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SET VERTEX COLORINGS AND JOINS OF GRAPHS

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Abstract. For a nontrivial connected graph G, let $c\colon V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. For a vertex v of G, the neighborhood color set $\mathrm{NC}(v)$ is the set of colors of the neighbors of v. The coloring c is called a set coloring if $\mathrm{NC}(u) \neq \mathrm{NC}(v)$ for every pair u, v of adjacent vertices of G. The minimum number of colors required of such a coloring is called the set chromatic number $\chi_s(G)$. A study is made of the set chromatic number of the join $G+H$ of two graphs G and H. Sharp lower and upper bounds are established for $\chi_s(G+H)$ in terms of $\chi_s(G)$, $\chi_s(H)$, and the clique numbers $\omega(G)$ and $\omega(H)$.

Keywords: neighbor-distinguishing coloring, set coloring, neighborhood color set

MSC 2010: 05C15

1. Introduction

Many methods have been introduced that use graph colorings to distinguish all vertices of a graph or the two vertices in each pair of adjacent vertices. Certainly the most common graph colorings used to distinguish every two adjacent vertices in a graph G are the proper colorings, where distinct colors are assigned to every two adjacent vertices of G. The minimum number of colors required in a proper coloring of G is the *chromatic number* $\chi(G)$. In [1] another vertex coloring of graphs for the purpose of distinguishing every two adjacent vertices of G which may require fewer than $\chi(G)$ colors was introduced.

For a nontrivial connected graph G, let $c\colon V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. For a set $S \subseteq V(G)$, define the *set* $c(S)$ *of colors of* S by

$$c(S) = \{c(v)\colon v \in S\}.$$

For a vertex v in a graph G, let $N(v)$ be the neighborhood of v (the set of all vertices adjacent to v in G). The *neighborhood color set* $\mathrm{NC}_c(v) = c(N(v))$ is the set of colors of the neighbors of v. (If the coloring c under consideration is clear, we write $\mathrm{NC}(v)$ for the neighborhood color set of v.) The coloring c is called *set neighbor-distinguishing* or simply a *set coloring* if $\mathrm{NC}(u) \neq \mathrm{NC}(v)$ for every pair u, v of adjacent vertices of G. The minimum number of colors required of such a coloring is called the *set chromatic number* of G and is denoted by $\chi_s(G)$. This concept was introduced and studied in [1] where it was observed that

$$1 \leqslant \chi_s(G) \leqslant \chi(G) \leqslant n$$

for every graph G of order n. To illustrate these concepts, we consider the graph G of Fig. 1. The chromatic number of G is $\chi(G) = 4$. In fact, the set chromatic number of G is $\chi_s(G) = 3$. Fig. 1 shows a set 3-coloring of G and so $\chi_s(G) \leqslant 3$. We now show that $\chi_s(G) \geqslant 3$. Suppose that there is a set 2-coloring c of G using the colors 1 and 2. Then $\mathrm{NC}(v) \in \{\{1\}, \{2\}, \{1, 2\}\}$ for each vertex v of G. This implies that $\mathrm{NC}(v_i) = \mathrm{NC}(v_j)$ for some integers i and j with $1 \leqslant i < j \leqslant 4$, which is impossible. Thus $\chi_s(G) = 3$, as claimed.

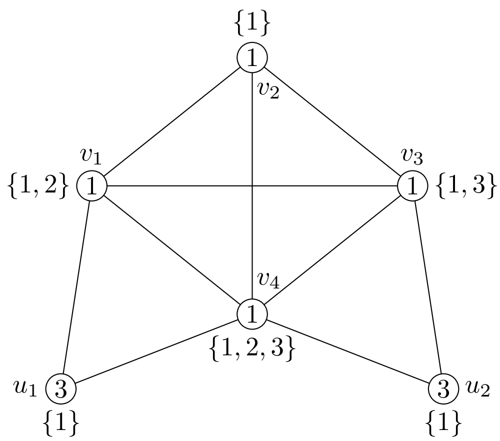

Figure 1. A set coloring of a graph.

If G is a connected graph of order n, then $\chi_s(G) = 1$ if and only if $\chi(G) = 1$ (in which case $G = K_1$) and $\chi_s(G) = n$ if and only if $\chi(G) = n$ (in which case $G = K_n$). It was shown in [1] that $\chi_s(G) = n - 1$ if and only if $\chi(G) = n - 1$ and that for each pair k, n of integers with $2 \leqslant k \leqslant n$, there is a connected graph G of order n with $\chi_s(G) = k$. The following observation will be useful to us.

Observation 1.1 ([1])**.** If u and v are two adjacent vertices in a graph G such that $N(u) - \{v\} = N(v) - \{u\}$, then $c(u) \neq c(v)$ for every set coloring c of G. Furthermore, if $S = N(u) - \{v\} = N(v) - \{u\}$, then $\{c(u), c(v)\} \nsubseteq c(S)$.

