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MONOTONE META-LINDELÖF SPACES

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Abstract. In this paper, we study the monotone meta-Lindelöf property. Relationships between monotone meta-Lindelöf spaces and other spaces are investigated. Behaviors of monotone meta-Lindelöf *GO*-spaces in their linearly ordered extensions are revealed.

Keywords: monotonically meta-Lindelöf, compact, point-countable, order, linearly ordered extension

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1. PRELIMINARIES

Monotone topological properties play an important role in the research of general topology (see [3]–[5], [7], [9] and [14]). In [3], the authors studied monotone Lindelöf spaces.

A space X is *monotonically Lindelöf* if for each open cover \mathcal{U} of X there exists a countable open cover $r(\mathcal{U})$ of X refining \mathcal{U} such that if \mathcal{U} and \mathcal{V} are open covers and \mathcal{U} refines \mathcal{V} , then $r(\mathcal{U})$ refines $r(\mathcal{V})$. Monotone Lindelöf spaces are Lindelöf, however a Lindelöf space may not be monotonically Lindelöf.

In this paper, we introduce the monotone meta-Lindelöf property which is weaker than monotone Lindelöfness but stronger than meta-Lindelöfness. Properties of monotone meta-Lindelöf spaces are investigated. Behaviors of monotone meta-Lindelöf *GO*-spaces in their linearly ordered extensions are revealed.

Recall that a generalized ordered space (*GO*-space) is a Hausdorff space X equipped with a linear order and having a base of convex sets (a set A is called convex if $x \in A$ for every x lying between two points of A). If the topology of X coincides with the open interval topology of the given linear order, we say that

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X is a linearly ordered topological space (*LOTS*). Čech showed that the class of *GO*-spaces is the same as the class of spaces that can be topologically embedded in some *LOTS* (see [10]).

If X is a *GO*-space and Y is a *LOTS* containing X as a subspace, and the order on X is inherited from the order on Y , then Y called a linearly ordered extension of X . If the *GO*-space X is closed (respectively, dense) in the *LOTS* Y , then Y is called the closed (respectively, dense) linearly ordered extension of X .

Throughout the paper, spaces are topological spaces and Hausdorff, mappings are continuous and surjective. Let \mathcal{U} and \mathcal{V} be open covers of the space X . If \mathcal{U} refines \mathcal{V} , we say that \mathcal{U} is a refinement of \mathcal{V} , denoted by $\mathcal{U} \prec \mathcal{V}$. A space X is meta-Lindelöf if every open cover \mathcal{U} of X has a point-countable open refinement \mathcal{V} . \mathbb{R} , \mathbb{Q} , \mathbb{P} and \mathbb{Z} denote the set of all real numbers, the set of all rational numbers, the set of all irrational numbers and the set of all integers respectively. The spaces $[0, \omega_1)$ and $[0, \omega_1]$ are the usual ordinal spaces unless specifically stated and the space $[0, 1]$ is the subspace of the real line \mathbb{R} . Other terms and symbols can be found in [6] and [10].

2. THE DEFINITION OF MONOTONE META-LINDELÖF SPACES

Definition 1. A space X is monotonically meta-Lindelöf if each open cover \mathcal{U} of X has a point-countable open refinement $r(\mathcal{U})$ such that if \mathcal{U} and \mathcal{V} are open covers and $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$. In this case, r is called a monotone meta-Lindelöf operator for the space X .

Proposition 1. *Spaces with a point-countable base are monotonically meta-Lindelöf.*

Proof. Let the space X have a point-countable base \mathcal{B} . For any open cover \mathcal{U} of X , put $r(\mathcal{U}) = \{B \in \mathcal{B} : \exists U \in \mathcal{U} \text{ such that } B \subset U\}$, then r is a monotone meta-Lindelöf operator for X . \square

Proposition 1 is not reversible (see Example 3).

Obviously,

(\diamond) monotone Lindelöf \Rightarrow monotone meta-Lindelöf \Rightarrow meta-Lindelöf.

Examples 1, 2 and Proposition 2 show that the implications in (\diamond) are not reversible.

