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SOME EXAMPLES OF CONTINUOUS IMAGES OF RADON-NIKODÝM COMPACT SPACES

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Abstract. We provide a characterization of continuous images of Radon-Nikodým compacta lying in a product of real lines and model on it a method for constructing natural examples of such continuous images.

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1. Introduction

This note contains some ideas to address the well known open problem whether the class of Radon-Nikodým (or briefly RN) compact spaces is stable under continuous images. In particular, we provide a method to construct a number of concrete compact spaces which seem to be natural candidates to be counterexamples, but we are unable to decide whether they are Radon-Nikodým compact or not. We think that the understanding of these examples could be a useful tool towards the solution of the problem.

2. Basic definitions and notation

We denote by $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ the set of all sequences of natural numbers, and we consider $0 \in \mathbb{N}$. If $s \in \mathbb{N}^{<\omega}$ is a finite sequence of natural numbers and $\sigma \in \mathcal{N}$, then $s < \sigma$ means that s is the initial segment of σ. We write $\mathcal{N}_s = \{\sigma \in \mathcal{N} \colon\ s < \sigma\}$.

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If K is a topological space, a map $d\colon K \times K \longrightarrow \mathbb{R}$ is said to ε-fragment K if for every (closed) set $L \subset K$ there exists a nonempty relative open subset U of L of d-diameter less than ε, that is, $\sup\{d(x,y)\colon x,y \in U\} < \varepsilon$. If d ε-fragments K for every ε we shall say that d fragments K.

Given a bounded family Δ of continuous functions over K we shall denote by d_Δ the uniform pseudometric over Δ, that is $d_\Delta(x,y) = \sup\{|f(x) - f(y)|\colon f \in \Delta\}$. When we view a compactum lying as a subset of a product of real lines, $K \subset \mathbb{R}^\Gamma$, we shall identify every element $\gamma \in \Gamma$ with the continuous function on K given by projection onto the γ-th coordinate. Thus, for a given $\Delta \subset \Gamma$, $d_\Delta(x,y) = \sup\{|x_\gamma - y_\gamma|\colon \gamma \in \Delta\}$.

Originally, RN compact spaces are defined as weak* compact subsets of dual Banach spaces with the Radon-Nikodým property. From an intrinsic topological point of view, they can be characterized as those compact spaces K for which there exists a lower semicontinuous metric $d\colon K \times K \longrightarrow \mathbb{R}$ which fragments K. The class of quasi Radon-Nikodým (qRN) compact spaces is a superclass of the class of RN compacta which is stable under continuous images. According to the definition of the first author [1], a compact space is qRN compact if there exists a lower semicontinuous map $f\colon K \times K \longrightarrow \mathbb{R}^+$ which is nonzero out of the diagonal and which fragments K. This is actually equivalent to other definitions given by Fabian, Heiler and Matouskova [6] and Reznichenko, as is shown in [9] and [2]. We refer to the recent survey of Fabian [5] on this subject, as well as to other articles containing different approaches or partial results concerning the problem of continuous images as [10], [7] or [3].

In this note, we will work with compact spaces viewed as closed subsets of Tychonoff cubes $[-1,1]^\Gamma$. Below we present the two known characterizations of both RN and qRN compact spaces in terms of these embeddings. Theorem 1 can be found in [8]. As concerns Theorem 2, the equivalence of (2) and (3) is proven in [6], and the equivalence of (1) with the others in [2].

Theorem 1. *A compact space K is* RN *compact if and only if there exists an embedding of K into the product of intervals $K \subset [-1,1]^\Gamma$ such that d_Γ fragments K.*

Theorem 2. *For a compact space K the following conditions are equivalent:*

(1) *K is quasi* RN *compact.*

(2) *There exists an embedding of K into the product of intervals $K \subset [-1,1]^\Gamma$ and a map $u\colon \Gamma \longrightarrow \mathcal{N}$ such that for every finite sequence s of naturals of length $n > 0$ the pseudometric $d_{u^{-1}(\mathcal{N}_s)}$ 2^{-n}-fragments K.*

(3) *For any embedding of K into a product of intervals $K \subset [-1,1]^\Gamma$ there exists a map $u\colon \Gamma \longrightarrow \mathcal{N}$ such that for every finite sequence s of naturals of length $n > 0$ the pseudometric $d_{u^{-1}(\mathcal{N}_s)}$ 2^{-n}-fragments K.*

The picture is that we have three classes of compact sets: RN compacta, their continuous images (ciRN) and quasi RN (qRN) compacta,

$$\mathrm{RN} \subset \mathrm{ciRN} \subset \mathrm{qRN}$$

and we do not know at all whether any of the inclusions is strict.