In [1] the set chromatic numbers of some well-known graphs (namely cycles, bipartite graphs, and complete multipartite graphs) were determined. Furthermore, several bounds were established for the set chromatic number of a graph G in terms of other graphical parameters, namely the chromatic number $\chi(G)$ and the clique number $\omega(G)$, which is the order of a largest complete subgraph (clique) in G. Some of these results are stated below.

Theorem 1.2 ([1])**.** *A nonempty graph G has set chromatic number 2 if and only if G is bipartite. Furthermore, if G is a 3-chromatic graph, then $\chi_s(G) = 3$.*

Theorem 1.3 ([1])**.** *For every graph G,*

$$\chi_s(G) \geqslant 1 + \lceil \log_2 \omega(G) \rceil. \tag{1}$$

Theorem 1.4 ([1])**.** *Let G be a graph. If v is a vertex of G, then*

$$\chi_s(G) - 1 \leqslant \chi_s(G - v) \leqslant \chi_s(G) + \deg v.$$

If e is an edge of G, then

$$|\chi_s(G) - \chi_s(G - e)| \leqslant 2.$$

Furthermore, if $e = uv$ is not a bridge in G such that the distance between u and v in $G - e$ is at least 4, then $|\chi_s(G) - \chi_s(G - e)| \leqslant 1$.

For two vertex-disjoint graphs G and H, the *join* $G + H$ of G and H is the graph whose vertex set is $V(G) \cup V(H)$ and whose edge set consists of $E(G) \cup E(H)$ together with all edges joining a vertex of G and a vertex of H. While $\chi(G+H) = \chi(G)+\chi(H)$ for every two graphs G and H, such is not the case for the set chromatic number. Our goal here is to study the set chromatic number of the join of two graphs G and H and establish sharp lower and upper bounds for $\chi_s(G + H)$. It is convenient to introduce some notation. For each integer k, let

$$\mathbb{N}_k = \{1, 2, \ldots, k\}.$$

For integers a and b with $a < b$, let

$$[a..b] = \{x \in \mathbb{Z} \colon\ a \leqslant x \leqslant b\}.$$

In particular, $[1..b] = \mathbb{N}_b$. We refer to the book [2] for graph theory notation and terminology not described in this paper.

2. Lower bounds for $\chi_s(G+H)$

We begin by presenting a lower bound for the set chromatic number $\chi_s(G+H)$ of two graphs G and H in terms of $\chi_s(G)$ and $\chi_s(H)$. The following lemma will be useful to us.

Lemma 2.1. *Let G and H be two graphs. If c is a set coloring of $G+H$, then c restricted to G is a set coloring of G.*

Proof. For a vertex v in G and a set coloring c of $G+H$, observe that

$$\mathrm{NC}(v) = c(N_G(v)) \cup c(V(H)) \tag{2}$$

and for every two adjacent vertices x and y of G, $\mathrm{NC}(x) \neq \mathrm{NC}(y)$. By (2), it follows that $c(N_G(x)) \neq c(N_G(y))$ and so c restricted to $V(G)$ is a set coloring of G. □

The following is an immediate consequence of Lemma 2.1.

Corollary 2.2. *For every two graphs G and H,*

$$\chi_s(G+H) \geqslant \max\{\chi_s(G), \chi_s(H)\}.$$

Next we present a necessary condition for graphs G and H such that the equality holds in Corollary 2.2.

Proposition 2.3. *If G and H are nonempty graphs, then*

$$\chi_s(G+H) > \max\{\chi_s(G), \chi_s(H)\}.$$

Proof. Suppose that $\chi_s(G+H) = \max\{\chi_s(G), \chi_s(H)\} = \chi_s(G) = k$ and let a set k-coloring $c\colon V(G+H) \to \mathbb{N}_k$ of $G+H$ be given. Since the restriction of c to G is a set coloring of G by Lemma 2.1, it follows that $c(V(G)) = \mathbb{N}_k$. Then $\mathrm{NC}(v) = \mathbb{N}_k$ for every vertex v in H. Hence no two vertices in H are adjacent. □

The converse of Proposition 2.3 does not hold in general. While there are graphs G for which $\chi_s(G+\overline{K}_n) = \chi_s(G)$, there are also graphs G for which $\chi_s(G+\overline{K}_n) > \chi_s(G)$. To see this, let $H = \overline{K}_n$ for some $n \geqslant 1$. For the graph C_5 of order 5, observe that $\chi_s(C_5) = 3$ since $\chi(C_5) = 3$. Consider the set 3-coloring c_1 of C_5 given by $c_1(v_i) = 1$ for $1 \leqslant i \leqslant 3$ and $c_1(v_i) = i-2$ for $i = 4, 5$ (see Fig. 2). Furthermore, observe that $\{1\} \subseteq \mathrm{NC}(v) \neq \mathbb{N}_3$ for every vertex v in C_5.