In a *LOTS*, monotone meta-Lindelöfness does not imply monotone Lindelöfness:

Example 1. Let $X = [0, 1] \times (0, 1)$ be equipped with the open interval topology of the lexicographical order. Then X is monotonically meta-Lindelöf, but not monotonically Lindelöf.

Proof. For each $t \in [0, 1]$, $\{t\} \times (0, 1)$ has a countable base \mathcal{B}_t , so X has a point-countable base $\mathcal{B} = \{B \in \mathcal{B}_t : t \in [0, 1]\}$. By Proposition 1, X is monotonically meta-Lindelöf. Since the open cover $\{\{t\} \times (0, 1) : t \in [0, 1]\}$ of X has no countable subcover X is not Lindelöf. So X is not monotonically Lindelöf. \square

In GO -spaces, monotone meta-Lindelöfness does not imply monotone Lindelöfness:

Example 2. The Michael line M (a GO -space) is monotonically meta-Lindelöf, but not monotonically Lindelöf.

Proof. Note that the Michael line M (the real line with the irrationals isolated and the rationals having their usual neighborhoods) has a point-countable base $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\} \cup \{\{p\} : p \in \mathbb{P}\}$. By Proposition 1 M is monotonically meta-Lindelöf. However M is not monotonically Lindelöf since it is not Lindelöf ([13]). \square

Example 3. The space $X = ([0, \omega_1) \times \mathbb{Z}) \cup \{(\omega_1, 0)\}$ equipped with the lexicographical-order topology is monotonically meta-Lindelöf, but without a point-countable base.

Proof. For any open cover \mathcal{U} of X , [3] noted that if

$$\alpha = \alpha(\mathcal{U}) = \min\{\alpha' \in [0, \omega_1) : (\langle \alpha', 0 \rangle, \langle \omega_1, 0 \rangle) \subset U \text{ for some } U \in \mathcal{U}\}$$

and

$$r(\mathcal{U}) = \{(\langle \alpha, 0 \rangle, \langle \omega_1, 0 \rangle)\} \cup \{\langle \beta, \kappa \rangle : \beta < \alpha \text{ and } \kappa \in \mathbb{Z} \text{ or } \beta = \alpha \text{ and } \kappa \leq 0\},$$

then r is a monotone Lindelöf operator. So r is also a monotone meta-Lindelöf operator. Since the point $\langle \omega_1, 0 \rangle$ has no countable neighborhood base, X has no point-countable base. \square

In Example 2.3 of [3], it is shown that $[0, \omega_1]$ is not monotonically Lindelöf. Note that in its proof, if r is assumed to be a monotone meta-Lindelöf operator for $[0, \omega_1]$, then $r(\mathcal{U}_\gamma)$ is a point-countable open refinement of \mathcal{U}_γ . Thus from the proof of Example 2.3 of [3], we can see that the following stronger result is true.

Proposition 2. *The compact LOTS $[0, \omega_1]$ is not a monotone meta-Lindelöf space.*

Corollary 1. *A monotonically meta-Lindelöf compact LOTS X is first countable.*

Proof. Let \prec be the linear order on X . If the compact LOTS X is not first countable, then it contains a closed subspace which is homomorphic to $[0, \omega_1]$: in fact, let $p \in X$ have no countable neighborhood base. Without loss of generality, we may assume that p has no immediate predecessor and is not the minimal element and any strict increasing sequence of $\{x \in X : x \prec p\}$ cannot be convergent to p .

Take $x_0 \in X$ such that $x_0 \prec p$. Start with x_0 , by transfinite induction, we can obtain a closed subset $F = \{x_\alpha \prec p : \alpha \in [0, \omega_1]\}$ of X where for each limit ordinal $\gamma \in [0, \omega_1]$, $x_\gamma = \sup\{x_\alpha : \alpha < \gamma\}$ (since X is a compact LOTS this can be done) and $x_\alpha \prec x_\beta$ whenever $\alpha < \beta$. Clearly F is homomorphic to $[0, \omega_1]$. By Proposition 3 (1), F (homomorphic to $[0, \omega_1]$) is monotonically meta-Lindelöf. This contradicts Proposition 2. \square

The compact LOTS $[0, \omega_1]$ in Proposition 2 is not connected. We will see that a connected compact LOTS may not imply monotone meta-Lindelöfness.