A key conceptual difference between RN and qRN is that there is no analogue of condition (3) in Theorem 2 for RN, referring to *any embedding* of K into a cube. This makes the class qRN easier to handle than RN in many aspects: in order to check that some space K is qRN compact we take any embedding of K and try to see whether there is a function $u\colon \Gamma \longrightarrow \mathcal{N}$ fulfilling the above mentioned condition (3). However, if we want to check that a certain space is RN compact we must find an appropriate embedding, probably different from the obvious ones. Such a difficulty is found, of course, in the problem of the continuous image.

3. A characterization of continuous images of RN compacta

The main result of this note, and the inspiration for the announced concrete examples, will be Theorem 5, where we give a similar characterization as those appearing in Theorems 1 and 2 for the intermediate class ciRN of the continuous images of Radon-Nikodým compacta.

Definition 3. A family $\Delta \subset C(K,[-1,1])$ of continuous functions over K called a Namioka family if the pseudometric d_Δ fragments K.

Using this concept, Theorem 1 can be restated saying that a compact space is RN if and only if there exists a Namioka family $\Delta \subset C(K)$ which separates the points of K. We shall need the fact that if a Namioka family exists, then indeed it can be chosen to be much bigger than simply a separating family.

Lemma 4. *Let K be an* RN *compactum. Then there exists a Namioka family Δ such that $C(K) = \overline{\bigcup_{n\in\mathbb{N}} n\Delta}^{\|\cdot\|}$.*

P r o o f. We consider Δ_0, a Namioka family over K, and then define Δ to be the set of all $f \in C(K)$ such that $|f(x) - f(y)| \leqslant d_{\Delta_0}(x,y)$ for all $x, y \in K$. Clearly $d_\Delta = d_{\Delta_0}$ so Δ is a Namioka family. On the other hand, $\bigcup_{n\in\mathbb{N}} n\Delta$ is a linear lattice of functions which separates the points of K, so by the Stone-Weierstrass Theorem, it is uniformly dense in $C(K)$. □

Theorem 5. *For a compact space $K \subset [a,b]^\Gamma$ the following conditions are equivalent:*

(1) *K is a continuous image of an* RN *compactum.*

(2) *There exists a function $u\colon \Gamma \longrightarrow \mathcal{N}$ and a family of compact sets $K_s \subset [a,b]^\Gamma$ for $s \in \mathbb{N}^{<\omega}$ such that*

  (a) *$d_{u^{-1}(\mathcal{N}_s)}$ fragments K_s for every s, and*

  (b) *if $length(s) = n > 0$, then for every $x \in K$ there exists $y \in K_s$ such that $d_{u^{-1}(\mathcal{N}_s)}(x,y) \leqslant 2^{-n}$.*

Of course, the use of the numbers 2^{-n} is inessential, we could have used any numbers ε_n converging to 0. We give an equivalent reformulation of Theorem 5 which will be more suitable for the purpose of making the proof more transparent, though it will be the previous statement that will be more relevant in further discussion.

Theorem 5 (b). *For a compact space $K \subset [a,b]^\Gamma$ the following conditions are equivalent:*

(1) *K is a continuous image of an* RN *compactum.*

(2) *For every $\varepsilon > 0$ there exists a countable decomposition $\Gamma = \bigcup\limits_m \Gamma_m^\varepsilon$ and compact sets $K_m^\varepsilon \subset [a,b]^\Gamma$ such that:*

  (a) *K_m^ε is fragmented by $d_{\Gamma_m^\varepsilon}$,*

  (b) *for every $m \in \mathbb{N}$ and $x \in K$ there is $y \in K_m^\varepsilon$ such that $d_{\Gamma_m^\varepsilon}(x,y) \leqslant \varepsilon$.*

Before passing to the proof, we state a lemma from [1] that we will often use.

Lemma 6. *Let $f\colon Q \longrightarrow S$ be a continuous surjection between compact spaces and $d\colon Q \times Q \longrightarrow \mathbb{R}$ a lower semicontinuous map that fragments Q. Then the map $\hat{d}(x,y) = \inf\{d(u,v)\colon f(u) = x, f(v) = y\}$ fragments S.*

P r o o f. Theorem 5 is easily seen to be equivalent to Theorem 5(b), we leave this to the reader. We shall prove Theorem 5(b). Without loss of generality, we suppose that $[a,b] = [-2,2]$.