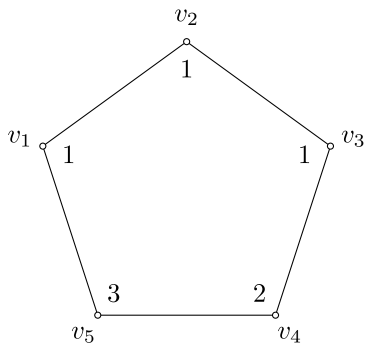

Figure 2. The graph C_5.

Define the 3-coloring c_2 of $C_5 + H$ by $c_2(v) = c_1(v)$ if $v \in V(C_5)$ and $c_2(v) = 1$ if $v \in V(H)$. Then

$$\mathrm{NC}_{c_2}(v) = \begin{cases} \mathrm{NC}_{c_1}(v) & \text{if } v \in V(C_5), \\ \mathbb{N}_3 & \text{if } v \in V(H). \end{cases}$$

Since c_2 is a set 3-coloring of $C_5 + H$, it follows that $\chi_s(C_5 + H) = \chi_s(C_5) = 3$. On the other hand, for the graph $F = C_5 + K_1$, observe that $F + K_1 = C_5 + K_2$. By Proposition 2.3, $\chi_s(F + K_1) > \chi_s(C_5) = \chi_s(F) = 3$. In fact, $\chi_s(F + H) = 4 = \chi_s(F) + \chi_s(H)$.

From the example above, we see that for a graph G, $\chi_s(G + \overline{K}_n) = \chi_s(G) = k$ if and only if there exists a set k-coloring c of G such that $\mathrm{NC}(v) \neq \mathbb{N}_k$ for every vertex v of G. However, it is not clear which graphs G have this property.

From Proposition 2.3, we saw that

$$\chi_s(G + H) > \max\{\chi_s(G), \chi_s(H)\}$$

if both G and H are nonempty. We now present a sharp lower bound for $\chi_s(G+H)$, where G and H are general graphs.

Theorem 2.4. *For every two graphs G and H,*

$$\chi_s(G + H) \geqslant \max\{\chi_s(G) + \lceil \log_2 \omega(H) \rceil, \chi_s(H) + \lceil \log_2 \omega(G) \rceil\}.$$

Proof. Suppose that $\chi_s(G+H) = l$ and let a set l-coloring of $G+H$ using the colors in $\mathbb{N}_l$ be given. It suffices to show that

$$\chi_s(G + H) \geqslant \chi_s(G) + \lceil \log_2 \omega(H) \rceil.$$

Permuting the colors assigned to the vertices of $G + H$, if necessary, we can obtain a set coloring $c\colon V(G + H) \to \mathbb{N}_l$ such that $c(V(G)) = \mathbb{N}_{l'}$ for some positive integer

$l' \leqslant l$. By Lemma 2.1, $l' \geqslant \chi_s(G)$. Therefore, the neighborhood color set of each vertex belonging to H contains $\mathbb{N}_{l'}$ as a subset. Since there are $2^{l-l'}$ subsets of $\mathbb{N}_l$ containing $\mathbb{N}_{l'}$ as a subset, it follows that

$$\omega(H) \leqslant 2^{l-l'}.$$

Hence

$$\lceil \log_2(\omega(H)) \rceil \leqslant l - l' \leqslant \chi_s(G+H) - \chi_s(G),$$

which implies that

$$\chi_s(G) + \lceil \log_2(\omega(H)) \rceil \leqslant \chi_s(G+H),$$

producing the desired result. □

To see that the bound in Theorem 2.4 is sharp, we construct graphs G_k and H_k with $\omega(G_k) = 2^{k-1} = \omega(H_k) + 1$ and $\chi_s(G_k) = \chi_s(H_k) = k$ for each integer $k \geqslant 3$. We start with the complete graph $F = K_{2^{k-1}}$ of order 2^{k-1} with $V(F) = \{v_1, v_2, \ldots, v_{2^{k-1}}\}$. Let $S_1, S_2, \ldots, S_{2^{k-1}}$ be the 2^{k-1} subsets of $\mathbb{N}_{k-1}$, where $|S_1| \leqslant |S_2| \leqslant \ldots \leqslant |S_{2^{k-1}}|$. Hence $S_1 = \emptyset$ and $S_{2^{k-1}} = \mathbb{N}_{k-1}$. For $2 \leqslant i \leqslant 2^{k-1}$, we add $|S_i|$ pendant edges at the vertex v_i, obtaining a graph G_k with $\omega(G_k) = 2^{k-1}$ and $\chi_s(G_k) = k$ by Theorem 1.3. This graph G_k was constructed in [1] to show that the bound given in Theorem 1.3 is sharp. The graph H_k is obtained from G_k by removing the vertex $v_{2^{k-1}}$ and the $k-1$ end-vertices adjacent to $v_{2^{k-1}}$. Observe that $\omega(H_k) = 2^{k-1} - 1$ and $\chi_s(H_k) = k$. The graphs G_4 and H_4 are shown in Fig. 3 together with set 4-colorings.

