Czechoslovak Mathematical Journal

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*Czechoslovak Mathematical Journal*, Vol. 59 (2009), No. 4, 909–927

Persistent URL: `http://dml.cz/dmlcz/140525`

**Czechoslovak Mathematical Journal, 59 (134) (2009), 909–927**

# HOLOMORPHY TYPES AND SPACES OF ENTIRE FUNCTIONS OF BOUNDED TYPE ON BANACH SPACES

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(Received April 1, 2008)

*Abstract.* In this paper spaces of entire functions of $\Theta$-holomorphy type of bounded type are introduced and results involving these spaces are proved. In particular, we "construct an algorithm" to obtain a duality result via the Borel transform and to prove existence and approximation results for convolution equations. The results we prove generalize previous results of this type due to B. Malgrange: Existence et approximation des équations aux dérivées partielles et des équations des convolutions. Annales de l'Institute Fourier (Grenoble) VI, 1955/56, 271–355; C. Gupta: Convolution Operators and Holomorphic Mappings on a Banach Space, Séminaire d'Analyse Moderne, 2, Université de Sherbrooke, Sherbrooke, 1969; M. Matos: Absolutely Summing Mappings, Nuclear Mappings and Convolution Equations, IMECC-UNICAMP, 2007; and X. Mujica: Aplicações $\tau(p;q)$-somantes e $\sigma(p)$-nucleares, Thesis, Universidade Estadual de Campinas, 2006.

*Keywords*: Banach spaces, holomorphy types, homogeneous polynomials, holomorphic functions, convolution operators, Borel transform, approximation and existence theorems

*MSC 2010*: 46G20, 46G25

## 1. Introduction

The starting point of this paper is the set of existence and approximation results for convolution equations due to B. Malgrange [8] and their generalizations to spaces of nuclear entire functions of bounded type due to C. Gupta [6].

In this context, special classes of homogeneous polynomials between Banach spaces play a crucial role. Historically, there are two abstract approaches to deal with special classes of polynomials, namely, holomorphy types and modules of homogeneous polynomials, which go back to L. Nachbin [14] and A. Pietsch [15], respectively. Us-

The first author is supported by Fapesp, project 07/50811-4. The second author is supported by CNPq.

ing modules of homogeneous polynomials, M. Matos [12] generalized the aforementioned results to modules of $(s;(r,q))$-quasi-nuclear polynomials and X. Mujica [13] proved results about the Borel transform and convolution operators for modules of $\sigma(p)$-nuclear polynomials.

In this paper we show how all these cases can be generalized using holomorphy types instead of modules of homogeneous polynomials. As usual, the Borel transform is used to obtain duality results. Furthermore, convolution operators on spaces of functions of $\Theta$-holomorphy type of bounded type are characterized and existence and approximation results for convolution equations are proved. The main point is to consider holomorphy types enjoying some special properties. In the fashion of Dineen [2], we identify the properties a holomorphy type must enjoy for the results to hold true (see Definitions 2.3 and 3.2). The construction of the spaces of functions of $\Theta$-holomorphy type of bounded type and the duality results using the Borel transform were also inspired by Dineen [2].

## 2. Holomorphy types

In this work, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ denotes the set $\mathbb{N}\cup\{0\}$. The letters $E$ and $F$ will always denote complex Banach spaces and $E'$ represents the topological dual of $E$. The space of all continuous $m$-homogeneous polynomials from $E$ into $F$ is denoted by $\mathcal{P}(^mE;F)$ and the space of all entire mappings from $E$ into $F$ is denoted by $\mathcal{H}(E;F)$. When $F=\mathbb{C}$ we write $\mathcal{P}(^mE)$ and $\mathcal{H}(E)$ instead of $\mathcal{P}(^mE;\mathbb{C})$ and $\mathcal{H}(E;\mathbb{C})$, respectively.

**Definition 2.1.** A *holomorphy type* $\Theta$ from $E$ to $F$ is a sequence of Banach spaces $(\mathcal{P}_\Theta(^mE;F))_{m=0}^\infty$, the norm on each of them being denoted by $\|\cdot\|_\Theta$, such that the following conditions hold true:

(1) Each $\mathcal{P}_\Theta(^mE;F)$ is a vector subspace of $\mathcal{P}(^mE;F)$.

(2) $\mathcal{P}_\Theta(^0E;F)$ coincides with $\mathcal{P}(^0E;F)=F$ as a normed vector space.

(3) There is a real number $\sigma\geqslant 1$ for which the following is true: given any $k\in\mathbb{N}_0$, $m\in\mathbb{N}_0$, $k\leqslant m$, $a\in E$, and $P\in\mathcal{P}_\Theta(^mE;F)$, we have

$$\hat{d}^kP(a)\in\mathcal{P}_\Theta(^kE;F),$$
$$\Bigl\|\frac{1}{k!}\hat{d}^kP(a)\Bigr\|_\Theta\leqslant\sigma^m\|P\|_\Theta\|a\|^{m-k}.$$

It is obvious that each inclusion $\mathcal{P}_\Theta(^mE;F)\subset\mathcal{P}(^mE;F)$ is continuous and $\|P\|\leqslant\sigma^m\|P\|_\Theta$.

**Definition 2.2.** Let $(\mathcal{P}_\Theta(^mE;F))_{m=0}^\infty$ be a holomorphy type from $E$ to $F$. A mapping $f\in\mathcal{H}(E;F)$ is said to be of $\Theta$-*holomorphy type of bounded type* if

(i) $(m!)^{-1}\hat{d}^m f(0) \in \mathcal{P}_\Theta({}^m E; F)$ for all $m \in \mathbb{N}_0$,
(ii) $\lim\limits_{m\to\infty} ((m!)^{-1}\|\hat{d}^m f(0)\|_\Theta)^{1/m} = 0$.

The vector subspace of $\mathcal{H}(E;F)$ of all such $f$ of $\Theta$-holomorphy type of bounded type is denoted by $\mathcal{H}_{\Theta b}(E;F)$.

**Remark 2.1.** The inequality $\|\cdot\| \leqslant \sigma^m \|\cdot\|_\Theta$ implies that each entire mapping $f$ of $\Theta$-holomorphy type of bounded type is an entire mapping of bounded type in the sense introduced by Gupta in [6], that is, $f$ is bounded on bounded subsets of $E$.

It is easy to see that the following result holds.

**Proposition 2.1.** *The space $\mathcal{P}_\Theta({}^m E; F)$ is contained in $\mathcal{H}_{\Theta b}(E;F)$ for each $m \in \mathbb{N}_0$.*

**Proposition 2.2.** *Let $f \in \mathcal{H}(E;F)$. Then $f \in \mathcal{H}_{\Theta b}(E;F)$ if and only if*
(i) *$(m!)^{-1}\hat{d}^m f(a) \in \mathcal{P}_\Theta({}^m E; F)$, for all $a \in E$ and $m \in \mathbb{N}_0$,*
(ii) *$\lim\limits_{m\to\infty} (\|(m!)^{-1}\hat{d}^m f(a)\|_\Theta)^{1/m} = 0$ for all $a \in E$.*

Proof. Given $f \in \mathcal{H}_{\Theta b}(E;F)$ and $a \in E$, let $\varepsilon > 0$ be such that $\varepsilon\|a\| < 1$. Considering $\sigma \geqslant 1$ as in condition (3) of Definition 2.1, there is $C(\varepsilon) > 0$ such that

$$\frac{1}{m!}\|\hat{d}^m f(0)\|_\Theta \leqslant C(\varepsilon)\Big(\frac{\varepsilon}{\sigma}\Big)^m$$

for all $m \in \mathbb{N}_0$. Since

$$\frac{1}{m!}\hat{d}^m f(a) = \sum_{k=0}^\infty \frac{1}{m!}\hat{d}^m\Big(\frac{1}{(m+k)!}\hat{d}^{m+k} f(0)\Big)(a)$$

and $\hat{d}^{m+k} f(0) \in \mathcal{P}_\Theta({}^{m+k} E; F)$ for all $m, k \in \mathbb{N}_0$, it follows by condition (3) of Definition 2.1 that

$$\hat{d}^m\Big(\frac{1}{(m+k)!}\hat{d}^{m+k} f(0)\Big)(a) \in \mathcal{P}_\Theta({}^m E; F)$$

and

$$\Big\|\frac{1}{m!}\hat{d}^m\Big(\frac{1}{(m+k)!}\hat{d}^{m+k} f(0)\Big)(a)\Big\|_\Theta \leqslant \sigma^{m+k}\Big\|\frac{1}{(m+k)!}\hat{d}^{m+k} f(0)\Big\|_\Theta \|a\|^k.$$