(1) $\Rightarrow$ (2): Let L be RN compact and let $\pi\colon L \longrightarrow K \subset [-2,2]^\Gamma$ be a continuous surjection. We take a Namioka family Δ on L as in Lemma 4, so that we view $L \subset [-1,1]^\Delta$. We fix $\varepsilon \in (0,1)$. For every $\gamma \in \Gamma$ we have a continuous function on L given by $z \mapsto \pi(z)_\gamma$, so since $C(L) = \overline{\bigcup\limits_{m\in\mathbb{N}} m\Delta}^{\|\cdot\|}$ we have that

for every $\gamma \in \Gamma$ there exist $m(\gamma) \in \mathbb{N}$ and $\delta(\gamma) \in \Delta$ such that $|\pi(z)_\gamma - m(\gamma) z_{\delta(\gamma)}| \leqslant \varepsilon$ for every $z \in L$.

We set

$$\begin{aligned}\Gamma_m^\varepsilon &= \{\gamma\colon\ m(\gamma) = m\},\\ K_m^\varepsilon &= \{x \in [-2,2]^\Gamma\colon\ \exists z \in L\colon\ m z_{\delta(\gamma)} = x_\gamma \forall \gamma \in \Gamma_m^\varepsilon\}.\end{aligned}$$

Each K_m^ε is compact, because it is the continuous image of L under the map $z \mapsto (mz_{\delta(\gamma)})_{\gamma\in\Gamma}$.

We check condition (b). If $x \in K$, then $x = \pi(z)$ for some $z \in L$, and we know that $|x_\gamma - m(\gamma)z_{\delta(\gamma)}| \leqslant \varepsilon$ for every $\gamma \in \Gamma$. We define y to be an element of $[-2,2]^\Gamma$ such that $y_\gamma = mz_{\delta(\gamma)}$ for $\gamma \in \Gamma_m^\varepsilon$. Then $y \in K_m^\varepsilon$ and $d_{\Gamma_m^\varepsilon}(x,y) \leqslant \varepsilon$.

For condition (a), we consider $\hat{K}_m^\varepsilon \subset [-2,2]^{\Gamma_m^\varepsilon}$, the projection of K_m^ε to the coordinates of Γ_m^ε. We prove that $d_{\Gamma_m^\varepsilon}$ fragments $\hat{K}_m^\varepsilon$ (this is equivalent to saying that $d_{\Gamma_m^\varepsilon}$ fragments K_m^ε). We have a continuous surjection $\varphi\colon L \longrightarrow \hat{K}_m^\varepsilon$ given by $\varphi(z) = (mz_{\delta(\gamma)})_{\gamma\in\Gamma_m^\varepsilon}$. Observe that $d_{\Gamma_n^\varepsilon}(\varphi(z),\varphi(z')) = m\sup_{\gamma\in\Gamma}|z_{\delta(\gamma)} - z'_{\delta(\gamma)}| \leqslant md_\Delta(z,z')$, so

$$d_{\Gamma_n^\varepsilon}(x,x') \leqslant m\inf\{d_\Delta(z,z')\colon\ \varphi(z) = x, \varphi(z') = x'\}.$$

Since d_Δ fragments L, the conclusion follows from Lemma 6.

(2) $\Rightarrow$ (1): Let us set $\varepsilon_n = 2^{-n}$. Without loss of generality we assume that $\Gamma_m^\varepsilon \cap \Gamma_{m'}^\varepsilon = \emptyset$ for $m \neq m'$. For $\gamma \in \Gamma$ and $n \in \mathbb{N}$, let $m_n(\gamma)$ be the only m such that $\gamma \in \Gamma_m^{\varepsilon_n}$.

We consider again $\hat{K}_m^\varepsilon \subset [a,b]^{\Gamma_m^\varepsilon}$, the projection of K_m^ε to the coordinates Γ_m^ε. The metric $d_{\Gamma_m^\varepsilon}$ fragments $\hat{K}_m^\varepsilon$ so this is an RN compactum.

Let $L \subset \prod\limits_{n\in\mathbb{N}}\prod\limits_{m\in\mathbb{N}} K_m^{\varepsilon_n}$ be the set consisting of all x for which there exists $g(x) \in [a,b]^\Gamma$ such that

$$d_{\Gamma_m^{\varepsilon_n}}(g(x), p_{mn}(x)) \leqslant \varepsilon_n$$

where $p_{mn}(x) \in K_m^{\varepsilon_n}$ is the coordinate of x in the factor $K_m^{\varepsilon_n}$.