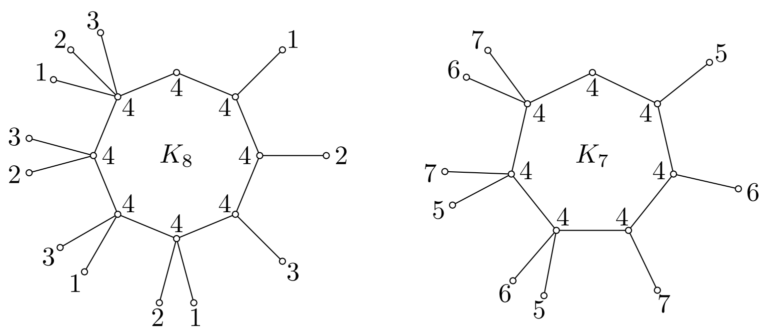

Figure 3. The graphs G_4 and H_4.

For two integers $k_1, k_2 \geqslant 3$, Theorem 2.4 implies that $\chi_s(G_{k_1} + H_{k_2}) \geqslant k_1 + k_2 - 1$. On the other hand, we obtain a set k_1-coloring of G_{k_1} using the colors $1, 2, \ldots, k_1$ such that the vertices belonging to $K_{2^{k_1-1}}$ are assigned the color k_1. Similarly, we obtain a set k_2-coloring of H_{k_2} using the colors $k_1, k_1+1, \ldots, k_1+k_2-1$ such that the vertices belonging to $K_{2^{k_2-1}-1}$ are assigned the color k_1. Combining these two colorings, we obtain a set (k_1+k_2-1)-coloring of $G_{k_1}+H_{k_2}$. Hence in this case,

$$\chi_s(G_{k_1}) + \lceil \log_2 \omega(H_{k_2}) \rceil = \chi_s(H_{k_2}) + \lceil \log_2 \omega(G_{k_1}) \rceil = \chi_s(G_{k_1} + H_{k_2}),$$

establishing the sharpness of the lower bound presented in Theorem 2.4.

3. On the set chromatic numbers of $G + K_p$

It is well known that $\chi(G + K_1) = \chi(G) + 1$ for every graph G. However, the analogous statement is not true for the set chromatic numbers since $\chi_s(C_5) = \chi_s(C_5 + K_1) = 3$, for example. On the other hand, if $\chi_s(G+K_1) \neq \chi_s(G)+1$, then only one possibility remains.

Proposition 3.1. *For every graph G,*

$$\chi_s(G) \leqslant \chi_s(G + K_1) \leqslant \chi_s(G) + 1.$$

Proof. Since the inequality $\chi_s(G) \leqslant \chi_s(G + K_1)$ is an immediate consequence of Corollary 2.2, we show that $\chi_s(G + K_1) \leqslant \chi_s(G) + 1$. Suppose that $\chi_s(G) = l$ and let c be a set l-coloring of G. Construct $G + K_1$ by adding a new vertex u to G and joining u to every vertex in G. Since the $(l + 1)$-coloring c' of $G + K_1$ defined by $c'(v) = c(v)$ if $v \in V(G)$ and $c'(u) = l+1$ is a set coloring, $\chi_s(G + K_1) \leqslant l+1 = \chi_s(G) + 1$. □

We now consider the set chromatic number of $G + K_p$ for all positive integers p.

Theorem 2.3. *For a graph G and a positive integer p,*

$$\chi_s(G) + p - 1 \leqslant \chi_s(G + K_p) \leqslant \chi_s(G) + p.$$

Proof. Since the result is true for $p = 1$ (by Proposition 3.1), we may assume that $p \geqslant 2$. Since $G + K_p = (G + K_{p-1}) + K_1$, it follows by repeated application of Proposition 3.1 that $\chi_s(G + K_p) \leqslant \chi_s(G) + p$. It therefore remains only to verify that $\chi_s(G + K_p) \geqslant \chi_s(G) + p - 1$.

Suppose that $\chi_s(G + K_p) = k$ and let c be a set k-coloring of $G + K_p$. Then $c|_{V(G)}$ is a set coloring of G by Lemma 2.1. Hence $|c(V(G))| \geqslant \chi_s(G)$. On the other hand, if x and y are distinct vertices in K_p, then $N(x) - \{y\} = N(y) - \{x\}$. Hence Observation 1.1 implies that each vertex in $V(K_p)$ must be assigned a distinct color, that is, $|c(V(K_p))| = p$. Furthermore, at most one of the p vertices in $V(K_p)$ can be assigned a color in $c(V(G))$. Hence

$$|c(V(G)) \cap c(V(K_p))| \leqslant 1$$

and so

$$\begin{aligned}\chi_s(G + K_p) &= |c(V(G))| + |c(V(K_p))| - |c(V(G)) \cap c(V(K_p))| \\ &\geqslant \chi_s(G) + p - 1,\end{aligned}$$

completing the proof. □

4. An upper bound for $\chi_s(G + H)$

While $\chi(G+H)$ equals $\chi(G)+\chi(H)$ for all graphs G and H, the number $\chi_s(G)+\chi_s(H)$ is not even an upper bound in general for $\chi_s(G + H)$.

