Belmesnaoui Aqzzouz; Redouane Nouira

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THE ORDER σ-COMPLETE VECTOR LATTICE OF AM-COMPACT OPERATORS

Belmesnaoui Aqzzouz, Sala Eljadida, and Redouane Nouira, Kénitra

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Abstract. We establish necessary and sufficient conditions under which the linear span of positive AM-compact operators (in the sense of Fremlin) from a Banach lattice E into a Banach lattice F is an order σ-complete vector lattice.

Keywords: AM-compact operator, order continuous norm, discrete vector lattice

MSC 2010: 46A40, 46B40, 46B42

1. Introduction and notations

A Banach lattice G is said to be a KB-space if every monotone sequence in the unit ball of G is convergent. Note that every KB-space has an order continuous norm (cf. [9], Theorem 2.4.2), but the norm of the Banach lattice c_0 is order continuous without the space being a KB-space. In [4], Aliprantis and Burkinshaw established that each weakly compact operator from an AL-space into a KB-space has a weakly compact modulus. This result was generalized by Chen and Wickstead [6] by proving that if E is an AL-space or F is an AM-space, then the space of weakly compact operators from E into F is a vector lattice. As a consequence, they obtained that the linear span of positive weakly compact operators forms an order complete vector lattice.

On the other hand, Wickstead gave several necessary and sufficient conditions under which the linear span of positive compact operators is an order σ-complete vector lattice ([11], Theorem 2.1). In the same way, Chen and Wickstead [6] studied the order structure of the linear span of positive weakly compact (resp. positive Dunford-Pettis) operators between Banach lattices and they proved analogous results to those for compact operators ([6], Theorem 3.5) (resp. ([6], Theorem 3.8)). Also, they gave an example of a compact and a compactly dominated operator whose modulus is not weakly compact. However, this modulus is Dunford-Pettis and AM-compact. We claim that we can construct one such that the modulus is neither AM-compact nor Dunford-Pettis.

It is natural to consider the corresponding problem for AM-compact operators between Banach lattices. Recall that the class of such operators was introduced by Fremlin in [7]. A regular operator T from a vector lattice E into a Banach lattice F is said to be AM-compact if it carries each order bounded subset of E onto a relatively compact subset of F. It is easy to see that each regular compact operator between two Banach lattices is AM-compact but the converse is false in general. In fact, the identity operator of the Banach lattice l^1 is AM-compact but it is not compact. Whenever E is an AM-space with unit, the class of AM-compact operators on E coincides with that of regular compact operators on E.

In the same vein as the papers [11] and [6], we give an example (Theorem 2.1) which shows the existence of an order complete Banach lattice E and a compact and compactly dominated operator T from E into an order complete Banach lattice F such that the modulus of T is neither AM-compact nor Dunford-Pettis. This justifies the claim of Wickstead and Chen announced in ([6], p. 405). Next, we characterize Banach lattices E and F such that the linear span of positive AM-compact operators, that we denote by $\mathcal{AM}^r(E,F)$, is an order σ-complete vector lattice (Theorem 2.3).

To state our results we need to fix some notation and recall some definitions that will be used in this paper. A vector lattice E is order σ-complete if every majorized countable nonempty subset of E has a supremum. Let E be a vector lattice, then for any two elements $x, y \in E$ with $x \leqslant y$, the set $[x, y] = \{z \in E\colon\ x \leqslant z \leqslant y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. An order ideal B is a solid subspace of a vector lattice E i.e. if $x \in B$ and $y \in E$ is such that $|y| \leqslant |x|$, then $y \in B$. A principal ideal is any order ideal generated by a subset containing only one element x; this order ideal will be denoted by I_x. A generalized sequence (x_α) is order convergent to $x \in E$ if there exists a generalized sequence (y_α) such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leqslant y_\alpha$ for each α, where the notation $y_\alpha \downarrow 0$ means that the sequence (y_α) is decreasing, its infimum exists and $\inf(y_\alpha) = 0$. A band is an order ideal which is order closed. The band generated by an element x is called a principal band that we denote by B_x.

A Banach lattice is a Banach space $(E, \| \|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. Finally, the topological dual E' of a Banach lattice E, endowed with the dual norm, is a Banach lattice. For terminology which is not explained, we refer the reader to the book of Zaanen [13].

2. Main results

For each vector lattice E, we define $E^+ = \{x \in E\colon\ 0 \leqslant x\}$. If E and F are two vector lattices, a linear mapping T from E into F is said to be positive if $T(x) \in F^+$ whenever $x \in E^+$. It is well known that each positive linear mapping from a Banach lattice into a normed vector lattice is continuous.

By operator we mean a bounded linear mapping from a Banach lattice E; into a Banach lattice F. Recall that Krengel [8] constructed two operators S and $T\colon\ l^2 \longrightarrow l^2$ such that T is compact but T is not an element of the linear span of positive operators from l^2; into l^2, and S is an element of the linear span of positive operators from l^2; into l^2 which is compact but its modulus $|S|$ is not compact; (observe that the modulus $|S|$ exists because the vector lattice l^2 is order complete).