Notice that this element $g(x)$ is uniquely determined by x since $p_{m_n(\gamma)n}(x)_\gamma \to g(x)_\gamma$ as $n \to \infty$.The space L is compact and the map $g\colon\ L \longrightarrow [a,b]^\Gamma$ is continuous (these two facts are easily checked by considering net convergence). Moreover, $g(L) \supset K$ (this follows from part (b) of condition (2) in the theorem). Since the class RN is closed under the operations of taking countable products and closed subspaces, L is RN compact and K is a continuous image of an RN compactum. $\square$

4. A way how to construct continuous images of RN compacta

We think now of a continuous image of an RN compactum as a compact space K that satisfies part (2) of Theorem 5. For simplicity, we shall assume that $[a,b] = [0,1]$, $\Gamma = \mathcal{N}$ and $u\colon\ \mathcal{N} \longrightarrow \mathcal{N}$ is just the identity map. We can think of Theorem 5 as giving a *constructive* process for producing continuous images of RN compacta in the following way:

Step 1. Begin with a family $\{K_s\colon s \in \mathbb{N}^{<\omega}\}$ of compact subsets of $[0,1]^{\mathcal{N}}$ such that $d_{\mathcal{N}_s}$ fragments K_s.

Step 2. For every s of length n, consider the 2^{-n}-$\mathcal{N}_s$-enlargement of K_s,

$$[K_s] = \{x \in [0,1]^{\mathcal{N}}\colon \exists y \in K_s \ d_{\mathcal{N}_s}(x,y) \leqslant 2^{-n}\},$$

and a compact set $L_s \subset [K_s]$.

Step 3. Finally, $K = \bigcap_{s\in\mathbb{N}^{<\omega}} L_s$ is a continuous image of an RN compactum.

This would be the general procedure but still it does not seem that much *constructive*: How can we get compacta K_s for step 1? And how can we get the sets L_s of step 2? Well, some canonical choices can help us in this task:

For step 1 we can begin with an RN compactum $L \subset [0,1]^{\mathcal{N}}$ which is fragmented by $d_{\mathcal{N}}$, and make $K_s = L$ for all s. We know plenty of concrete examples of such objects as we shall describe later. For step 2, simply observe that the sets $[K_s]$ described above are themselves compact, so we can take $L_s = [K_s]$, in our case

$$L_s = \{x \in [0,1]^{\mathcal{N}}\colon \exists y \in L\colon d_{\mathcal{N}_s}(x,y) \leqslant 2^{-n}\}.$$

After this we shall denote the resulting ciRN compactum by $\tilde{L} = \bigcap_{s\in\mathbb{N}^{<\omega}} L_s \subset [0,1]^{\mathcal{N}}$. Notice that it only depends on the RN compactum by $L \subset [0,1]^{\mathcal{N}}$ that we took as the starting point of our construction. It follows from the proof of Theorem 5 that $\tilde{L}$ is indeed a continuous image of some closed subset of the countable power $L^{\mathbb{N}}$.

A trivial example of a compact space $L \subset [0,1]^{\mathcal{N}}$ which is fragmented by $d_{\mathcal{N}}$ is a scattered compactum. Recall that a compactum is scattered if every nonempty subset contains an isolated point. Scattered compacta are indeed fragmented by any metric, even the discrete metric. Scattered compacta, being totally disconnected, are typically found as compact subsets of the Cantor cube $L \subset \{0,1\}^{\mathcal{N}}$ and in this case, it follows immediately that also $\tilde{L} \subset \{0,1\}^{\mathcal{N}}$ is totally disconnected. It is a known fact that continuous images of RN compacta (indeed all qRN compacta) which are totally disconnected are RN compacta. Thus, scattered compacta seem not to be good starting points for our procedure if we are looking for candidates to be counterexamples to the problem of the continuous images.

As we mentioned in the previous paragraph, the problem of the continuous images has a affirmative solution in the case when the image is totally disconnected. So we focus rather on connected compacta, where the problem becomes harder. We shall obtain our connected RN compactum by taking the convex hull of a scattered compactum:

Proposition 7. *Let $S \subset \{0,1\}^{\mathcal{N}}$ be a scattered compactum and let $\overline{\mathrm{co}}(S) \subset [0,1]^{\mathcal{N}}$ be the closure of its convex hull in $[0,1]^{\mathcal{N}}$. Then $d_{\mathcal{N}}$ fragments $\overline{\mathrm{co}}(S)$.*

We notice that it is not known in general whether the closed convex hull of an RN compact space is again RN compact (when such an operation makes sense and produces a compact set). This is indeed a particular instance of the problem of the continuous images, cf. [8].