Recall that *the long line* Z is the space $Z = [0, \omega_1) \times [0, 1)$ with the open interval topology generated by the lexicographical order. Obviously, Z is countably compact but not compact. By Theorem 9.2 of [1] Z is not meta-Lindelöf. The space $Z^* = Z \cup \{\omega_1\}$ is called *the extended long line* (that is, for any $z \in Z$, $z < \omega_1$ and Z^* is equipped with the open interval topology, equivalently, Z^* is the one-point compactification of Z) (see [13]).

Corollary 2. *The connected compact LOTS Z^* is not monotonically meta-Lindelöf.*

Proof. Note that Z^* is not first countable since the point ω_1 has no countable neighborhood base. So by Corollary 1, Z^* is not monotonically meta-Lindelöf. \square

To be clear at a glance, we give the following diagram, the implications are not reversible.

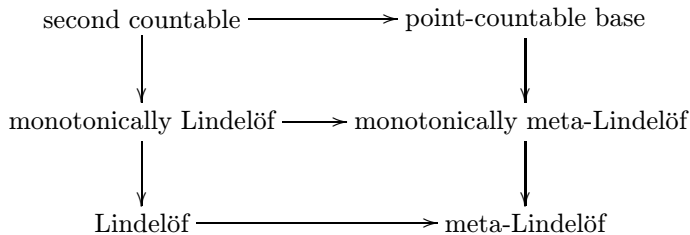


Diagram (*).

Recall that a space X is said to have calibre ω_1 if a point-countable family of non-empty open subsets is countable [11].

Remark 1. If X has calibre ω_1 and each $x \in X$ has an open neighborhood U_x with a point-countable base, then the properties in Diagram (*) are equivalent.

In fact, let X be meta-Lindelöf and \mathscr{W} be a point-countable open refinement of $\mathscr{U} = \{U_x : x \in X\}$. For each $W \in \mathscr{W}$, take a $U \in \mathscr{U}$ such that $W \subset U$. Put $\mathscr{B}_W = \{W \cap B : B \in \mathscr{B}_U\}$, where \mathscr{B}_U is a point-countable base of U . Then $\mathscr{B} = \bigcup\{\mathscr{B}_W : W \in \mathscr{W}\}$ is a point-countable base for X . Since X has calibre ω_1 , \mathscr{B} is a countable base for X .

Recall that a mapping $f: X \rightarrow Y$ is an s -mapping if for every $y \in Y$, $f^{-1}(y)$ is separable.

Proposition 3.

- (1) *Monotone meta-Lindelöfness is hereditary for closed subspaces;*
- (2) *monotone meta-Lindelöfness is preserved by open s -mappings.*

Proof. (1) Let the space X be monotonically meta-Lindelöf and r be a monotone meta-Lindelöf operator for X . Suppose that $F \subset X$ is closed. For any open cover \mathscr{U}_F of F , there exists a family \mathscr{U} of open subsets of X such that $\mathscr{U}_F = \{U \cap F : U \in \mathscr{U}\}$. Put $\mathscr{U}' = \{U \cup (X - F) : U \in \mathscr{U}\}$ and $r_F(\mathscr{U}_F) = \{W \cap F : W \in r(\mathscr{U}')\}$, then r_F is a monotone meta-Lindelöf operator for F .

(2) Let $f: X \rightarrow Y$ be an open s -mapping, X be monotonically meta-Lindelöf and r_X be a monotone meta-Lindelöf operator for X . For any open cover \mathscr{U} of Y , put $r_Y(\mathscr{U}) = \{f(W) : W \in r_X(f^{-1}(\mathscr{U}))\}$. For any $y \in Y$, since $f^{-1}(y)$ is separable and $r_X(f^{-1}(\mathscr{U}))$ is point-countable, $\{G \in r_X(f^{-1}(\mathscr{U})) : G \cap f^{-1}(y) \neq \emptyset\}$ is countable. So $r_Y(\mathscr{U})$ is a point-countable open refinement of \mathscr{U} . Clearly r_Y is a monotone meta-Lindelöf operator for the space Y . □

Remark 2.