Furthermore,

$$\begin{aligned}\frac{1}{m!}\|\hat{d}^m f(a)\|_\Theta &\leqslant \sum_{k=0}^\infty \sigma^{m+k}\|\frac{1}{(m+k)!}\hat{d}^{m+k} f(0)\|_\Theta \|a\|^k \\ &\leqslant C(\varepsilon)\sum_{k=0}^\infty \sigma^{m+k}\Big(\frac{\varepsilon}{\sigma}\Big)^{m+k}\|a\|^k = C(\varepsilon)\frac{\varepsilon^m}{1-\varepsilon\|a\|},\end{aligned}$$

since $\varepsilon\|a\| < 1$. Therefore

$$\lim_{m\to\infty}\Big(\frac{1}{m!}\|\hat{d}^m f(a)\|_\Theta\Big)^{1/m} = 0.$$

$\square$

For each $\varrho > 0$ we define a natural seminorm on $\mathcal{H}_{\Theta b}(E;F)$ by

$$f \in \mathcal{H}_{\Theta b}(E;F) \to \|f\|_{\Theta,\varrho} = \sum_{m=0}^{\infty}\frac{\varrho^m}{m!}\|\hat{d}^m f(0)\|_\Theta < \infty.$$

Condition (ii) of Definition 2.2 implies that $\|\cdot\|_{\Theta,\varrho}$ is a well defined seminorm. Now we are considering on $\mathcal{H}_{\Theta b}(E;F)$ the topology generated by the seminorms $\|\cdot\|_{\Theta,\varrho}$, $\varrho > 0$. This topology is denoted by $\tau_\Theta$ and it is Hausdorff.

**Proposition 2.3.** *$[\mathcal{H}_{\Theta b}(E;F), \tau_\Theta]$ is a Fréchet space.*

Proof. Since the sequence $(\|\cdot\|_{\Theta,n})_{n\in\mathbb{N}}$ generates the topology of $\mathcal{H}_{\Theta b}(E;F)$, it follows that this topological vector space is a metrizable locally convex space. We consider a Cauchy sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{H}_{\Theta b}(E;F)$. This implies that $(\hat{d}^m f_k(0))_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{P}_\Theta(^mE;F)$ for all $m \in \mathbb{N}_0$. Hence there is $P_m \in \mathcal{P}_\Theta(^mE;F)$, $m \in \mathbb{N}_0$, such that

$$\lim_{k\to\infty}\frac{1}{m!}\hat{d}^m f_k(0) = P_m.$$

For every $\varrho > 0$ there is $0 \leqslant M_\varrho < \infty$ such that $\|f_k\|_{\Theta,\varrho} \leqslant M_\varrho$ for all $k \in \mathbb{N}$. It follows that

$$\Big\|\frac{1}{m!}\hat{d}^m f_k(0)\Big\|_\Theta \leqslant \frac{M_\varrho}{\varrho^m}$$

for all $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Hence we have $\|P_m\|_\Theta \leqslant M_\varrho/\varrho^m$ for all $m \in \mathbb{N}_0$, and we can write

$$\lim_{m\to\infty}\|P_m\|_\Theta^{1/m} \leqslant \frac{1}{\varrho}$$

for all $\varrho > 0$. This implies that

$$\lim_{m\to\infty}\|P_m\|_\Theta^{1/m} = 0$$

and

$$f(x) := \sum_{m=0}^{\infty} P_m(x)$$

belongs to $\mathcal{H}_{\Theta b}(E;F)$. For every $\varepsilon > 0$ and $\varrho > 0$ we have $k(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{m=0}^{\infty}\varrho^m\Big\|\frac{1}{m!}\hat{d}^m f_k(0) - \frac{1}{m!}\hat{d}^m f_j(0)\Big\|_\Theta < \varepsilon$$

for all $k, j \geqslant k(\varepsilon)$. Now we pass to the limit for $j$ tending to infinity and have

$$\sum_{m=0}^{\infty} \varrho^m \Big\| \frac{1}{m!} \hat{d}^m f_k(0) - P_m \Big\|_\Theta \leqslant \varepsilon$$

for all $k \geqslant k(\varepsilon)$. Thus $(f_k)_{k=1}^{\infty}$ converges to $f$ in the topology of $\mathcal{H}_{\Theta b}(E; F)$. $\square$

If $f$ belongs to $\mathcal{H}_{\Theta b}(E; F)$ we may consider *Taylor's polynomial of $f$ at 0 with degree $n$*, given by

$$\tau_{n,f,0}(x) = \sum_{k=0}^{n} \frac{1}{k!} \hat{d}^k f(0)(x)$$

for all $x \in E$. Since, for each $\varrho > 0$, we have

$$\|f - \tau_{n,f,0}\|_{\Theta,\varrho} = \sum_{k=n+1}^{\infty} \frac{\varrho^k}{k!} \|\hat{d}^k f(0)\|_\Theta,$$

hence $(\tau_{n,f,0})_{n=1}^{\infty}$ converges to $f$ in the topology of $\mathcal{H}_{\Theta b}(E; F)$.

**Definition 2.3.** Let $(\mathcal{P}_\Theta(^m E; F))_{m=0}^{\infty}$ be a holomorphy type from $E$ to $F$. The holomorphy type $\Theta$ is said to be a *$\pi_1$-holomorphy type* if the following conditions hold:

(1) $\|\varphi^m \otimes b\|_\Theta = \|\varphi\|^m \|b\|$ for all $\varphi \in E'$, $b \in F$ and $m \in \mathbb{N}_0$;

(2) for each $m \in \mathbb{N}_0$, $\mathcal{P}_f(^m E; F)$ is dense in $(\mathcal{P}_\Theta(^m E; F), \|\cdot\|_\Theta)$, where $\mathcal{P}_f(^m E; F)$ denotes the space of all $m$-homogeneous polynomials of finite type.

**Example 2.1.** (a) C. Gupta in [6], M. Matos in [12] and X. Mujica in [13] proved that sequences of spaces of nuclear polynomials, $(s; (r, q))$-quasi-nuclear polynomials and $\sigma(p)$-nuclear polynomials, from $E$ to $F$, satisfy conditions (1) and (2) of Definition 2.3. Hence, each of these sequences is a $\pi_1$-holomorphy type from $E$ to $F$.

(b) More generally, let $(\mathcal{P}_\Theta(^m E; F))_{m=0}^{\infty}$ be a holomorphy type from $E$ to $F$ satisfying (1) of Definition 2.3 and such that $\mathcal{P}_f(^m E; F)$ is contained in $\mathcal{P}_\Theta(^m E; F)$ for each $m$ in $\mathbb{N}$. If we denote the closure of $\mathcal{P}_f(^m E; F)$ for the topology of $\mathcal{P}_\Theta(^m E; F)$ by $\overline{\mathcal{P}_f(^m E; F)}^{\Theta}$, then the sequence $\big(\overline{\mathcal{P}_f(^m E; F)}^{\Theta}\big)_{m=0}^{\infty}$ is a $\pi_1$-holomorphy type.

**Proposition 2.4.** *For each $\pi_1$-holomorphy type from $E$ to $F$, $\mathcal{P}_N(^m E; F)$ is contained in $\mathcal{P}_\Theta(^m E; F)$ and the inclusion mapping is continuous.*

Proof. If $P \in \mathcal{P}_N(^m E; F)$ then $P = \sum\limits_{i=1}^{\infty} {x'_i}^m \otimes y_i$, where $(x'_i)_{i=1}^{\infty} \subset E'$, $(y_i)_{i=1}^{\infty} \subset F$ and $\sum\limits_{i=1}^{\infty} \|x'_i\|^m \|y_i\| < \infty$. Let $P_n = \sum\limits_{i=1}^{n} {x'_i}^m \otimes y_i$, then for $n > j$ we have

$$\|P_n - P_j\|_\Theta \leqslant \sum_{i=j+1}^{n} \|{x'_i}^m \otimes y_i\|_\Theta = \sum_{i=j+1}^{n} \|x'_i\|^m \|y_i\|.$$

Hence $(P_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{P}_{\Theta}(^mE;F)$ and converges to $Q \in \mathcal{P}_{\Theta}(^mE;F)$. Since

$$\|P_n - Q\| \leqslant \sigma^m \|P_n - Q\|_{\Theta},$$

we have that $P = Q \in \mathcal{P}_{\Theta}(^mE;F)$. It is obvious that the inclusion is continuous. □

The next result is a trivial consequence of the previous proposition.