Proof. Let $P(S)$ denote the space of regular Borel probability measures on S endowed with the weak* topology. Recall that every continuous function $f\colon S \longrightarrow [0,1]$ induces a continuous function $\hat{f}\colon P(S) \longrightarrow [0,1]$ given by $\hat{f}(\mu) = \int f \,\mathrm{d}\mu$. Applying this fact to every coordinate function over S, we find a natural continuous function $g\colon P(S) \longrightarrow [0,1]^{\mathcal{N}}$ whose image is precisely $g(P(S)) = \overline{\mathrm{co}}(S)$. To get convinced about this fact, notice that $g(\delta_s) = s$ for every $s \in S$ (δ_s denotes the corresponding Dirac measure) and g commutes with convex linear combinations, and recall the well know fact that $P(S) = \overline{\mathrm{co}}\{\delta_s\colon\ s \in S\}$. Now, we can view $P(S)$ as a weak* compact subset of $C(S)^*$ where $C(S)$ is the space of continuous functions over S. Since S is scattered, $C(S)$ is an Asplund space and this implies that any weak* compact subset of $C(S)^*$, and in particular $P(S)$, is fragmented by the norm of $C(S)^*$ (cf. [4]). We observe the following: for every $x, y \in P(S)$,

$$d_{\mathcal{N}}(g(\mu), g(\nu)) \leqslant \|\mu - \nu\| = \sup\{|h(\mu) - h(\nu)|\colon\ h \in C(S), \|h\| \leqslant 1\}.$$

Hence, the pseudometric $d_{\mathcal{N}}(g(\mu), g(\nu))$ fragments $P(S)$ as well, and making use of Lemma 6, we conclude that $d_{\mathcal{N}}$ fragments $g(P(S)) = \overline{\mathrm{co}}(S)$. $\square$

The map g appearing in the proof of Proposition 7 is one-to-one provided that the linear span of the coordinate functions is dense in $C(S)$, and in that case we would have $\overline{\mathrm{co}}(S) = P(S)$. This will happen if for instance the coordinate functions form an algebra of functions over S, by the Stone-Weierstrass theorem.

Summarizing, the ciRN compacta that we are proposing are those obtained in the following way:

Step 0. Begin with a scattered compactum $S \subset \{0,1\}^{\mathcal{N}}$.

Step 1. Consider its closed convex hull $L(S) = \overline{\mathrm{co}}(S) \subset [0,1]^{\mathcal{N}}$.

Step 2. Finally, take the ciRN compactum

$$\tilde{L}(S) = \bigcap_{s \in \mathbb{N}^{<\omega}} L(S)_s = \bigcap_{s \in \mathbb{N}^{<\omega}} \{x \in [0,1]^{\mathcal{N}}\colon\ \exists y \in L(S)\ d_{\mathcal{N}_s}(x,y) \leqslant 2^{-n}\},$$

where n is the length of s.

Problem 8. *Let $S \subset \{0,1\}^{\mathcal{N}}$ be a compact scattered space. Is the space $\tilde{L}(S)$ an* RN *compact?*

There are a couple of cases when we know that the answer to this question is affirmative. One case occurs if the weight of S is less than $\mathfrak{b}$ [2]. This will not interfere if we take S to be of weight the continuum, or at least weight $\mathfrak{b}$, which on the other hand is the natural choice. The other case is when S is Eberlein compact, which implies that $\tilde{L}(S)$ is Eberlein compact as well. We do not know much more about these two cases, and those which can be obtained by mixing them up.

5. The space of almost increasing functions

We promised concrete examples and we are going to describe a very concrete one in this section. The only variable on which the space $\tilde{L}(S)$ depends is on the choice of some scattered compact $S \subset \{0,1\}^{\mathcal{N}}$, which by the above remarks we should take care not to be Eberlein compact. One natural example of such is an ordinal interval $[0,\alpha]$ endowed with the order topology. The way of viewing a scattered compactum of this type as a subset of $\{0,1\}^{\mathcal{N}}$ is to fix a well order $\prec$ on $\mathcal{N}$ and to declare

$$S = \left\{x \in \{0,1\}^{\mathcal{N}} \colon\ \forall i \prec j\ x_i \leqslant x_j\right\}.$$

It is easy to check that the closed convex hull of this is nothing else than

$$\overline{\mathrm{co}}(S) = \left\{x \in [0,1]^{\mathcal{N}} \colon\ \forall i \prec j\ x_i \leqslant x_j\right\}.$$

It requires a little bit more work to realize that $\tilde{L}(S)$ has also a nice description as the set of "almost increasing functions". We consider the following distance defined on $\mathcal{N}$: $d(\sigma,\tau) = 1/2^{\min(n\colon\, \sigma_n \neq \tau_n)-1}$.