Theorem 4.1. *For every two graphs G and H,*

$$\chi_s(G + H) \leqslant \chi_s(G) + \chi_s(H) + 1.$$

Proof. Let $\chi_s(G) = k$ and $\chi_s(H) = l$. Suppose that $c_G\colon V(G) \to \mathbb{N}_k$ and $c_H\colon V(H) \to \mathbb{N}_l$ are set colorings of G and H, respectively.

If $\mathrm{NC}_{c_G}(v) \neq \mathbb{N}_k$ for every vertex v in G, then let c'_H be an l-coloring of H defined by $c'_H(v) = c_H(v) + k$ for every v in H and define a coloring c_1 of $G + H$ by

$$c_1(v) = \begin{cases} c_G(v) & \text{if } v \in V(G), \\ c'_H(v) & \text{if } v \in V(H). \end{cases}$$

Thus c_1 uses $k + l$ colors. We show that c_1 is a set coloring of $G + H$. Let x and y be adjacent vertices in $G + H$. Observe that for every vertex v in G,

$$\mathrm{NC}_{c_G}(v) = \mathrm{NC}_{c_1}(x) - [(k + 1)..(k + l)].$$

If $x, y \in V(G)$, then observe that $\mathrm{NC}_{c_G}(x) \neq \mathrm{NC}_{c_G}(y)$ and so $\mathrm{NC}_{c_1}(x) \neq \mathrm{NC}_{c_1}(y)$. A similar argument applies for the case with $x, y \in V(H)$.

Hence suppose that $x \in V(G)$ and $y \in V(H)$. Since y is adjacent to every vertex in G, it follows that $\mathbb{N}_k \subseteq \mathrm{NC}_{c_1}(y)$. On the other hand, since $\mathrm{NC}_{c_G}(x) \neq \mathbb{N}_k$ by assumption, $\mathbb{N}_k \nsubseteq \mathrm{NC}_{c_1}(x)$ and so $\mathrm{NC}_{c_1}(x) \neq \mathrm{NC}_{c_1}(y)$.

Thus c_1 is a set $(k+l)$-coloring of $G+H$ and so $\chi_s(G+H) \leqslant k+l = \chi_s(G)+\chi_s(H)$. Similarly, if $\mathrm{NC}_{c_H}(v) \neq \mathbb{N}_l$ for every vertex v in H, then $\chi_s(G+H) \leqslant \chi_s(G)+\chi_s(H)$.

Hence assume now that there are vertices $u^* \in V(G)$ and $v^* \in V(H)$ such that $\mathrm{NC}_{c_G}(u^*) = \mathbb{N}_k$ and $\mathrm{NC}_{c_H}(v^*) = \mathbb{N}_l$. Then let c''_H be an $(l+1)$-coloring of H defined by $c''_H(v) = c_H(v)+k$ if $v \in V(H) - \{v^*\}$ and $c''_H(v^*) = k+l+1$. Observe that c''_H is a set $(l+1)$-coloring. Let c_2 be a coloring of $G+H$ given by

$$c_2(v) = \begin{cases} c_G(v) & \text{if } v \in V(G), \\ c''_H(v) & \text{if } v \in V(H). \end{cases}$$

Thus c_2 uses $k+l+1$ colors. We show that c_2 is a set coloring. Let x and y be adjacent vertices in $G+H$.

Observe that if $x, y \in V(G)$ or $x, y \in V(H)$, then an argument similar to that used before implies that $\mathrm{NC}_{c_2}(x) \neq \mathrm{NC}_{c_2}(y)$, since $c_2|_{V(G)} = c_G$ and $c_2|_{V(H)} = c''_H$ are set colorings of G and H, respectively.

We now consider the case where $x \in V(G)$ and $y \in V(H)$. If y is not adjacent to v^*, then notice that $k+l+1 \notin \mathrm{NC}_{c_2}(y)$, while $k+l+1 \in \mathrm{NC}_{c_2}(x)$. Hence $\mathrm{NC}_{c_2}(x) \neq \mathrm{NC}_{c_2}(y)$. If y is adjacent to v^*, then $\mathrm{NC}_{c_H}(y) \neq \mathrm{NC}_{c_H}(v^*) = \mathbb{N}_l$. Hence there exists an integer $i^* \in \mathbb{N}_l - \mathrm{NC}_{c_H}(y)$, that is, there is a color $i^* \in \mathbb{N}_l$ such that no vertex colored i^* in H by c_H is adjacent to y. Since v^* is adjacent to y, it follows that $c_H(v^*) \neq i^*$ and so every vertex in H that is colored i^* by c_H is now colored i^*+k in $G+H$ by c_2. This implies that $i^*+k \notin \mathrm{NC}_{c_2}(y)$, while $i^*+k \in \mathrm{NC}_{c_2}(x)$. Hence $\mathrm{NC}_{c_2}(x) \neq \mathrm{NC}_{c_2}(y)$.