**Corollary 2.1.** *For each $\pi_1$-holomorphy type from $E$ to $F$, $\mathcal{H}_{Nb}(E;F)$ is contained in $\mathcal{H}_{\Theta b}(E;F)$ and the inclusion mapping is continuous.*

**Proposition 2.5.** *If $\Theta$ is a $\pi_1$-holomorphy type from $E$ to $F$, then the vector subspace $S$ of $\mathcal{H}_{\Theta b}(E;F)$ generated by $\{e^{\varphi}b\colon\ \varphi \in E',\, b \in F\}$ is dense in $\mathcal{H}_{\Theta b}(E;F)$.*

Proof. It is clear that $e^{\varphi}b \in \mathcal{H}_{\Theta b}(E;F)$ for all $\varphi \in E'$ and $b \in F$. Since $(\tau_{m,f,0})_{m=1}^{\infty}$ converges to $f$ in the topology of $\mathcal{H}_{\Theta b}(E;F)$, it is enough to prove that $\mathcal{P}_{\Theta}(^mE;F) \subseteq \overline{S}$ for all $m \in \mathbb{N}$, where $\overline{S}$ is the closure of $S$ in the topology of $\mathcal{H}_{\Theta b}(E;F)$. We show this by induction on $m$. For $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $\varphi \in E'$ and $b \in F$, we have

$$e^{\lambda\varphi}b = \sum_{j=0}^{\infty} \frac{\lambda^j \varphi^j b}{j!}$$

converging in $\mathcal{H}_{\Theta b}(E;F)$ and

$$\lim_{|\lambda|\to 0} \Big\| \frac{e^{\lambda\varphi}b - b}{\lambda} - \varphi b \Big\|_{\Theta,\varrho} = \lim_{|\lambda|\to 0} |\lambda| \Big\| \sum_{j=2}^{\infty} \frac{1}{j!} \lambda^{j-2} \varphi^j b \Big\|_{\Theta,\varrho} = 0$$

for every $\varrho > 0$, since

$$\begin{aligned}\Big\| \sum_{j=2}^{\infty} \frac{1}{j!} \lambda^{j-2} \varphi^j b \Big\|_{\Theta,\varrho} &= \sum_{j=2}^{\infty} \varrho^j |\lambda|^{j-2} \Big\| \frac{\varphi^j b}{j!} \Big\|_{\Theta} \\ &\leqslant \sum_{j=2}^{\infty} \varrho^j \Big\| \frac{\varphi^j b}{j!} \Big\|_{\Theta} = \Big\| \sum_{j=2}^{\infty} \frac{\varphi^j b}{j!} \Big\|_{\Theta,\varrho} < \infty\end{aligned}$$

for $|\lambda| \leqslant 1$. Hence $\varphi b \in \overline{S}$. Now, if we suppose that $\varphi^j b \in \overline{S}$ for $j = 1, \ldots, m-1$, $\varphi \in E'$ and $b \in F$, then we have

$$\begin{aligned}&\lim_{|\lambda|\to 0} \Big\| \frac{1}{\lambda^m} \Big( e^{\lambda\varphi}b - \sum_{j=0}^{m-1} \frac{1}{j!} \lambda^j \varphi^j b \Big) - \frac{\varphi^m b}{m!} \Big\|_{\Theta,\varrho} \\ &\quad = \lim_{|\lambda|\to 0} |\lambda| \Big\| \sum_{j=m+1}^{\infty} \frac{1}{j!} \lambda^{j-m} \varphi^j b \Big\|_{\Theta,\varrho} = 0.\end{aligned}$$

Hence $\varphi^m b \in \overline{S}$ and, consequently, $\mathcal{P}_f(^mE;F) \subseteq \overline{S}$. Since the closure of $\mathcal{P}_f(^mE;F)$ in $\mathcal{H}_{\Theta b}(E;F)$ is $\mathcal{P}_{\Theta}(^mE;F)$, we have proved that $\mathcal{P}_{\Theta}(^mE;F) \subseteq \overline{S}$ for all $m \in \mathbb{N}$. □

Now if we suppose that $\Theta$ is a $\pi_1$-holomorphy type from $E$ to $F$, then we can define the *Borel transform*

$$\mathcal{B}_\Theta\colon\ [\mathcal{P}_\Theta(^mE;F)]' \to \mathcal{P}(^mE';F')$$

given by $\mathcal{B}_\Theta T(\varphi)(y) = T(\varphi^m y)$ for all $T \in [\mathcal{P}_\Theta(^mE;F)]'$, $\varphi \in E'$ and $y \in F$. It is clear that $\mathcal{B}_\Theta$ is well-defined and linear. By (1) of Definition 2.3, we have that $\mathcal{B}_\Theta$ is continuous and $\|\mathcal{B}_\Theta T\| \leqslant \|T\|$ and using (2) of Definition 2.3, we obtain the injectivity of $\mathcal{B}_\Theta$.

We denote by $\mathcal{P}_{\Theta'}(^mE';F')$ the range of $\mathcal{B}_\Theta$ in $\mathcal{P}(^mE';F')$ and define the norm in $\mathcal{P}_{\Theta'}(^mE';F')$ by $\|\mathcal{B}_\Theta T\|_{\Theta'} = \|T\|$. Thus $([\mathcal{P}_\Theta(^mE;F)]', \|\cdot\|)$ is isomorphic isometrically to $(\mathcal{P}_{\Theta'}(^mE';F'), \|\cdot\|_{\Theta'})$.

Now we have an interesting result involving the Borel transform.

**Proposition 2.6.** *Let $(\mathcal{P}_\Theta(^mE;F))_{m=0}^\infty$ be a $\pi_1$-holomorphy type from $E$ to $F$. If the Borel transform*

$$\mathcal{B}_\Theta\colon\ ([\mathcal{P}_\Theta(^mE;F)]', \|\cdot\|) \to (\mathcal{P}(^mE';F'), \|\cdot\|)$$

*is a topological isomorphism onto its range, then $\mathcal{P}_N(^mE;F) = \mathcal{P}_\Theta(^mE;F)$ as sets, and the identity mapping $\mathcal{P}_N(^mE;F) \to \mathcal{P}_\Theta(^mE;F)$ is a topological isomorphism. Here we are considering the usual norm on $\mathcal{P}(^mE';F')$.*

P r o o f. We use the canonical notation of (symmetric) projective tensor products used by K. Floret in [5]. Let $\Phi_\Theta = i_\Theta \circ \Phi_N$, where $i_\Theta$ denotes the inclusion $\mathcal{P}_N(^mE;F) \hookrightarrow \mathcal{P}_\Theta(^mE;F)$ and $\Phi_N\colon \tilde{\otimes}_{\pi_s}^{m,s} E' \tilde{\otimes}_\pi F \longrightarrow \mathcal{P}_N(^mE;F)$ is defined by $\Phi_N((x' \otimes \ldots \otimes x') \otimes y) = x'(\cdot)^m y$. We have that the Borel transform $\mathcal{B}_\Theta$ is the transpose of $\Phi_\Theta$. Since $\mathcal{B}_\Theta$ is a topological isomorphism, it follows by a classical result of Banach [1], page 146, Théorème 2, that $\Phi_\Theta$ is surjective. Hence $i_\Theta$ is surjective and the results follow. $\square$

**Definition 2.4.** Let $\Theta$ be a $\pi_1$-holomorphy type from $E$ to $\mathbb{C}$. If $T \in [\mathcal{H}_{\Theta b}(E)]'$, then the *Borel transform of* $T$, denoted by $\mathcal{B}T$, is the function on $E'$ defined by $\mathcal{B}T(\varphi) = T(e^\varphi) \in \mathbb{C}$.