- (1) *Monotone meta-Lindelöfness is not hereditary for open subspaces:* the space X in Example 3 has an open subspace $[0, \omega_1) \times \{0\}$ homomorphic to the space $[0, \omega_1)$ which is countably compact but not compact. By Theorem 9.2 of [1], $[0, \omega_1) \times \{0\}$ is not meta-Lindelöf and thus not monotonically meta-Lindelöf.
- (2) *Monotone meta-Lindelöfness is not preserved by open mappings:* the first countable T_0 -space $[0, \omega_1)$ is an image of a metric space X under an open mapping (see 4.2.D of [6]). Since the metric space X has a point-countable base, X is monotonically meta-Lindelöf, but $[0, \omega_1)$ is not.
- (3) *Separable (hence countable) monotone meta-Lindelöf spaces are monotone Lindelöf:* this follows the fact that in separable spaces, point-countable family of open sets is countable.
- (4) *Monotone meta-Lindelöfness is not productive:* the Sorgenfrey line S (the real line with half-open intervals of the form $[a, b)$ as a basis for the topology) is a

separable GO -space. By Proposition 3.1 of [3], S is monotonically Lindelöf (so monotonically meta-Lindelöf). However, $S \times S$ is not Lindelöf since it has a closed non-Lindelöf subspace $\{\langle x, -x \rangle : x \in S\}$. Since $S \times S$ is separable it is not meta-Lindelöf and thus not monotonically meta-Lindelöf.

Example 4. The preimage of a monotone meta-Lindelöf space under a perfect mapping need not to be monotonically meta-Lindelöf.

Proof. Let $X = [0, \omega_1] \times [0, 1]$ and $p: X \rightarrow [0, 1]$ be the projection onto the second coordinate. Clearly f is perfect. By Proposition 1, $[0, 1]$ is monotonically meta-Lindelöf. Since X has a closed subspace $[0, \omega_1] \times \{0\}$ homomorphic to $[0, \omega_1]$ which is not monotonically meta-Lindelöf (see Proposition 2), X is not monotonically meta-Lindelöf. \square

By Proposition 2, the compact $LOTS$ $[0, \omega_1]$ is not monotonically meta-Lindelöf. We will show that there exists a compact space Y which is neither monotonically meta-Lindelöf nor a GO -space (so not a $LOTS$).

Proposition 4. Let $Y = X \cup \{p\}$ ($p \notin X$) be the one-point compactification of the discrete space X of cardinality of ω_1 . Then

- (1) Y is not monotonically meta-Lindelöf;
- (2) Y is not a GO -space.

Proof. (1) Assume that Y is monotonically meta-Lindelöf and r is a monotone meta-Lindelöf operator. Then the open cover $\mathcal{U}_0 = \{Y \setminus \{x\} : x \in X\}$ of Y has its point-countable refinement $r(\mathcal{U}_0)$.

Put $\mathcal{U}'_0 = \{U \in \mathcal{U}_0 : \exists V \in r(\mathcal{U}_0) \text{ such that } p \in V \subset U\}$, then \mathcal{U}'_0 is countable since $Y \setminus V$ is finite with $p \in V \in r(\mathcal{U}_0)$. Obviously $\mathcal{U}_1 = \mathcal{U}_0 \setminus \mathcal{U}'_0$ is still an open cover of Y .

Put $\mathcal{U}'_1 = \{U \in \mathcal{U}_1 : \exists V \in r(\mathcal{U}_1) \text{ such that } p \in V \subset U\}$, then \mathcal{U}'_1 is countable and $\mathcal{U}_2 = \mathcal{U}_1 \setminus \mathcal{U}'_1$ is an open cover of Y .