Theorem 9. *In this case, we can describe the space $\tilde{L}(S)$ as follows:*

$$\tilde{L}(S) = \left\{x \in [0,1]^{\mathcal{N}} \colon\ \forall \sigma \prec \tau\ x_\sigma \leqslant x_\tau + d(\sigma,\tau)\right\}.$$

Proof. It is enough to observe that for $x \in [0,1]^{\mathcal{N}}$ and $\varepsilon > 0$, the following two conditions are equivalent:

(1) For every $\sigma \prec \tau$, $x_\sigma \leqslant x_\tau + 2\varepsilon$.
(2) There exists $y \in L(S)$ such that $|x_\sigma - y_\sigma| \leqslant \varepsilon$ for all σ.

Namely, if condition (1) holds, we can define $y_\sigma = \inf\{1, x_\tau + \varepsilon\colon\ \tau \succeq \sigma\}$. $\square$

6. RN quotients of a qRN compactum

It is not only that we do not know whether every qRN compactum is RN, moreover we do not know the answer to such a question as whether every qRN compactum has an RN quotient of the same weight. It is shown in [2] that every qRN compactum is a subspace of a product of $\mathfrak{d}$ many RN compact spaces, and also that every qRN compactum of weight less than $\mathfrak{b}$ is RN compact ($\mathfrak{d}$ and $\mathfrak{b}$ denote the domination and bounding cardinal numbers, following the notation in [11]). As a corollary one gets:

Proposition 10. *Let K be a* qRN *compactum of weight κ.*
(1) *If $\kappa > \mathfrak{d}$, then K has an* RN *quotient of weight λ for every $\lambda < \kappa$.*
(2) *If $cf(\kappa) > \mathfrak{d}$, then K has an* RN *quotient of weight κ.*
(3) *Every quotient of K of weight less than $\mathfrak{b}$ is* RN *compact.*

Observe that the previous statements give no information about a compact space K of weight $\mathfrak{c}$ under CH. We do not know whether the space of almost increasing functions is RN compact, but at least we can show the following result.

Theorem 11. *Let $K \subset [0,1]^{\mathcal{N}}$ be the compact space of almost increasing functions associated to a well order of $\mathcal{N}$.*
(1) *For every cardinal $\lambda < \mathfrak{c}$, K has an* RN *quotient of weight λ.*
(2) *If $\mathfrak{c}$ is a regular cardinal, then K has an* RN *quotient of weight $\mathfrak{c}$.*
(3) *If the well order $(\prec)$ is chosen so that every infinite $(\prec)$-interval is dense in the Baire space $\mathcal{N}$, then K has an* RN *quotient of weight $\mathfrak{c}$.*

Before starting the proof, we introduce some auxiliary definitions and results. For $\sigma \in \mathbb{N}^{\mathbb{N}}$, a $\prec$-increasing sequence $\tau^* : \tau^1 \prec \tau^2 \prec \dots$ such that $d(\sigma, \tau^n) \leqslant 2^{-n-2}$ is called a σ-good sequence. For two sequences τ^*, σ^* in $\mathcal{N}$ we write $\tau^* \prec \sigma^*$ if $\tau^i \prec \sigma^j$ for all i, j. A family Z of such sequences is called separated if for any $\zeta^*, \xi^* \in Z$, either $\zeta^* \prec \xi^*$ or $\xi^* \prec \zeta^*$, so that $(\prec)$ is a well order on Z.

Given a $(\prec)$-increasing sequence τ^* in $\mathcal{N}$ we produce a continuous function $\varphi[\tau^*]\colon K \longrightarrow [0,1]$ as follows. First, we consider the map $\Phi\colon [0,1]^{\mathbb{N}} \longrightarrow [0,1]^{\mathbb{N}}$ which consists in associating with each sequence (x^n) a sequence (y^n) with the property that $|y^n - y^{n+1}| \leqslant 2^{-n}$, constructing it recursively: given y^n, we choose y^{n+1} to be the closest number to x^{n+1} that satisfies $|y^n - y^{n+1}| \leqslant 2^{-n}$; in a formula:

$$\Phi((x^k)_{k\in\mathbb{N}})_{n+1} = y^{n+1} = x^{n+1} + \operatorname{sign}(y^n - x^{n+1}) \cdot \max(|y^n - x^{n+1}| - 2^{-n}, 0).$$