The function $\mathcal{B}T$ is well-defined since $e^\varphi \in \mathcal{H}_{\Theta b}(E)$ for all $\varphi \in E'$.

**Definition 2.5.** Let $(\mathcal{P}_\Theta(^mE))_{m=0}^\infty$ be a $\pi_1$-holomorphy type from $E$ to $\mathbb{C}$. We say that $f \in \mathcal{H}(E')$ is of $\Theta'$-exponential type if $\hat{d}^m f(0) \in \mathcal{P}_{\Theta'}(^mE')$ for all $m \in \mathbb{N}_0$, and there are $C \geqslant 0$, $c > 0$ such that

$$\|\hat{d}^m f(0)\|_{\Theta'} \leqslant Cc^m$$

for all $m \in \mathbb{N}_0$.

The vector space of all these functions is denoted by $\mathrm{Exp}_{\Theta'}(E')$.

**Theorem 2.1.** *If $\Theta$ is a $\pi_1$-holomorphy type from $E$ to $\mathbb{C}$, then the mapping*

$$\mathcal{B}\colon\ [\mathcal{H}_{\Theta b}(E)]' \to \mathrm{Exp}_{\Theta'}(E'),$$

*given by $\mathcal{B}T(\varphi) = T(e^{\varphi})$ for all $T \in [\mathcal{H}_{\Theta b}(E)]'$ and $\varphi \in E'$, establishes an algebraic isomorphism between these spaces.*

P r o o f. Let $T \in [\mathcal{H}_{\Theta b}(E)]'$ and $\varphi \in E'$, then

$$\mathcal{B}T(\varphi) = T(e^{\varphi}) = T\biggl(\sum_{m=0}^{\infty} \frac{1}{m!}\varphi^m\biggr) = \sum_{m=0}^{\infty} \frac{1}{m!} T(\varphi^m).$$

If we define $T_m = T|_{\mathcal{P}_\Theta({}^mE)} \in [\mathcal{P}_\Theta({}^mE)]'$, then there is $P'_m \in \mathcal{P}_{\Theta'}({}^mE')$ such that $T_m(\varphi^m) = P'_m(\varphi)$ with $\|T_m\| = \|P'_m\|_{\Theta'}$ for all $m \in \mathbb{N}_0$. If $f \in \mathcal{H}_{\Theta b}(E)$, then there are $C \geqslant 0$ and $\varrho > 0$ such that

$$|T(f)| \leqslant C\|f\|_{\Theta,\varrho}.$$

In particular, for each $Q_m \in \mathcal{P}_\Theta({}^mE)$ we have

$$|T_m(Q_m)| \leqslant C\varrho^m \|Q_m\|_\Theta$$

and

$$\|P'_m\|_{\Theta'} = \|T_m\| = \sup_{\|Q_m\|_\Theta \leqslant 1} |T_m(Q_m)| \leqslant C\varrho^m$$

for all $m \in \mathbb{N}_0$. Since

$$\mathcal{B}T(\varphi) = \sum_{m=0}^{\infty} \frac{1}{m!} T(\varphi^m) = \sum_{m=0}^{\infty} \frac{1}{m!} P'_m(\varphi),$$

it follows by Definition 2.5 that $\mathcal{B}T \in \mathrm{Exp}_{\Theta'}(E')$.

The linearity is clear and the injectivity is a consequence of Proposition 2.5. Thus we only have to prove that $\mathcal{B}$ is surjective.

Let $H \in \mathrm{Exp}_{\Theta'}(E')$, then by Definition 2.5

$$H(\varphi) = \sum_{m=0}^{\infty} \frac{1}{m!} P'_m(\varphi)$$

with $P'_m \in \mathcal{P}_{\Theta'}({}^mE')$, and there are $C \geqslant 0$, $\varrho > 0$ such that

$$\|P'_m\|_{\Theta'} \leqslant C\varrho^m$$

for all $m \in \mathbb{N}_0$. Let $H_m \in [\mathcal{P}_\Theta(^m E)]'$ such that $H_m(\varphi^m) = P'_m(\varphi)$, and $\|H_m\| = \|P'_m\|_{\Theta'}$ for all $\varphi \in E'$, and $m \in \mathbb{N}_0$. For $f \in \mathcal{H}_{\Theta b}(E)$ we define

$$T_H(f) = \sum_{m=0}^{\infty} \frac{1}{m!} H_m(\hat{d}^m f(0)).$$

Hence

$$\begin{aligned}|T_H(f)| &\leqslant \sum_{m=0}^{\infty} \frac{1}{m!} \|H_m\| \|\hat{d}^m f(0)\|_\Theta = \sum_{m=0}^{\infty} \frac{1}{m!} \|P'_m\|_{\Theta'} \|\hat{d}^m f(0)\|_\Theta \\ &\leqslant C \sum_{m=0}^{\infty} \frac{\varrho^m}{m!} \|\hat{d}^m f(0)\|_\Theta = C\|f\|_{\Theta,\varrho}.\end{aligned}$$

Therefore $T_H \in [\mathcal{H}_{\Theta b}(E)]'$ and

$$\mathcal{B}T_H(\varphi) = T_H(e^\varphi) = \sum_{m=0}^{\infty} \frac{1}{m!} H_m(\varphi^m) = \sum_{m=0}^{\infty} \frac{1}{m!} P'_m(\varphi) = H(\varphi)$$

for all $\varphi \in E'$, that is, $\mathcal{B}T_H = H$. □

**Example 2.2.** (a) C. Gupta proved in [6] that if $E'$ has the $\lambda$-bounded approximation property, then the Borel transform $\mathcal{B}_N$ establishes an isometric isomorphism between $[\mathcal{P}_N(^m E; F)]'$ and $\mathcal{P}(^m E'; F')$.

(b) M. Matos proved in [12] that if $E'$ has the $\lambda$-bounded approximation property, then the Borel transform $\mathcal{B}_{\tilde{N},(s;(r,q))}$ establishes an isometric isomorphism between $[\mathcal{P}_{\tilde{N},(s;(r,q))}(^m E)]'$ and $\mathcal{P}_{(s',m(r';q'))}(^m E')$, where $\mathcal{P}_{\tilde{N},(s;(r,q))}(^m E)$ denotes the space of all $(s;(r,q))$-quasi-nuclear $m$-homogeneous polynomials on $E$ and $\mathcal{P}_{(s',m(r';q'))}(^m E')$ denotes the space of all $(s',m(r';q'))$-summing $m$-homogeneous polynomials on $E'$.

(c) X. Mujica proved in [13] that if $E'$ has the $\lambda$-bounded approximation property and $F$ is reflexive, then the Borel transform $\mathcal{B}_{\sigma(p)}$ establishes an isometric isomorphism between $[\mathcal{P}_{\sigma(p)}(^m E; F)]'$ and $\mathcal{P}_{\tau(p)}(^m E'; F')$, where $\mathcal{P}_{\sigma(p)}(^m E; F)$ denotes the space of all $\sigma(p)$-nuclear $m$-homogeneous polynomials from $E$ into $F$, and $\mathcal{P}_{\tau(p)}(^m E'; F')$ denotes the space of all $\tau(p)$-summing $m$-homogeneous polynomials from $E'$ into $F'$. In particular, the result follows when $F$ is equal to $\mathbb{C}$.

In the same way, Theorem 2.1 was obtained in all these cases in the corresponding references.

For more details on the indexes $s$, $r$, $q$, $s'$, $r'$, $q'$ and $p$, see the corresponding references.

## 3. Convolution operators on $\mathcal{H}_{\Theta b}(E)$

We need some notation to introduce the convolution operators. If $a \in E$ and $f \in \mathcal{H}_{\Theta b}(E)$, we denote by $\tau_a f$ the complex function on $E$ defined by $(\tau_a f)(x) = f(x-a)$ for all $x \in E$. By Proposition 2.2 we have $\tau_a f \in \mathcal{H}_{\Theta b}(E)$.