Suppose that for each $i < \omega$ we have obtained an open cover \mathcal{U}_i of Y and countable $\mathcal{U}'_i \subset \mathcal{U}_i$ with $\mathcal{U}_{i+1} \prec \mathcal{U}_i$ and $\mathcal{U}'_i \cap \mathcal{U}'_j = \emptyset$ for $i \neq j$. Put $\mathcal{U}_\omega = \mathcal{U}_0 \setminus \bigcup \{\mathcal{U}'_i : i < \omega\}$, then for each $i < \omega$ the open cover \mathcal{U}_ω of Y refines \mathcal{U}_i . Thus $r(\mathcal{U}_\omega) \prec r(\mathcal{U}_i)$. So for each $i < \omega$, we can take $V \in r(\mathcal{U}_\omega)$, $V_i \in r(\mathcal{U}_i)$ and $U_i \in \mathcal{U}'_i$ such that $p \in V \subset V_i \subset U_i$. This contradicts the finiteness of $Y \setminus V$.

(2) Assume that Y is a GO -space. Then it is easy to see the compact GO -space Y is a $LOTS$. Let \prec be the linear order on Y . Note that p has no countable neighborhood base.

Similar to Corollary 1, we can take a closed subspace $F = \{y_\alpha \prec p : \alpha \in [0, \omega_1]\}$ of Y , where for each limit ordinal $\gamma \in [0, \omega_1]$, $y_\gamma = \sup\{y_\alpha : \alpha < \gamma\}$, and $y_\alpha \prec y_\beta$

whenever $\alpha < \beta$, such that F is homomorphic to $[0, \omega_1]$. Obviously $y_{\omega_1} \preceq p$. If $y_{\omega_1} \prec p$, then $U = \{y \in Y : y_{\omega_1} \prec y\} \ni p$ is open and $Y \setminus U$ is infinite, a contradiction. If $y_{\omega_1} = p$, take a limit ordinal α with $0 < \alpha < \omega_1$. Then $U = \{y \in Y : y_\alpha \prec y\} \ni p$ is open and $Y \setminus U$ is infinite, a contradiction. \square

3. LINEARLY ORDERED EXTENSIONS OF MONOTONE META-LINDELÖF GO -SPACES

Lemma 1. *For a GO -space X , the following are equivalent:*

- (1) X is monotonically meta-Lindelöf;
- (2) each open cover \mathcal{U} of X by convex sets has a point-countable open refinement $r(\mathcal{U})$ such that if \mathcal{U} and \mathcal{V} are open covers of X by convex sets and $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$;
- (3) same as (2), but each member of $r(\mathcal{U})$ is a convex set.

Proof. Note that any non-empty subset G of the GO -space X can be uniquely represented as $G = \bigcup\{S_i : i \in I\}$, where each S_i is a convex component of G and if the set G is open, then each S_i is open. Moreover, if $G \subset G'$, where $G' = \bigcup\{S'_i : i \in I'\}$ and $\{S'_i : i \in I'\}$ is the set of all convex components of G' , then $\{S_i : i \in I\} \prec \{S'_i : i \in I'\}$. Then the proof is obvious. \square

Let X be a GO -space with the topology τ and λ be the usual open interval topology on X . Put

$$(\dagger) \quad R = \{x \in X : [x, \rightarrow) \in \tau \setminus \lambda\} \quad \text{and} \quad L = \{x \in X : (\leftarrow, x] \in \tau \setminus \lambda\}.$$

Define $X^* \subset X \times \mathbb{Z}$ as follows:

$$X^* = (X \times \{0\}) \cup (R \times \{k \in \mathbb{Z} : k < 0\}) \cup (L \times \{k \in \mathbb{Z} : k > 0\}).$$

Let X^* have the open interval topology generated by the lexicographical order. Then $e : X \rightarrow X^*$ defined by $e(x) = \langle x, 0 \rangle$ is an order-preserving homeomorphism from X onto the closed subspace $X \times \{0\}$ of X^* . So the space X^* is a closed linearly ordered extension of X .

It is well known that if \mathcal{P} is paracompactness (respectively, metrizability, Lindelöfness or quasi-developability), then a GO -space X has \mathcal{P} if and only if its closed linearly ordered extension X^* has \mathcal{P} . Now we have

Proposition 5. *For a GO-space X , the following are equivalent:*

- (1) X is monotonically meta-Lindelöf;
- (2) the closed linearly ordered extension X^* of X is monotonically meta-Lindelöf.