Notice that Φ is continuous because this recursive formula is continuous on x_{n+1} and y_n, so y^{n+1} depends continuously on $x_1, \dots, x_{n+1}$. For $x \in K \subset [0,1]^{\mathcal{N}}$, we also write $\Phi[\tau^*](x) = \Phi((x_{\tau^n})_n)$. Clearly, the image of Φ consists of convergent sequences so one can define

$$\varphi[\tau^*](x) = \lim \Phi((x_{\tau^n})_{n\in\mathbb{N}}).$$

The function $\varphi[\tau^*]$ is continuous on K. The reason is that for every m, the map $x \mapsto \Phi[\tau^*](x)_n$ is continuous, and $\varphi[\tau^*](x)$ is the uniform limit of this sequence of continuous functions, $|\Phi[\tau^*](x)_n - \Phi[\tau^*](x)_{n+1}| \leqslant 2^{-n}$.

Lemma 12. *Let $\sigma \in \mathcal{N}$ and let Z be a separated family of σ-good sequences. Then $\mathcal{F} = \{\varphi[\zeta^*]\colon\ \zeta^* \in Z\}$ is a Namioka family.*

P r o o f. Given $m \in \mathbb{N}$ and L a closed subset of K we will try to find a nonempty open set V of L of uniform diameter less than 2^{3-m} for the uniform metric associated with the family $\mathcal{F}$.

For every $n \in \mathbb{N}$, $\sigma|n = (\sigma_0, \ldots, \sigma_{n-1})$, and we denote by $d_n = d_{\mathcal{N}_{\sigma|n}}$ the corresponding pseudometric. Since d_n 2^{-n}-fragments K we can find a nonempty open set $U \subset L$ with d_n-diameter less than or equal to 2^{-n}, for every $n \leqslant m$. On the other hand, $\zeta^n \in \mathcal{N}_{\sigma|n}$ for every $\zeta \in Z$ because $d(\zeta^n, \sigma) \leqslant 2^{-n-1}$. This means that for every $\zeta \in Z$ and every $n \leqslant m$, the set $\{x_{\zeta^n}\colon\ x \in U\}$ lies in an interval, say I_{ζ^n}, of diameter less than or equal to 2^{-n}.

Remark A. We note that, if the case were that $\Phi[\zeta^*](x)_n \in I_{\zeta^m}$ for every $x \in U$ and every $\zeta \in Z$ then we would be done, because this would imply that for every $x \in U$, $\varphi[\zeta^*](x)$ lies in an interval of diameter at most $1/2^{m-2}$, namely $I_{\zeta^m} + 1/2^{m-1}$.

If the condition expressed in the remark fails, we begin a procedure of reducing U and changing the intervals I_n as follows. If it is not the case it means that there exists $x \in U$ and some ζ such that $\Phi[\zeta^*](x)_m \notin I_{\zeta^m}$. We choose ζ_1^* to be the $(\prec)$-minimum ζ^* with this property and fix the corresponding $x \in U$. In particular, we will have that $\Phi[\zeta^*](x)_n \neq x_{\zeta^n}$. By the definition of Φ, if this happens it is because there exists some $j < n$ such that $|x_{\zeta^j} - x_{\zeta^{j+1}}| > 2^{-j}$. Actually, since x is almost increasing and $d(\zeta^j, \zeta^{j+1}) \leqslant 2^{-j-1}$ (for ζ^* is σ-good), it must be the case that $x_{\zeta^j} + 2^{-j} < x_{\zeta^{j+1}}$.

We define

$$\begin{aligned} U_1 = \{y \in U\colon\ |y_{\zeta^m} - x_{\zeta^m}| < 2^{-m-2} \ &\text{ and } \ |y_{\zeta^j} - x_{\zeta^j}| < 2^{-m-5} \\ &\text{ and } \ |y_{\zeta^{j+1}} - x_{\zeta^{j+1}}| < 2^{-m-5}\} \end{aligned}$$

Claim B. For every $y \in U_1$ and every $\xi^* \succ \zeta^*$ we have $x_{\zeta^j} + 2^{-j-1} < y_{\xi^j}$.

Proof of the claim:

$$x_{\zeta^j} + 2^{-j} < x_{\zeta^{j+1}} \leqslant y_{\zeta^{j+1}} + 2^{-m-5} \leqslant y_{\xi^j} + d(\zeta^{j+1}, \xi^j) + 2^{-m-5} \leqslant y_{\xi^j} + 2^{-j-1}.$$

Notice that for every $y \in U_1$ we have:

- If $\xi^* < \zeta^*$, then $\Phi[\xi^*](y)_m \in I_m$,
- $|\Phi[\zeta^*](y)_m - \Phi[\zeta^*](x)_m| < 2^{-m-2}$.