Therefore, c_2 is a set $(k+l+1)$-coloring of $G+H$ and we obtain $\chi_s(G+H) \leqslant k+l+1 = \chi_s(G)+\chi_s(H)+1$. □

We next show that the upper bound in Theorem 4.1 is sharp. We have seen in Theorem 1.3 that $\chi_s(G) \geqslant 1 + \lceil \log_2 \omega(G) \rceil$. Furthermore, for each integer $k \geqslant 2$ there exists a graph G with $\chi_s(G) = k$ and $\omega(G) = 2^{k-1}$, that is,

$$\chi_s(G) = 1 + \log_2 \omega(G).$$

In particular, if $\chi_s(G) \geqslant 3$, then $\chi_s(G) < \omega(G)$. The following lemma will be useful to us.

Lemma 4.2. *Let $k \geqslant 3$ be an integer and suppose that G is a graph with $\chi_s(G) = k$ and $\omega(G) = 2^{k-1}$. Then for every set k-coloring of G, each clique in G of order 2^{k-1} is monochromatic.*

Proof. Let $\omega = \omega(G)$ and suppose that H is a clique in G of order ω with $V(H) = \{v_1, v_2, \ldots, v_\omega\}$. Let c be a set k-coloring of G. Since $k < \omega$, some vertices in $V(H)$ are assigned the same color. Without loss of generality, let $c(v_1) = c(v_2) = 1$. We show that H is monochromatic, for otherwise, say, $c(v_\omega) = 2$. Then $\{1, 2\} \subseteq \mathrm{NC}(v_i)$ for $1 \leqslant i \leqslant \omega - 1$. Since there are 2^{k-2} subsets of $\mathbb{N}_k$ containing 1 and 2, it follows that $\omega - 1 \leqslant 2^{k-2}$. However, this implies that

$$2^{k-1} = \omega \leqslant 2^{k-2} + 1,$$

which occurs only when $k \leqslant 2$, a contradiction. □

Theorem 4.3. *For each integer $k \geqslant 3$, there is a connected graph G such that*

$$\chi_s(G) = k \quad \text{and} \quad \chi_s(G + G) = 2k + 1.$$

P r o o f. Let $k \geqslant 3$ be an integer. We now construct a connected graph G as follows. Let $S_1, S_2, \ldots, S_{2^{k-1}}$ be the 2^{k-1} subsets of $\mathbb{N}_{k-1}$, where $|S_1| \leqslant |S_2| \leqslant \ldots \leqslant |S_{2^{k-1}}|$. Hence $S_1 = \emptyset$ and $S_{2^{k-1}} = \mathbb{N}_{k-1}$. Then the graph F_1 is obtained from $K_{2^{k-1}}$ with $V(K_{2^{k-1}}) = \{v_1, v_2, \ldots, v_{2^{k-1}}\}$ by adding $|S_i|$ new vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,|S_i|}$ and joining them to v_i for each i ($2 \leqslant i \leqslant 2^{k-1}$). Hence F_1 is a connected graph of order

$$2^{k-1} + \sum_{i=1}^{k-1} i \cdot \binom{k-1}{i}$$

and we observe that $F_1 \cong G_k$, where G_k is the graph with $\omega(G_k) = 2^{k-1}$ and $\chi_s(G_k) = k$ mentioned after Theorem 2.4. Let F_2 be a vertex-disjoint copy of F_1 with the vertices $v_{2^{k-1}+1}, v_{2^{k-1}+2}, \ldots, v_{2^k}$ forming $K_{2^{k-1}}$ and $w_{i,1}, w_{i,2}, \ldots, w_{i,|S_i|}$ being the end-vertices adjacent to the vertex $v_{2^{k-1}+i}$ for $2 \leqslant i \leqslant 2^{k-1}$. Then the graph G is obtained from F_1 and F_2 by (i) removing the vertices $u_{2,1}$ and $w_{2^{k-1},k-1}$ and (ii) joining v_2 and v_{2^k}. Fig. 4 shows the graph G for $k = 4$. Hence G is a connected graph of order

$$2^k + 2\Bigg[\sum_{i=1}^{k-1} i \cdot \binom{k-1}{i}\Bigg] - 2$$

and $\omega(G) = 2^{k-1}$.