Proof. (2) \Rightarrow (1). By Proposition 3 the closed subspace $X \times \{0\}$ of X^* is monotonically meta-Lindelöf. So X is monotonically meta-Lindelöf.

(1) \Rightarrow (2). We will identify X with the subspace $X \times \{0\}$ of X^* .

Let \mathcal{U} be an open cover of X^* by convex sets. Then $\mathcal{U}_X = \{U \cap X : U \in \mathcal{U}\}$ is an open cover of X by convex sets. By Lemma 1, \mathcal{U}_X has point-countable open refinement $r_X(\mathcal{U}_X)$ consisting of convex sets of X , where r_X is a monotone meta-Lindelöf operator for X . For a convex set S of X , put

$$I(S) = \{x \in S : \exists a, b \in S \text{ with } a < x < b\},$$

$$S^\sim = \{\langle x, k \rangle \in X^* : x \in I(S)\} \cup \{\langle x, 0 \rangle : x \in S \setminus I(S)\}$$

and

$$\mathcal{S}^\sim = \{S^\sim : S \in r_X(\mathcal{U}_X)\}.$$

For any $S^\sim \in \mathcal{S}^\sim$ with $S \in r_X(\mathcal{U}_X)$, there exists a $U \in \mathcal{U}$ such that $S \subset U$. Since S is an open convex set and $U \subset X^*$ is convex, S^\sim is open and $S^\sim \subset U$ (see Lemma 3.2 (b), (c) of [10]).

Let $r(\mathcal{U}) = \mathcal{S}^\sim \cup \{\{\langle x, k \rangle\} : \langle x, k \rangle \in X^* \setminus X\}$. Since $r_X(\mathcal{U}_X)$ is point-countable and each $\{\langle x, k \rangle\}$ with $k \neq 0$ is open, $r(\mathcal{U})$ is a point-countable open cover of X^* refining \mathcal{U} . If \mathcal{U} and \mathcal{V} are open covers of X^* by convex sets and $\mathcal{U} \prec \mathcal{V}$, then $r_X(\mathcal{U}_X) \prec r_X(\mathcal{V}_X)$. For any $S \in r_X(\mathcal{U}_X)$, there exists a $T \in r_X(\mathcal{V}_X)$ such that the convex sets S and T satisfy $S \subset T$ and thus by Lemma 3.2 (a) of [10] $S^\sim \subset T^\sim$. So $r(\mathcal{U}) \prec r(\mathcal{V})$. By Lemma 1, X^* is monotonically meta-Lindelöf. \square

If \mathcal{P} is “a continuous separating family”, then the Michael line M and the Sorgenfrey line S have \mathcal{P} ([8]), M^* has \mathcal{P} ([2]), but S^* does not have \mathcal{P} ([2], [12]). For comparison we have

Corollary 3. *For the Sorgenfrey line S and the Michael line M , their closed linearly ordered extensions S^* and M^* are monotonically meta-Lindelöf.*

For a GO-space X , let R and L be defined as in (†). Put

$$\ell(X) = (X \times \{0\}) \cup (R \times \{-1\}) \cup (L \times \{1\}).$$

Equip $\ell(X)$ with the open interval topology generated by the lexicographical order. Then the space $\ell(X)$ has a dense subspace $X \times \{0\}$ which is homeomorphic to the space X . So the space $\ell(X)$ is a dense linearly ordered extension of X .

Example 5. There exists a monotone meta-Lindelöf GO -space X for which its dense linearly ordered extension $\ell(X)$ is not monotonically meta-Lindelöf.

Proof. Define a topology on the linearly ordered set $X = [0, \omega_1]$ with a base as follows: points of $[0, \omega_1]$ are isolated and ω_1 has the neighborhoods of the form $[\alpha, \omega_1]$, $\alpha < \omega_1$. For an open cover \mathcal{U} of X , put $\alpha_{\mathcal{U}} = \min\{\alpha: [\alpha, \omega_1] \subset U \text{ for some } U \in \mathcal{U}\}$ and $r(\mathcal{U}) = \{[\alpha_{\mathcal{U}}, \omega_1]\} \cup \{\{\beta\}: \beta < \alpha_{\mathcal{U}}\}$. Then r is a monotone meta-Lindelöf operator for the GO -space X .