Therefore, if in addition $\Phi[\xi^*](y)_m \in I_{\xi^m}$ for every $\xi^* > \zeta^*$, the proof would be complete, on the same grounds as in Remark *A*. If not, we must repeat the procedure of reduction and pass to a new open set U_2, and look now for the minimum $\zeta_2^* > \zeta^*$ such that $\Phi[\zeta_2^*](x)_n \notin I_{\zeta_2^n}$, etc. In order to conclude, we observe that the reduction procedure cannot be repeated infinitely many times, so that for some $k \in \mathbb{N}$, an open set U_k of small $\mathcal{F}$ diameter will be found. The reason for the impossibility of infinite repetitions is precisely the inequality of *Claim B*. There are only finitely many possible values for the natural number $j \leqslant m$, so for some j, after infinitely many steps we would have a sequence $x^{(1)}_{\zeta(1)} < x^{(2)}_{\zeta(2)} < \ldots$ of numbers in $[0,1]$ with $x^{(n)}_{\zeta(n)} + 2^{-j-1} < x^{(n+1)}_{\zeta(n+1)}$. This is a contradiction which completes the proof of the lemma. $\square$

Lemma 13. *Let $\sigma \in \mathcal{N}$, let Z be a separated family of σ-good sequences of cardinality κ, and $\mathcal{F} = \{\varphi[\zeta^*]\colon \zeta \in Z\}$. Consider the quotient space L of K induced by the equivalence relation ($x \sim y \iff \forall f \in \mathcal{F} f(x) = f(y)$). Then L is an* RN *compactum of weight κ.*

P r o o f. It is clear that L is RN compact since $\mathcal{F}$ is a Namioka family. The point is to prove that the weight of L is not less than κ. To this aim we shall show that actually L contains a copy of the compact space $P[0,\kappa]$ of probability measures on the ordinal interval $[0,\kappa]$. Remember that this space can be identified with

$$P[0,\kappa] = \{f\colon [0,\kappa] \longrightarrow [0,1]\colon \alpha \prec \beta \Rightarrow f(\alpha) \leqslant f(\beta)\}.$$

Let $Z = \{\zeta^*(\alpha)\colon \alpha \prec \kappa\}$ be an $(\prec)$-increasing enumeration of Z. With every $f \in P[0,\kappa]$ we associate a function $\psi(f) \in K$ defined as

- $\psi(f)(\sigma) = 0$ if $\sigma \prec \zeta^1(0)$
- $\psi(f)(\sigma) = f(\alpha)$ if $\zeta^1(\alpha) \preceq \sigma \prec \zeta^1(\alpha+1)$
- $\psi(f)(\sigma) = 1$ if $\sigma \succ \zeta^1$ for all $\zeta \in Z$.

Then $\psi\colon P[0,\kappa] \longrightarrow K$ is a one-to-one continuous map. The key fact is that after composing it with the quotient map onto L,

$$\psi\colon P[0,\kappa] \longrightarrow K \longrightarrow L$$

the function is still one-to-one. The reason is that $\varphi[\zeta^*(\alpha)](\psi(f)) = f(\alpha)$ for every $\alpha \prec \kappa$ and every $f \in P[0,\kappa]$. $\square$

Proof of Theorem 11: In the case of item (3), it is clear that for any $\sigma \in \mathcal{N}$ we can find a separated σ-good family of size $\mathfrak{c}$, since actually we can find a σ-good sequence inside any infinite $(\prec)$-interval. For (1) and (3), we shall show that for every cardinal $\lambda \leqslant \mathfrak{c}$ there is a separated family of σ-good sequences of cardinality $cf(\lambda)$. Let $o\colon \mathcal{N} \longrightarrow \mathrm{Ord}$ be the function which associates with every $\sigma \in \mathcal{N}$ the ordinal which indicates its position in the well order $(\prec)$. Let us pick a point

$$\sigma \in \bigcap_{o(\tau)<\lambda} \mathrm{cl}_{\mathcal{N}}\{\xi\colon\ o(\tau) \prec o(\xi) \prec \lambda\},$$

where $\mathrm{cl}_{\mathcal{N}}(\cdot)$ indicates the closure in the Baire space of irrationals $\mathcal{N}$ with its usual metric topology. Such a point σ exists because $\mathcal{N}$ is hereditarily Lindelöf and we suppose that $cf(\lambda) > \omega$. It is clear that we can find a separated family Z of σ-good sequences of size $cf(\lambda)$, all of them below the ordinal position λ.

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