**Definition 3.1.** A continuous linear mapping $\mathcal{O}\colon \mathcal{H}_{\Theta b}(E) \to \mathcal{H}_{\Theta b}(E)$ is called a *convolution operator on* $\mathcal{H}_{\Theta b}(E)$ if it is translation invariant, that is, for all $a \in E$ and $f \in \mathcal{H}_{\Theta b}(E)$, $\mathcal{O}(\tau_a f) = \tau_a(\mathcal{O}f)$.

If we denote the set of all convolution operators on $\mathcal{H}_{\Theta b}(E)$ by $\mathcal{A}_\Theta$, then $\mathcal{A}_\Theta$ is an algebra with unity under the usual vector space operations and under composition as multiplication.

**Proposition 3.1.** *Let $a \in E$ and $f \in \mathcal{H}_{\Theta b}(E)$. Then*

(i) *$\hat{d}^m f(\cdot)(a) \in \mathcal{H}_{\Theta b}(E)$ and*

$$\hat{d}^m f(x)(a) = \sum_{k=0}^{\infty} \frac{1}{k!} \widehat{d^{m+k} f(0) x^k}(a)$$

*in the sense of the topology of $\mathcal{H}_{\Theta b}(E)$ for all $m \in \mathbb{N}_0$;*

(ii)

$$(\tau_{-a} f)(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(x)(a)$$

*in the sense of the topology of $\mathcal{H}_{\Theta b}(E)$.*

Proof. (i) It is known (see Nachbin [14], page 29) that the following pointwise equalities are true:

$$\hat{d}^m f(x)(a) = \sum_{k=0}^{\infty} \frac{1}{k!} \widehat{d^{m+k} f(0) x^k}(a) = \sum_{k=0}^{\infty} \frac{1}{k!} \widehat{d^{m+k} f(0) a^m}(x).$$

Since $\hat{d}^m f(0) \in \mathcal{P}_\Theta({}^m E)$ for all $m \in \mathbb{N}_0$, it follows by condition (3) of Definition 2.1 that $\widehat{d^{m+k} f(0) a^m} \in \mathcal{P}_\Theta({}^k E)$ and

$$\left\| \widehat{d^{m+k} f(0) a^m} \right\|_\Theta \leqslant \frac{\sigma^{m+k} m! k!}{(m+k)!} \|\hat{d}^{m+k} f(0)\|_\Theta \|a\|^m.$$

Hence

$$\begin{aligned} &\lim_{k\to\infty} \left( \left\| \frac{1}{k!} \widehat{d^{m+k} f(0) a^m} \right\|_\Theta \right)^{1/k} \\ &\quad \leqslant \lim_{k\to\infty} \left( m! \sigma^{m+k} \left\| \frac{1}{(m+k)!} \hat{d}^{m+k} f(0) \right\|_\Theta \|a\|^m \right)^{1/k} \\ &\quad = \sigma \lim_{k\to\infty} \left( (\sigma\|a\|)^m m! \left\| \frac{1}{(m+k)!} \hat{d}^{m+k} f(0) \right\|_\Theta \right)^{1/k} = 0 \end{aligned}$$

for all $m \in \mathbb{N}_0$, and this shows that $\hat{d}^m f(\cdot)(a) \in \mathcal{H}_{\Theta b}(E)$. Now, for $\varrho > 0$ we have

$$\begin{aligned}
&\left\| \hat{d}^m f(\cdot)(a) - \sum_{k=0}^{\nu} \frac{1}{k!} \left[ \widehat{d^{m+k} f(0)(\cdot)^k} \right](a) \right\|_{\Theta,\varrho} \\
&\quad \leqslant \sum_{k=\nu+1}^{\infty} \sigma^{m+k} m! \varrho^k \left\| \frac{1}{(m+k)!} \hat{d}^{m+k} f(0) \right\|_{\Theta} \|a\|^m \\
&\quad = \frac{m! \|a\|^m}{\varrho^m} \sum_{k=\nu+1}^{\infty} \frac{(\varrho\sigma)^{m+k}}{(m+k)!} \left\| \hat{d}^{m+k} f(0) \right\|_{\Theta} \\
&\quad = \frac{\|a\|^m m!}{\varrho^m} \left\| \sum_{k=\nu+1}^{\infty} \frac{1}{(m+k)!} \hat{d}^{m+k} f(0) \right\|_{\Theta,\varrho\sigma}.
\end{aligned}$$

Since the last member of the inequality goes to zero as $\nu$ tends to infinity, we have proved (i).

(ii) For $\varrho > 0$ we have

$$\begin{aligned}
&\left\| \tau_{-a} f - \sum_{m=0}^{v} \frac{1}{m!} \hat{d}^m f(\cdot)(a) \right\|_{\Theta,\varrho} \\
&\quad = \sum_{k=0}^{\infty} \frac{\varrho^k}{k!} \left\| \sum_{m=v+1}^{\infty} \frac{1}{m!} \widehat{d^{m+k} f(0) a^m} \right\|_{\Theta} \\
&\quad \leqslant \sum_{k=0}^{\infty} \varrho^k \sum_{m=v+1}^{\infty} \frac{1}{k! m!} \left\| \widehat{d^{m+k} f(0) a^m} \right\|_{\Theta} \\
&\quad \leqslant \sum_{k=0}^{\infty} \sum_{m=v+1}^{\infty} \frac{\varrho^k \sigma^{m+k}}{(m+k)!} \|\hat{d}^{m+k} f(0)\|_{\Theta} \|a\|^m.
\end{aligned}$$

Since

$$\lim_{m\to\infty} \left( \frac{1}{m!} \|\hat{d}^m f(0)\|_{\Theta} \right)^{1/m} = 0,$$

given $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$\frac{1}{m!} \|\hat{d}^m f(0)\|_{\Theta} \leqslant C(\varepsilon) \varepsilon^m$$

for all $m \in \mathbb{N}_0$. If we choose $\varepsilon > 0$ such that $\sigma\varepsilon\varrho < 1$ and $\|a\|\sigma\varepsilon < 1$, then we have

$$\begin{aligned}
&\left\| \tau_{-a} f - \sum_{m=0}^{v} \frac{1}{m!} \hat{d}^m f(\cdot)(a) \right\|_{\Theta,\varrho} \\
&\quad \leqslant \sum_{k=0}^{\infty} \sum_{m=v+1}^{\infty} \varrho^k \sigma^{m+k} \|a\|^m C(\varepsilon) \varepsilon^{m+k} \\
&\quad \leqslant C(\varepsilon) \left( \sum_{k=0}^{\infty} (\sigma\varepsilon\varrho)^k \right) \left( \sum_{m=v+1}^{\infty} (\|a\|\sigma\varepsilon)^m \right).
\end{aligned}$$

Since the last member of the inequality goes to zero as $\nu$ tends to infinity, we have proved (ii). $\square$

If $T \in [\mathcal{H}_{\Theta b}(E)]'$, then there are $C \geqslant 0$ and $\varrho > 0$ such that

$$|T(f)t| \leqslant C\|f\|_{\Theta,\varrho} \tag{3.1}$$

for all $f \in \mathcal{H}_{\Theta b}(E)$. For each $P \in \mathcal{P}_{\Theta}(^mE)$ with $A \in \mathcal{L}_s(^mE)$ (the space of all $m$-linear symmetric mappings on $E^m$) such that $P = \hat{A}$, we may define the polynomial

$$\begin{aligned} T(\widehat{A(\cdot)^k})\colon\ E &\longrightarrow \mathbb{C} \\ y &\longmapsto T(A(\cdot)^k y^{m-k}) \end{aligned}$$

which belongs to $\mathcal{P}(^{m-k}E)$ for each $k \in \mathbb{N}_0$, $k \leqslant m$.

Now we are interested in polynomials of the type $T(\widehat{A(\cdot)^k})$ that have a certain stability property:

**Definition 3.2.** Let $(\mathcal{P}_{\Theta}(^mE))_{m=0}^{\infty}$ be a holomorphy type from $E$ to $\mathbb{C}$. The holomorphy type $\Theta$ is said to be a $\pi_2$-*holomorphy type* if $T(\widehat{A(\cdot)^k}) \in \mathcal{P}_{\Theta}(^{m-k}E)$ and

$$\|T(\widehat{A(\cdot)^k})\|_{\Theta} \leqslant C\varrho^k\|P\|_{\Theta}$$

for all $k \in \mathbb{N}_0$, $k \leqslant m$, $T \in [\mathcal{H}_{\Theta b}(E)]'$ and $P \in \mathcal{P}_{\Theta}(^mE)$.