Note that $\ell(X)$ can actually be constructed from $[0, \omega_1]$ by inserting a predecessor $\langle \alpha, -1 \rangle$ at each limit ordinal α less than ω_1 . These inserted predecessors in $\ell(X)$ play the role of the limit ordinals in $[0, \omega_1]$. So it is clear that $\ell(X)$ is homeomorphic to the space $[0, \omega_1]$ which is not monotonically meta-Lindelöf by Proposition 2. Hence $\ell(X)$ is not monotonically meta-Lindelöf. \square

Note that for the Michael line M , the space $\ell(M) = (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \{-1, 1\})$ with the open interval topology generated by the lexicographical order. Let

$$(\ddagger) \quad M_1 = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{1\}) \quad \text{and} \quad M_{-1} = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{-1\})$$

be subspaces of $\ell(M)$.

Lemma 2. *Let M_1 and M_{-1} be the subspaces of $\ell(M)$ defined in (\ddagger) . Then for any open convex set S of M_1 , there exists a minimal interval I_S of $\ell(M)$ such that $S = I_S \cap M_1$. For any open convex set S of M_{-1} , an analogous conclusion holds.*

Proof. For an open convex set S of M_1 , S must be one of the following six intervals of M_1 (for $x, y \in M_1$, by $[x, y)_{M_1}$ or $(x, y)_{M_1}$ we mean an interval of M_1 with endpoints x and y).

- (1) $S = [\langle p, 1 \rangle, \langle p', 1 \rangle)_{M_1}$, $p, p' \in \mathbb{P}$;
- (2) $S = [\langle p, 1 \rangle, \langle q', 0 \rangle)_{M_1}$, $p \in \mathbb{P}$, $q' \in \mathbb{Q}$;
- (3) $S = (\langle p, 1 \rangle, \langle p', 1 \rangle)_{M_1}$, $p, p' \in \mathbb{P}$;
- (4) $S = (\langle p, 1 \rangle, \langle q', 0 \rangle)_{M_1}$, $p \in \mathbb{P}$, $q' \in \mathbb{Q}$;
- (5) $S = (\langle q, 0 \rangle, \langle p', 1 \rangle)_{M_1}$, $q \in \mathbb{Q}$, $p' \in \mathbb{P}$;
- (6) $S = (\langle q, 0 \rangle, \langle q', 0 \rangle)_{M_1}$, $q, q' \in \mathbb{Q}$.

Correspondingly, take the minimal interval I_S of $\ell(M)$ such that $S = I_S \cap M_1$ as follows.

- (1) $I_S = [\langle p, 1 \rangle, \langle p', -1 \rangle)$ for case (1);
- (2) $I_S = [\langle p, 1 \rangle, \langle q', 0 \rangle)$ for case (2);
- (3) $I_S = (\langle p, 1 \rangle, \langle p', -1 \rangle)$ for case (3);
- (4) $I_S = (\langle p, 1 \rangle, \langle q', 0 \rangle)$ for case (4);

- (5) $I_S = (\langle p, 1 \rangle, \langle p', -1 \rangle)$ for case (5);
- (6) $I_S = (\langle q, 0 \rangle, \langle q', 0 \rangle)$ for case (6).

Obviously, for any open convex set S of M_{-1} , an analogous conclusion holds. \square

Proposition 6. *For the Sorgenfrey line S and the Michael line M , their dense linearly ordered extensions $\ell(S)$ and $\ell(M)$ are monotonically meta-Lindelöf.*

Proof. Note that the space $\ell(S)$ is the set $\mathbb{R} \times \{-1, 0\}$ equipped with the open interval topology generated by the lexicographical order. Clearly $\ell(S)$ has a countable dense subset $\mathbb{Q} \times \{0\}$. So by Proposition 3.1 of [3] the separable space $\ell(S)$ is monotonically Lindelöf and thus monotonically meta-Lindelöf.