To give our example which is a simple modification of an example of Abramovich and Wickstead ([1], Theorem 4 (i)), we need to recall from [1] the following operators and equalities:

Let S_n be an operator on the 2^n-dimensional Euclidean space $l_2^{2^n}$ such that $\|S_n\| = 1$ and $\|\!|S_n|\!\| = 2^n$ and let J_n be the operator which embeds $l_2^{2^n}$ into $L^2[0,1]$ defined by the following formula:

$$J_n(x_1, x_2, x_3, \ldots, x_{2^n}) = \sum_{k=1}^{2^n} x_k \chi_{[(k-1)/2^n, k/2^n]}$$

Also, we will need the operator Q_n defined in [1] as an operator from $L^2([0,1])$ into $J_n(l_2^{2^n})$ by the following formula:

$$Q_n(f) = 2^n \sum_{k=1}^{2^n} \biggl(\int_{(k-1)/2^n}^{k/2^n} f \,\mathrm{d}\mu \biggr) \chi_{[(k-1)/2^n, k/2^n]}$$

where μ denotes the Lebesgue measure on $[0,1]$.

In [1] and [6], it has been established that for each $n \in \mathbb{N}$, we have

$$\|J_n \circ S_n\| = 1 \text{ and } \|\!|J_n \circ S_n|\!\| = 2^{n/2}$$

and that

$$|J_n| \circ |S_n|(x_k) = J_n \circ |S_n|(x_k) = \biggl(\sum_{k=1}^{2^n} x_k \biggr) \chi_{[0,1]} = 2^{n/2}(x_k)(v_n)\chi_{[0,1]}$$

where $v_n = 2^{-n/2}(1, 1, \ldots, 1) \in l_2^{2^n}$, so that $\|v_n\|_2 = 1$.

Also, they proved that

$$\|J_n \circ S_n \circ J_n^{-1} \circ Q_n\| = 2^{n/2}$$

while

$$\||J_n \circ S_n \circ J_n^{-1} \circ Q_n\|| = \|J_n \circ |S_n| \circ J_n^{-1} \circ Q_n\| = 2^n$$

In ([6], p. 405), Chen and Wickstead claimed that there exists a compactly dominated compact operator with a modulus which is neither Dunford-Pettis nor AM-compact. The following Theorem confirms their claim.

Theorem 2.1. *There exist a Banach lattice E, an order complete Banach lattice F and two compact operators S and T from E into Fsuch that $S < T$ and $-S < T$ but the modulus of S is neither AM-compact nor Dunford-Pettis.*

Proof. Let (r_n) be the sequence of Rademacher functions on $[0,1]$ defined by

$$r_n(x) = \mathrm{Sign}(\sin(2^n \pi x)) \quad \text{for each} \quad x \in [0,1].$$

Consider the Banach lattices $E = L^2([0,1])$ and $F = l^\infty(L^2([0,1]))$. Define two operators S and T from E into F by the following formulas:

$$S(f) = (2^{-n} J_n \circ S_n \circ J_n^{-1} \circ Q_n(f r_n^+))_{n=1}^{+\infty}$$

and

$$T(f) = \left(\left(\int_0^1 f(x)\,\mathrm{d}x \right) \cdot \chi_{[0,1]} \right)_{n=1}^{+\infty}.$$

A simple check shows that S and T are compact, $-S$, $S \leqslant T$ and that

$$|S|(f) = \left(\left(\int_0^1 f(x) r_n^+\,\mathrm{d}x \right) \chi_{[0,1]} \right)_{n=1}^{+\infty}.$$

It is well known that $(r_m^+) \subset [0, \chi_{[0,1]}]$, and that (r_m^+) converges weakly to $\frac{1}{2}$. We have just to prove that $(|S|(r_m^+))$ does not admit any convergent subsequence.

Observe that

$$|S|(r_m^+) = \left(\left(\int_0^1 r_m^+ \cdot r_n^+\,\mathrm{d}x \right) \cdot \chi_{[0,1]} \right)_{n=1}^{+\infty}.$$

If such sequence converges, then its limit l is given by

$$l = \left(\left(\frac{1}{2} \int_0^1 r_n^+\,\mathrm{d}x \right) \cdot \chi_{[0,1]} \right)_{n=1}^{+\infty}.$$

Since

$$|S|(r_m^+) - l = \left(\left(\int_0^1 \left(r_m^+ \cdot r_n^+ - \frac{1}{2} r_n^+\right) \mathrm{d}x\right) \cdot \chi_{[0,1]}\right)_{n=1}^{+\infty}$$

for $n = m$, we have

$$\int_0^1 \left(r_m^+ \cdot r_n^+ - \frac{1}{2} r_n^+\right) \mathrm{d}x = \frac{1}{2} \int_0^1 r_n^+ \,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2^m}\right) \geqslant \frac{1}{4}.$$

This shows that $(|S|(r_m^+))$ does not admit any convergent subsequence.

The above Theorem proves that the subspace of AM-compact operators is not necessary a vector sublattice. To characterise Banach lattices for which this remains true, we need to recall the following definitions:

A vector lattice equipped with a vector topology is said to be a locally convex solid lattice if zero admits a fundamental system of convex and solid neighborhoods.