We first show that $\chi_s(G) = k$. Observe that $\chi_s(G) \geqslant k$ by Theorem 1.3. On the other hand, let $R_1, R_2, \ldots, R_{2^{k-1}}$ and $T_1, T_2, \ldots, T_{2^{k-1}}$ be the 2^{k-1} subsets of $\mathbb{N}_k$ containing 1 and 2, respectively, where $|R_1| \leqslant |R_2| \leqslant \ldots \leqslant |R_{2^{k-1}}|$ and $|T_1| \leqslant |T_2| \leqslant \ldots \leqslant |T_{2^{k-1}}|$. Hence $R_1 = \{1\}$, $T_1 = \{2\}$, and $R_{2^{k-1}} = T_{2^{k-1}} = \mathbb{N}_k$. Then the coloring $c^*\colon V(G) \to \mathbb{N}_k$ of G such that

$$c^*(v_i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant 2^{k-1}, \\ 2 & \text{if } 2^{k-1} + 1 \leqslant i \leqslant 2^k \end{cases}$$

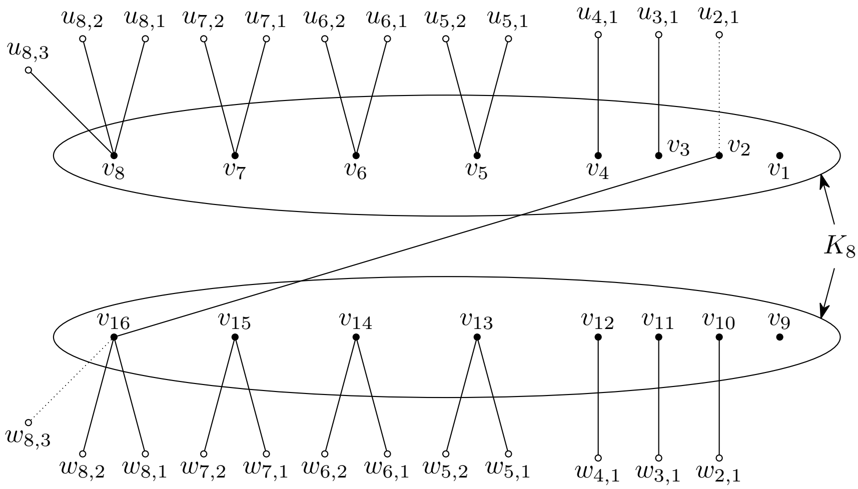

Figure 4. The graph G in the proof of Theorem 4.3 for $k = 4$.

and that the end-vertices are assigned colors such that

$$\mathrm{NC}(v_i) = R_i \quad \text{and} \quad \mathrm{NC}(v_{2^{k-1}+i}) = T_i$$

for $1 \leqslant i \leqslant 2^{k-1}$ is a set k-coloring. Therefore, $\chi_s(G) = k$.

We now show that c^* is a unique set k-coloring of G (up to the permutation of colors). Suppose that c is an arbitrary set k-coloring of G, say $c\colon V(G) \to A$, where $A = \{a_1, a_2, \ldots, a_k\}$. By Lemma 4.2, we may assume that $c(v_i) = a_1$ for $1 \leqslant i \leqslant 2^{k-1}$. Then $\mathrm{NC}(v_i) = A_i$ for $1 \leqslant i \leqslant 2^{k-1}$, where $A_1, A_2, \ldots, A_{2^{k-1}}$ are the 2^{k-1} subsets of A containing a_1 and $|A_1| \leqslant |A_2| \leqslant \ldots \leqslant |A_{2^{k-1}}|$. Hence $\mathrm{NC}(v_1) = A_1 = \{a_1\}$, $\mathrm{NC}(v_{2^{k-1}}) = A_{2^{k-1}} = A$, and without loss of generality we may assume that $\mathrm{NC}(v_2) = A_2 = \{a_1, a_2\}$. Hence $c(v_{2^k}) = a_2$. Since v_{2^k} belongs to a clique of order $2^{k-1} = \omega(G)$, it follows again by Lemma 4.2 that $c(v_{2^{k-1}+i}) = a_2$ for $1 \leqslant i \leqslant 2^{k-1}$, and furthermore, $\mathrm{NC}(v_{2^{k-1}+i}) = B_i$ for $1 \leqslant i \leqslant 2^{k-1}$, where $B_1, B_2, \ldots, B_{2^{k-1}}$ are the 2^{k-1} subsets of A containing a_2 and $|B_1| \leqslant |B_2| \leqslant \ldots \leqslant |B_{2^{k-1}}|$. However, this implies that c is the coloring c^* discussed before with the colors renamed (and possibly some v_i's relabeled). In particular, observe that there are two vertices (namely $v_{2^{k-1}}$ and v_{2^k}) whose neighborhood color set must be A.