To show that the space $\ell(M) = (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \{-1, 1\})$ is monotonically meta-Lindelöf, let the space M_r be \mathbb{R} with the topology defined as follows:

- Each $q \in \mathbb{Q}$ has a neighborhood base consisting of the usual open intervals;
- each $p \in \mathbb{P}$ has a neighborhood base consisting of the sets $[p, x)$, $x \in \mathbb{R}$.

Clearly the *GO*-space M_r is separable. So by Proposition 3.1 of [3], M_r is monotonically meta-Lindelöf. It is obvious that the subspace $M_1 = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{1\})$ of $\ell(M)$ is homeomorphic to M_r . So the space M_1 is monotonically meta-Lindelöf.

Similarly, let the space M_l be \mathbb{R} equipped with the topology: each $q \in \mathbb{Q}$ has a neighborhood base consisting of the usual open intervals and each $p \in \mathbb{P}$ has a neighborhood base consisting of the sets $(x, p]$, $x \in \mathbb{R}$. Then the *GO*-space M_l is monotonically meta-Lindelöf and M_l is homeomorphic to the subspace $M_{-1} = (\mathbb{Q} \times \{0\}) \cup (\mathbb{P} \times \{-1\})$ of $\ell(M)$.

Let \mathcal{U} be an open cover of $\ell(M)$ by convex sets. Then $\mathcal{U}_1 = \{U \cap M_1 : U \in \mathcal{U}\}$ is an open cover of M_1 by convex sets. By Lemma 1, \mathcal{U}_1 has a point-countable open refinement $r_1(\mathcal{U}_1)$ by convex sets, where r_1 is a monotone meta-Lindelöf operator for M_1 .

Similarly the open cover $\mathcal{U}_{-1} = \{U \cap M_{-1} : U \in \mathcal{U}\}$ of M_{-1} by convex sets has a point-countable open refinement $r_{-1}(\mathcal{U}_{-1})$ by convex sets, where r_{-1} is a monotone meta-Lindelöf operator for M_{-1} .

For any $S \in r_1(\mathcal{U}_1)$, there exists a $U \in \mathcal{U}$ such that $S \subset U \cap M_1 \in \mathcal{U}_1$. Since S is an open convex set of M_1 , by Lemma 2 there exists a minimal interval I_S of $\ell(M)$ such that $S = I_S \cap M_1$.

Claim. $I_S \subset U$.

Proof. Let $x \in I_S$. If $x \in I_S \cap M_1$, then $x \in U \cap M_1 \subset U$; if $x \in I_S \setminus M_1$, then $x = \langle p_0, 0 \rangle$ or $x = \langle p_0, -1 \rangle$, where $p_0 \in \mathbb{P}$, and there exist $q_1, q_2 \in \mathbb{Q}$ with $q_1 < q_2$ such that $x \in (\langle q_1, 0 \rangle, \langle q_2, 0 \rangle)$ and $\langle q_1, 0 \rangle, \langle q_2, 0 \rangle \in I_S \cap M_1$. So the points $\langle q_1, 0 \rangle$ and

$\langle q_2, 0 \rangle$ belong to U . Since U is a convex set of $\ell(M)$ we know that $x \in U$. Thus $I_S \subset U$.

Put $\mathcal{S}_1 = \{I_S : S \in r_1(\mathcal{U}_1)\}$. Then the cover \mathcal{S}_1 of M_1 by open convex sets of $\ell(M)$ refines \mathcal{U} . By the point-countability of $r_1(\mathcal{U}_1)$, \mathcal{S}_1 is point-countable.

Similarly, we can obtain a point-countable open cover \mathcal{S}_{-1} of M_{-1} by convex sets of $\ell(M)$ refining \mathcal{U} . Put

$$r(\mathcal{U}) = \mathcal{S}_1 \cup \mathcal{S}_{-1} \cup \{\{ \langle p, 0 \rangle \} : p \in \mathbb{P}\}.$$

Then $r(\mathcal{U})$ is a point-countable open refinement of \mathcal{U} by convex sets and r is a monotone meta-Lindelöf operator for $\ell(M)$. By Lemma 1 $\ell(M)$ is monotonically meta-Lindelöf. \square

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