^{(n)} - z$ share 0 CM, we get a contradiction.

Subcase 2.3. Suppose that $c \neq 1$ and $a_n \neq c$. Then (3.32) can be rewritten as $f^{(n+1)} + (a_n - c) f^{(n)} = (1 - c) z$. This leads to

$$f^{(n)}(z) = \frac{(1-c)z}{a_n - c} + \frac{c-1}{(a_n - c)^2}, \tag{3.66}$$
$$f = \frac{(1-c)z^{n+1}}{(n+1)!(a_n - c)} + \frac{(c-1)z^n}{n!(a_n - c)^2} + P_{n-1}(z).$$

From (3.66) and the condition that $f - z$ and $f^{(n)} - z$ share 0 CM we get a contradiction.

Subcase 2.4. Suppose that $c \neq 1$ and $a_n = c$. Then (3.32) can be rewritten as $f^{(n+1)} = (1 - a_n) z$. Consequently,

$$f^{(n)}(z) = \frac{1 - a_n}{2} \cdot z^2 + c_0, \quad f = \frac{1 - a_n}{(n+2)!} \cdot z^{n+2} + P_n(z), \tag{3.67}$$

where P_n is a polynomial with degree $\gamma_{P_n} \leqslant n$. From (3.67) and the condition that $f - z$ and $f^{(n)} - z$ share 0 CM we get a contradiction.

Theorem 1.2 is thus completely proved.

P r o o f of Corollary 1.5. First, from the assumptions of Corollary 1.5 we have (1.6), where Q is an entire function. From (1.6), Lemma 2.5 and the assumptions of Corollary 1.5 we have

$$T(r, \mathrm{e}^Q) \leqslant 2T(r, f) + O(\log T(r, f) + \log r) \ (r \notin E,\ r \to \infty). \tag{3.68}$$

From (3.68) and Lemma 2.6 we have

$$T(r, \mathrm{e}^Q) \leqslant 2T(2r, f) + O(\log T(2r, f) + \log r + \log 2) \ (r \geqslant r_0), \tag{3.69}$$

where r_0 is a sufficiently large positive number. From (3.69), Definition 1.1 and the condition $\mu(f) < \infty$ we get

$$\sigma(\mathrm{e}^Q) = \mu(\mathrm{e}^Q) \leqslant \mu(f) < \infty. \tag{3.70}$$

From (3.70) we see that Q is a polynomial. If f is a polynomial or Q is a constant, then (1.6) can be rewritten as (3.32). Next we suppose that Q is a nonconstant polynomial, and suppose that f, and so $f^{(k)}$, is a transcendental entire function, where k ($\geqslant 1$) is an arbitrary positive integer. Proceeding as in Theorem 1.2 we get (3.33). From (3.33) and $\mu(f) < \infty$ we get $\mu(f^{(n)}) < \infty$. If $\mu(f^{(n)}) > 1$, from (1.6) and (i) of Theorem 1.1 we get $\mu(f^{(n)}) = \infty$. From this and $\mu(f) = \mu(f^{(n)})$ we get $\mu(f) = \infty$, which is impossible. If $\mu(f^{(n)}) \leqslant 1$, then from (1.6) and (ii) of Theorem 1.1 and in the same manner as in the proof of Theorem 1.2 we get (3.41)–(3.57), and so we get a contradiction. Finally, from (3.32) and in the same manner as in the proof of Theorem 1.2 we get (i) and (ii) of Theorem 1.2, and Corollary 1.5 is thus completely proved.

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