Here $C$ and $\varrho$ are as in (3.1).

**Definition 3.3.** If $T \in [\mathcal{H}_{\Theta b}(E)]'$ and $f \in \mathcal{H}_{\Theta b}(E)$, we define the *convolution product* $T * f$ by

$$(T * f)(x) = T(\tau_{-x}f),$$

for all $x \in E$.

**Theorem 3.1.** *If $(\mathcal{P}_{\Theta}(^mE))_{m=0}^{\infty}$ is a $\pi_2$-holomorphy type, $T \in [\mathcal{H}_{\Theta b}(E)]'$ and $f \in \mathcal{H}_{\Theta b}(E)$, then $T * f \in \mathcal{H}_{\Theta b}(E)$ and the mapping $T*$ defines a convolution operator on $\mathcal{H}_{\Theta b}(E)$.*

P r o o f. By Proposition 3.1, for all $x \in E$ we have

$$\begin{aligned} (T * f)(x) = T(\tau_{-x}f) &= T\Bigg(\sum_{m=0}^{\infty}\frac{1}{m!}\hat{d}^m f(x)\Bigg) \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{\infty}\frac{1}{k!}T(\widehat{d^{k+m}f(0)(\cdot)^k})(x). \end{aligned} \tag{3.2}$$

By Definition 3.2 we have $T(\widehat{d^{k+m}f(0)(\cdot)^k}) \in \mathcal{P}_\Theta(^mE)$ for all $m \in \mathbb{N}_0$ and

$$\left\|T(\widehat{d^{k+m}f(0)(\cdot)^k})\right\|_\Theta \leqslant C\varrho^k \|\hat{d}^{m+k}f(0)\|_\Theta.$$

For $\varrho_0 > \varrho$ we can write

$$\begin{aligned}
&\left\|\sum_{k=0}^{\infty} \frac{1}{k!} T\left(\widehat{d^{k+m}f(0)(\cdot)^k}\right)\right\|_\Theta \\
&\qquad \leqslant \sum_{k=0}^{\infty} \frac{1}{k!} \left\|T\left(\widehat{d^{k+m}f(0)(\cdot)^k}\right)\right\|_\Theta \leqslant \sum_{k=0}^{\infty} \frac{1}{k!} C\varrho^k \|\hat{d}^{m+k}f(0)\|_\Theta \\
&\qquad \leqslant \sum_{k=0}^{\infty} \frac{1}{k!} C\varrho_0^k \|\hat{d}^{m+k}f(0)\|_\Theta \\
&\qquad \leqslant C\frac{m!}{\varrho_0^m} \sum_{k=0}^{\infty} \frac{2^{m+k}}{(m+k)!} \varrho_0^{m+k} \|\hat{d}^{m+k}f(0)\|_\Theta \\
&\qquad = C\frac{m!}{\varrho_0^m} \left\|\sum_{k=m}^{\infty} \frac{1}{k!} \hat{d}^k f(0)\right\|_{\Theta,2\varrho_0} \leqslant C\frac{m!}{\varrho_0^m} \|f\|_{\Theta,2\varrho_0} < \infty.
\end{aligned}$$

This means that

$$P_m = \sum_{k=0}^{\infty} \frac{1}{k!} T\left(\widehat{d^{k+m}f(0)(\cdot)^k}\right)$$

belongs to $\mathcal{P}_\Theta(^mE)$ and

$$\|P_m\|_\Theta \leqslant C\frac{m!}{\varrho_0^m} \|f\|_{\Theta,2\varrho_0}. \tag{3.3}$$

Hence

$$\lim_{m\to\infty} \left(\frac{1}{m!}\|P_m\|_\Theta\right)^{1/m} \leqslant \frac{1}{\varrho_0}$$

for all $\varrho_0 > \varrho$. This implies that

$$\lim_{m\to\infty} \left(\frac{1}{m!}\|P_m\|_\Theta\right)^{1/m} = 0.$$

Therefore, it follows from (3.2) that

$$(T * f) = \sum_{m=0}^{\infty} \frac{1}{m!} P_m$$

belongs to $\mathcal{H}_{\Theta b}(E)$.

It is clear that $T*$ is linear. For $\varrho_1 > 0$ it follows from (3.3) that

$$\begin{aligned}\|T*f\|_{\Theta,\varrho_1} &= \sum_{m=0}^{\infty} \frac{\varrho_1^m}{m!}\|P_m\|_\Theta \\ &\leqslant \sum_{m=0}^{\infty} \frac{\varrho_1^m}{m!}\frac{Cm!}{(\varrho_1+\varrho)^m}\|f\|_{\Theta,2(\varrho_1+\varrho)} \\ &\leqslant C\Big(\sum_{m=0}^{\infty}\frac{\varrho_1^m}{(\varrho_1+\varrho)^m}\Big)\|f\|_{\Theta,2(\varrho_1+\varrho)}.\end{aligned}$$

Thus $T*$ is continuous. Now we have

$$\begin{aligned}(T*\tau_a f)(x) &= T(\tau_{-x}\circ\tau_a f) = T(\tau_{-x+a}f) \\ &= (T*f)(-(-x+a)) = (T*f)(x-a) \\ &= \tau_a(T*f)(x)\end{aligned}$$

for all $x, a \in E$. This completes the proof that $T*$ is a convolution operator. $\square$

Now we are able to characterize all convolutions operators on $\mathcal{H}_{\Theta b}(E)$.

**Theorem 3.2.** *Let $(\mathcal{P}_\Theta(^mE))_{m=0}^\infty$ be a $\pi_2$-holomorphy type and define a mapping $\Gamma_\Theta$ from $\mathcal{A}_\Theta$ into $[\mathcal{H}_{\Theta b}(E)]'$ by $\Gamma_\Theta(\mathcal{O})(f) = (\mathcal{O}f)(0)$ for each $\mathcal{O}$ in $\mathcal{A}_\Theta$ and $f$ in $\mathcal{H}_{\Theta b}(E)$. Then $\Gamma_\Theta$ is a linear bijection between $\mathcal{A}_\Theta$ and $[\mathcal{H}_{\Theta b}(E)]'$.*

Proof. We define a mapping $\bar\Gamma_\Theta$ from $[\mathcal{H}_{\Theta b}(E)]'$ into $\mathcal{A}_\Theta$ by $\bar\Gamma_\Theta(T)(f) = T*f$ for all $T$ in $[\mathcal{H}_{\Theta b}(E)]'$ and $f$ in $\mathcal{H}_{\Theta b}(E)$. This linear mapping is well-defined by Theorem 3.1. Now we have

$$[(\bar\Gamma_\Theta\circ\Gamma_\Theta)(\mathcal{O})](f) = [\bar\Gamma_\Theta(\Gamma_\Theta(\mathcal{O}))](f) = \Gamma_\Theta(\mathcal{O})*f,$$

but for all $x \in E$ we have

$$(\Gamma_\Theta(\mathcal{O})*f)(x) = \Gamma_\Theta(\mathcal{O})(\tau_{-x}f) = \mathcal{O}(\tau_{-x}f)(0) = \tau_{-x}(\mathcal{O}f)(0) = (\mathcal{O}f)(x).$$

Hence

$$[(\bar\Gamma_\Theta\circ\Gamma_\Theta)(\mathcal{O})](f) = \mathcal{O}f$$

and $\bar\Gamma_\Theta\circ\Gamma_\Theta$ is the identity mapping on $\mathcal{A}_\Theta$. Also we have

$$(\Gamma_\Theta\circ\bar\Gamma_\Theta)(T)(f) = \Gamma_\Theta(\bar\Gamma_\Theta T)(f) = (\bar\Gamma_\Theta T)(f)(0) = (T*f)(0) = T(f).$$

Thus $\Gamma_\Theta\circ\bar\Gamma_\Theta$ is the identity mapping on $[\mathcal{H}_{\Theta b}(E)]'$ and this completes the proof. $\square$

**Definition 3.4.** Let $(\mathcal{P}_\Theta(^mE))_{m=0}^\infty$ be a $\pi_2$-holomorphy type. For $T_1, T_2 \in [\mathcal{H}_{\Theta b}(E)]'$ we define the *convolution product of* $T_1$ and $T_2$ *in* $[\mathcal{H}_{\Theta b}(E)]'$ by

$$T_1 * T_2 := \Gamma_\Theta(\mathcal{O}_1 \circ \mathcal{O}_2) \in [\mathcal{H}_{\Theta b}(E)]',$$

where $\mathcal{O}_1 = T_1*$ and $\mathcal{O}_2 = T_2*$.