We next consider set colorings of $G + G$. In particular, we will show that $\chi_s(G + G) = 2k + 1$. Let G and G' be the two copies of G in $G + G$. Note that $\chi_s(G + G) \leqslant \chi_s(G) + \chi_s(G') + 1 = 2k + 1$ by Theorem 4.1. To show that $\chi_s(G + G') \geqslant 2k + 1$, assume, to the contrary, that $\chi_s(G + G') = l \leqslant 2k$ and let $c\colon V(G + G') \to \mathbb{N}_l$ be a set l-coloring of $G + G'$. Let $\mathcal{C} = c(V(G))$ and $\mathcal{C}' = c(V(G'))$ and without loss of generality, assume that $|\mathcal{C}| \leqslant |\mathcal{C}'|$. By Lemma 2.1, observe that $c|_{V(G)}$ and $c|_{V(G')}$ are set colorings of G and G', respectively. Since $\chi_s(G) = \chi_s(G') = k$, it then follows that $k \leqslant |\mathcal{C}| \leqslant |\mathcal{C}'| \leqslant l \leqslant 2k$. We now consider three cases.

Case 1: $|\mathcal{C}'| \geqslant k+2$, say $\mathbb{N}_{k+2} \subseteq \mathcal{C}'$. Then the neighborhood color set of each vertex in G contains $\mathbb{N}_{k+2}$ as a subset. Since there are $2^{l-(k+2)}$ subsets of $\mathbb{N}_l$ containing $\mathbb{N}_{k+2}$ as a subset and G contains 2^{k-1} vertices that are mutually adjacent, it follows that $2^{l-(k+2)} \geqslant 2^{k-1}$. Thus $l \geqslant 2k+1$, which is a contradiction.

Case 2: $|\mathcal{C}| = k$, say $\mathcal{C} = \mathbb{N}_k$. Then $c|_{V(G)}$ is a set k-coloring and we may assume, without loss of generality, that c is defined so that $c|_{V(G)} = c^*$, where c^* is the set k-coloring of G discussed earlier. Let a be an arbitrary color in $\mathbb{N}_k$ and observe that there exist adjacent vertices x and y in G such that either $\mathrm{NC}_{c^*}(x) - \mathrm{NC}_{c^*}(y) = \{a\}$ or $\mathrm{NC}_{c^*}(y) - \mathrm{NC}_{c^*}(x) = \{a\}$. Then $a \notin \mathcal{C}'$, since otherwise $\mathrm{NC}_c(x) = \mathrm{NC}_c(y)$, contradicting the fact that c is a set coloring. Therefore, $\mathcal{C} \cap \mathcal{C}' = \emptyset$ and so $l = 2k$ and $\mathcal{C}' = [(k+1)..2k]$. Furthermore, by an earlier observation, there exists a vertex z in G such that $\mathrm{NC}_{c^*}(z) = \mathbb{N}_k$. Similarly, since $c'^* = c|_{V(G')}$ is a set k-coloring of G', it follows that there exists a vertex z' in G' such that $\mathrm{NC}_{c'^*}(z') = [(k+1)..2k]$. However, this implies that $\mathrm{NC}_c(z) = \mathrm{NC}_c(z') = \mathbb{N}_{2k}$, which is impossible since z and z' are adjacent in $G + G'$.

Case 3: $|\mathcal{C}| = |\mathcal{C}'| = k+1$, say $\mathcal{C} = \mathbb{N}_{k+1}$. Then the neighborhood color set of every vertex v in G' contains $\mathbb{N}_{k+1}$ as a subset. Since there are $2^{l-(k+1)}$ subsets of $\mathbb{N}_l$ containing $\mathbb{N}_{k+1}$ as a subset and G' contains 2^{k-1} vertices that are mutually adjacent, say the vertices $z'_1, z'_2, \ldots, z'_{2^{k-1}}$ form $K_{2^{k-1}}$ in G', it follows that $2^{l-(k+1)} \geqslant 2^{k-1}$, that is, $l = 2k$. Thus we may assume that $\mathcal{C}' = [k..2k]$. Furthermore, observe that the neighborhood color set of one of the 2^{k-1} vertices is $\mathbb{N}_l = \mathbb{N}_{2k}$, say $\mathrm{NC}_c(z'_1) = \mathbb{N}_{2k}$.

Now, since there are 2^{k-1} subsets of $\mathbb{N}_{2k}$ containing $[k..2k]$ as a subset and G contains 2^{k-1} vertices that are mutually adjacent, say the vertices $z_1, z_2, \ldots, z_{2^{k-1}}$ form $K_{2^{k-1}}$ in G, we may apply an argument similar to that used above to show that the neighborhood color set of one of the 2^{k-1} vertices is $\mathbb{N}_{2k}$, say $\mathrm{NC}_c(z_1) = \mathbb{N}_{2k}$. However, this is impossible since z_1 and z'_1 are adjacent in $G + G'$ and c is a set coloring.

Hence none of the three cases occurs. We now conclude that $\chi_s(G + G) = 2k+1$. □

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