It is easy to see that $[\mathcal{H}_{\Theta b}(E)]'$ is an algebra under this convolution product with unity $\delta$ given by $\delta(f) = f(0)$ for all $f$ in $\mathcal{H}_{\Theta b}(E)$, and the convolution product satisfies $(T_1 * T_2) * f = T_1 * (T_2 * f)$.

**Theorem 3.3.** *If $(\mathcal{P}_\Theta(^mE))_{m=0}^\infty$ is a $\pi_1$-$\pi_2$-holomorphy type, then the Borel transform is an algebra isomorphism between $[\mathcal{H}_{\Theta b}(E), \tau_\Theta]'$ and $\mathrm{Exp}_{\Theta'}(E')$.*

Proof. By Theorem 2.1, $\mathcal{B}$ is a vector space isomorphism between $[\mathcal{H}_{\Theta b}(E), \tau_\Theta]'$ and $\mathrm{Exp}_{\Theta'}(E')$. We only have to show that the multiplication operation is preserved. For $T_1$ and $T_2$ in $[\mathcal{H}_{\Theta b}(E)]'$ and $\varphi$ in $E'$ we have

$$\begin{aligned}\mathcal{B}(T_1 * T_2)(\varphi) &= (T_1 * T_2)(e^\varphi) = [(\mathcal{O}_1 \circ \mathcal{O}_2)(e^\varphi)](0) = [T_1 * (T_2 * e^\varphi)](0)\\ &= T_1(T_2 * e^\varphi) = T_1(e^\varphi T_2(e^\varphi)) = T_1(e^\varphi)T_2(e^\varphi)\\ &= \mathcal{B}T_1(\varphi) \cdot \mathcal{B}T_2(\varphi).\end{aligned}$$

Hence $\mathcal{B}(T_1 * T_2) = \mathcal{B}T_1 \cdot \mathcal{B}T_2$ as we wanted to prove. $\square$

**Example 3.1.** C. Gupta in [6], M. Matos in [12] and X. Mujica in [13] proved that if $E'$ has the $\lambda$-bounded approximation property, then the sequences of spaces of nuclear polynomials, $(s;(r,q))$-quasi-nuclear polynomials and $\sigma(p)$-nuclear polynomials from $E$ to $\mathbb{C}$ satisfy the condition of Definition 3.2. Hence, each of these sequences is a $\pi_2$-holomorphy type from $E$ to $\mathbb{C}$.

Results of the type of Theorems 3.2 and 3.3 were obtained to the corresponding cases in the corresponding references.

## 4. Approximation and existence theorems

Division theorems for entire functions play a fundamental role in proving the approximation and existence theorems for convolution equations. In order to obtain division theorems involving the Borel transform we need to introduce a new concept.

**Definition 4.1.** Let $U$ be an open subset of $E$ and $\mathcal{F}(U)$ a collection of holomorphic functions from $U$ into $\mathbb{C}$. We say that $\mathcal{F}(U)$ is *closed under division* if for each $f$ and $g$ in $\mathcal{F}(U)$ with $g \neq 0$ and $h = f/g$ a holomorphic function on $U$, we have $h$ in $\mathcal{F}(U)$.

The quotient notation $h = f/g$ means that $f(x) = h(x) \cdot g(x)$ for all $x \in U$.

It is not easy to prove division results in the sense of Definition 4.1 for the spaces of entire functions of exponential type. Some examples were obtained by C. Gupta in [6] and by M. Matos in [12] for the spaces $\mathrm{Exp}(E)$ and $\mathrm{Exp}_{(s,m(r;q))}(E)$, respectively.

The next useful result was proved by Gupta in [6].

**Lemma 4.1.** *Let $U$ be an open connected subset of $E$. Let $f$ and $g$ be holomorphic functions on $U$, with $g$ no identically zero, such that for any affine subspace $S$ of $E$ of dimension one and for any connected component $S'$ of $S \cap U$ on which $g$ is not identically zero, the restriction $f|_{S'}$ is divisible by the restriction $g|_{S'}$, with the quotient being holomorphic in $S'$. Then $f$ is divisible by $g$ and the quotient is holomorphic on $U$.*

**Theorem 4.1.** *Let $(\mathcal{P}_\Theta(^mE))_{m=0}^\infty$ be a $\pi_1$-$\pi_2$-holomorphy type. If $\mathrm{Exp}_{\Theta'}(E')$ is closed under division and $T_1, T_2 \in [\mathcal{H}_{\Theta b}(E)]'$ are such that $T_2 \neq 0$ and $T_1(P \exp \varphi) = 0$ whenever $T_2 * P \exp \varphi = 0$ with $\varphi \in E'$ and $P \in \mathcal{P}_\Theta(^mE)$, $m \in \mathbb{N}_0$, then $\mathcal{B}T_1$ is divisible by $\mathcal{B}T_2$ with the quotient being an element of $\mathrm{Exp}_{\Theta'}(E')$.*

Proof. Let $S$ be a one dimensional affine subspace of $E$. It is clear that $S$ is of the form $\{\varphi_1 + t\varphi_2;\ t \in \mathbb{C}\}$, where $\varphi_1, \varphi_2 \in E'$ are fixed. We suppose that $t_0$ is a zero of order $k$ of

$$g_2(t) = \mathcal{B}(T_2)(\varphi_1 + t\varphi_2) = T_2(\exp(\varphi_1 + t\varphi_2)).$$

Then we have

$$T_2(\varphi_2^j \exp(\varphi_1 + t_0\varphi_2)) = 0$$

for each $j < k$, and this implies

$$\begin{aligned} &T_2 * \varphi_2^j \exp(\varphi_1 + t_0\varphi_2) \\ &\quad = \sum_{m=0}^{j} \binom{j}{m} \varphi_2^{j-m} \exp(\varphi_1 + t_0\varphi_2) T_2(\varphi_2^m \exp(\varphi_1 + t_0\varphi_2)) = 0 \end{aligned}$$

for each $j < k$. Hence it follows that $T_1(\varphi_2^j \exp(\varphi_1 + t_0\varphi_2)) = 0$ for all $j < k$, and this implies that $t_0$ is a zero of order at least $k$ of $g_1(t) = \mathcal{B}T_1(\varphi_1 + t\varphi_2)$. Therefore $\mathcal{B}T_1|_S$ is divisible by $\mathcal{B}T_2|_S$ and the quotient is holomorphic on $S$. By Lemma 4.1 we have that $\mathcal{B}T_1$ is divisible by $\mathcal{B}T_2$ on $E'$ and the quotient is entire. Therefore it follows by Definition 4.1 that the quotient is in $\mathrm{Exp}_{\Theta'}(E')$. □

**Theorem 4.2.** *If $(\mathcal{P}_\Theta(^mE))_{m=0}^\infty$ is a $\pi_1$-$\pi_2$-holomorphy type, $\mathrm{Exp}_{\Theta'}(E')$ is closed under division and $\mathcal{O}$ is in $\mathcal{A}_\Theta$, then the vector subspace of $\mathcal{H}_{\Theta b}(E)$ generated by* *the exponential polynomial solutions of the homogeneous equation $\mathcal{O}$ =0 is dense in the closed subspace of all solutions of the homogeneous equation, that is, the vector subspace of $\mathcal{H}_{\Theta b}(E)$ generated by*

$$\mathcal{L} =\{P\exp\varphi;\ P\in\mathcal{P}_{\Theta}(^mE),\ m\in\mathbb{N}_0,\ \varphi\in E',\ \mathcal{O}(P\exp\varphi)=0\}$$

*is dense in*

$$\ker\mathcal{O}=\{f\in\mathcal{H}_{\Theta b}(E);\ \mathcal{O}f=0\}.$$

P r o o f. If $\mathcal{O}$ is equal to 0, the result follows by Proposition 2.5. Let $\mathcal{O}$ be different from 0. By Theorem 3.2, there is $T$ in $[\mathcal{H}_{\Theta b}(E)]'$, $T$ different from 0, such that $\mathcal{O}=T*$. Now we suppose that $X$ in $[\mathcal{H}_{\Theta b}(E)]'$ is such that $X|_{\mathcal{L}}=0$. Thus by Theorem 4.1, there is $h$ in $\mathrm{Exp}_{\Theta'}(E')$ such that $\mathcal{B}(X)=h\cdot\mathcal{B}(T)$. By Theorem 3.3, there is $S$ in $[\mathcal{H}_{\Theta b}(E)]'$ such that $h=\mathcal{B}(S)$ and $\mathcal{B}(X)=\mathcal{B}(S)\cdot\mathcal{B}(T)=\mathcal{B}(S*T)$. Hence $X=S*T$ and for each $f$ in $\ker\mathcal{O}$ we have $X*f=S*(T*f)=0$ and $X(f)=(X*f)(0)=0$. We have shown that every $X$ in $[\mathcal{H}_{\Theta b}(E)]'$ vanishing on the vector subspace of $\mathcal{H}_{\Theta b}(E)$ generated by $\mathcal{L}$ vanishes on $\ker\mathcal{O}$. Now the result follows as a consequence of the Hahn-Banach Theorem. $\square$

**Theorem 4.3.** *If $(\mathcal{P}_{\Theta}(^mE))_{m=0}^{\infty}$ is a $\pi_1$-$\pi_2$-holomorphy type, $\mathrm{Exp}_{\Theta'}(E')$ is closed under division and $\mathcal{O}$ is in $\mathcal{A}_{\Theta}$, then its transpose mapping $\mathcal{O}^t$ has the following properties:*
(i) *$\mathcal{O}^t([\mathcal{H}_{\Theta b}(E)]')$ is the orthogonal of $\ker\mathcal{O}$ in $[\mathcal{H}_{\Theta b}(E)]'$.*
(ii) *$\mathcal{O}^t([\mathcal{H}_{\Theta b}(E)]')$ is closed for the weak star topology in $[\mathcal{H}_{\Theta b}(E)]'$ defined by $\mathcal{H}_{\Theta b}(E)$.*

P r o o f. If $\mathcal{O}$ is equal to 0, the result is clear. Let $\mathcal{O}$ be different from 0 and $T$ in $[\mathcal{H}_{\Theta b}(E)]'$ such that $\mathcal{O}=T*$. For each $X$ in $\mathcal{O}^t([\mathcal{H}_{\Theta b}(E)]')$ there is $S$ in $[\mathcal{H}_{\Theta b}(E)]'$ satisfying $X=\mathcal{O}^t(S)$. Hence, for each $f$ in $\ker\mathcal{O}$ we have $X(f)=\mathcal{O}^t(S)(f)=S(\mathcal{O}f)=0$, and then $\mathcal{O}^t([\mathcal{H}_{\Theta b}(E)]')$ is contained in the orthogonal of $\ker\mathcal{O}$. Conversely, if $X$ is in the orthogonal of $\ker\mathcal{O}$, by Theorem 4.1 there is $h$ in $\mathrm{Exp}_{\Theta'}(E')$ such that $\mathcal{B}(X)=h\cdot\mathcal{B}(T)$ and by Theorem 3.3 there is $S$ in $[\mathcal{H}_{\Theta b}(E)]'$ such that $h=\mathcal{B}(S)$ and $\mathcal{B}(X)=\mathcal{B}(S)\cdot\mathcal{B}(T)=\mathcal{B}(S*T)$. Hence $X=S*T$, and for each $f$ in $\mathcal{H}_{\Theta b}(E)$ we have

$$\begin{aligned}X(f)&=(S*T)(f)=((S*T)*f)(0)=(S*(T*f))(0)\\&=S(T*f)=S(\mathcal{O}f)=\mathcal{O}^t(S)(f)\end{aligned}$$

and this implies that $X$ is equal to $\mathcal{O}^t(S)$ and belongs to $\mathcal{O}^t([\mathcal{H}_{\Theta b}(E)]')$. Thus (i) is proved.

Now we note that the orthogonal of $\ker \mathcal{O}$ is equal to

$$\bigcap_{f\in\ker\mathcal{O}} \{T \in [\mathcal{H}_{\Theta b}(E)]';\, T(f) = 0\}.$$

Since for each $f$ in $\mathcal{H}_{\Theta b}(E)$, the set $\{T \in [\mathcal{H}_{\Theta b}(E)]';\, T(f) = 0\}$ is closed in weak star topology, (ii) is proved. □

Our last result concerns the existence of solutions of convolution equations. The following result due to Dieudonné and Schwartz (see [7], page 308) is needed.

**Lemma 4.2.** *If $E$ and $F$ are Fréchet spaces and $u\colon\ E \longrightarrow F$ is a linear continuous mapping, then the following conditions are equivalent:*

(a) $u(E) = F$;

(b) $u^t\colon\ F' \longrightarrow E'$ *is injective and $u^t(F')$ is closed in the weak star topology of $E'$ defined by $E$.*

**Theorem 4.4.** *If $(\mathcal{P}_\Theta(^mE))_{m=0}^\infty$ is a $\pi_1$-$\pi_2$-holomorphy type, $\mathrm{Exp}_{\Theta'}(E')$ is closed under division and $\mathcal{O}$ is a non zero convolution operator, then $\mathcal{O}(\mathcal{H}_{\Theta b}(E))$ is equal to $\mathcal{H}_{\Theta b}(E)$.*

P r o o f. By Proposition 2.3, $\mathcal{H}_{\Theta b}(E)$ is a Fréchet space. By Lemma 4.2 (b) and by Theorem 4.3 (ii), it is enough to show that $\mathcal{O}^t$ is injective. Since $\mathcal{O} = T*$ for some $T$ in $[\mathcal{H}_{\Theta b}(E)]'$, hence for all $S$ in $[\mathcal{H}_{\Theta b}(E)]'$ and $f$ in $\mathcal{H}_{\Theta b}(E)$ we have $(\mathcal{O}^tS)(f) = S(\mathcal{O}f) = S(T * f) = (S * T)(f)$. Thus $\mathcal{O}^tS = S * T$ and if $\mathcal{O}^tS = 0$, then $S * T = 0$ and $\mathcal{B}(S * T) = 0$. Since $\mathcal{O} = T*$ is non zero it follows that $\mathcal{B}T$ is non zero and since $\mathcal{B}(S * T) = \mathcal{B}S \cdot \mathcal{B}T$, we get $\mathcal{B}S = 0$. Hence $S = 0$ and $\mathcal{O}^t$ is injective. □

**Example 4.1.** In conclusion, if $E'$ has the $\lambda$-bounded approximation property, then the holomorphy types

$$(\mathcal{P}_N(^mE))_{m=0}^\infty,\ (\mathcal{P}_{\tilde{N},(s;(r,q))}(^mE))_{m=0}^\infty \ \text{ and } \ (\mathcal{P}_{\sigma(p)}(^mE))_{m=0}^\infty$$

are $\pi_1$-$\pi_2$-holomorphy types, and the spaces $\mathrm{Exp}(E')$ and $\mathrm{Exp}_{(s',m(r';q'))}(E')$ are closed under division. In particular, we obtain Theorems 4.1, 4.2 and 4.4 for the cases of C. Gupta [6] and M. Matos [12].

**Open problems.** The following open problems related to the subject of this paper seem to be interesting:

(1) Is the space $\mathrm{Exp}_{\tau_{(p)}}(E)$ of X. Mujica [13] closed under division? If yes, existence and approximation results (Theorems 4.2 and 4.4) hold true for convolution equations on $\mathcal{H}_{\sigma(p)}(^mE)$.

(2) Is there an "algorithm" similar to the one in this paper for spaces of functions of a given type and order? Results in this direction will generalize those of A. Martineau [9], M. Matos [10], [11] and V. Fávaro [3], [4].

**Acknowledgements.** The authors thank Mário C. Matos and Jorge Mujica for theirs continuous help and advice while writing this paper. The authors also want to thank G. Botelho for his useful suggestions.

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