If E' is the topological dual of E, the absolute weak topology $|\sigma|(E, E')$ is the locally convex solid topology on E generated by the family of lattice seminorms $\{P_f \colon f \in E'\}$ where $P_f(x) = |f|(|x|)$ for each $x \in E$. Similarly, $|\sigma|(E', E)$ is the locally convex solid topology on E' generated by the family of lattice seminorms $\{P_x \colon x \in E\}$ where $P_x(f) = |f|(|x|)$ for each $f \in E'$. For more information about locally convex solid topologies, we refer the reader to the book of Aliprantis and Burkinshaw [2].

Let us recall that if an operator $T \colon E \longrightarrow F$ between two Banach lattices is positive (i.e. $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E), then its dual operator $T' \colon F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

We will need the following lemma which is a consequence of Grothendieck's Theorem ([10], Theorem 3, p. 51).

Lemma 2.2. *Let E and F be two Banach lattices and let $T \colon E \longrightarrow F$ be an operator. Then for each $x \in E^+$, $T([-x, x])$ is norm precompact in F if and only if $T'(B_{F'})$ is precompact for $|\sigma|(E', E)$ in E', where $B_{F'}$ is the closed unit ball of the topological dual F' of F.*

Recall that the norm of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges in norm to 0. For example, the norm of the Banach lattice l^1 is order continuous but the norm of the Banach lattice l^∞ is not.

A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the sublattice generated by x. The vector lattice E is discrete if it admits a complete disjoint system of discrete elements. For example, the Banach lattice l^1 is discrete but $C([0, 1])$ is not.

Now, we are in position to establish our principal result.

Theorem 2.3. *For Banach lattices E and F the following assertions are equivalent:*

1. *One of the following two conditions holds:*
   a) *F has an order continuous norm.*
   b) *The topological dual E' is discrete and F is order σ-complete.*
2. *The vector lattice F is order σ-complete and if $0 \leqslant S \leqslant T$ with $T \in \mathcal{AM}^r(E,F)$, then $S \in \mathcal{AM}^r(E,F)$.*
3. *$\mathcal{AM}^r(E,F)$ is an order σ-complete vector lattice.*
4. *Any increasing order bounded sequence in $\mathcal{AM}^r(E,F)$ has a supremum.*

Proof. $1 \Longrightarrow 2$. The implication $a \Longrightarrow \mathbf{2}$ is just a theorem of Fremlin [7].

For the implication $b \Longrightarrow \mathbf{2}$, let S and T be operators from E into F such that $0 \leqslant S \leqslant T$ and T is AM-compact. Then for each $x \in E^+$, $T([0,x])$ is norm precompact in F, and hence $T'(B_{E'})$ is precompact for the weak absolute topology $|\sigma|(E',E)$ (Lemma 2.2). Since $0 \leqslant S' \leqslant T'$, it results from Theorem 3.1.b of [5] that $S'(B_{F'})$ is also precompact for the weak absolute topology $|\sigma|(E',E)$. A second application of Lemma 2.2 gives the result.

The implications $2 \Longrightarrow 3$ and $3 \Longrightarrow 4$ are clear.

It remains to prove the implication $4 \Longrightarrow 1$. First, observe that by the same proof as Theorem 2.1 of Wickstead [11], we can show that the vector lattice F is σ-order complete. Now, aiming at contradiction, we assume that E is not discrete and that the norm of F is not order continuous. It follows from the proof of Theorem 1 of Wickstead [12] that F contains a sublattice H which is isomorphic to l^∞ and there exists a positive projection P from F into H. We denote by (e_n) the family of discrete elements of H and we put $e = \sup\{e_n \colon n \in \mathbb{N}\}$ and denote by P_n the projection of H onto the principal band generated by e_n in H.

It follows from Corollary 21.13 of [2], that there exist $\Phi \in (E')^+$ and a sequence (Φ_n) in $[-\Phi,\Phi]$ such that (Φ_n) converges to 0 for the weak topology $\sigma(E',E)$ but does not converge to 0 for the absolute weak topology $|\sigma|(E',E)$.

By passing to a subsequence, we can find $\alpha \in \mathbb{R}^+$ and y_n in $[0,y]$ such that

$$|\Psi_n(y_n)| > \alpha,\ |\Psi_n(y_k)| < \frac{\alpha}{k} \ \text{ and } \ |\Psi(y_k - y_n)| < \frac{\alpha}{4} \ \text{ for each } k < n.$$

Consider the operators S_n and T defined from E into F by the following formulas:

$$S_n(x) = \sum_{k=1}^{n} (\Phi + \Phi_k)(x) \cdot e_n$$

and

$$T(x) = 2\Phi(x)e \text{ for each } x \in E.$$

Clearly $0 \leqslant S_n \uparrow\leqslant T$ and $T \in \mathcal{AM}^r(E,F)$. Assume that the sequence (S_n) admits an operator S as a supremum in $\mathcal{AM}^r(E,F)$. It is evident to see that $P \circ S \in \mathcal{AM}^r(E,H)$ and that $0 \leqslant S_n \uparrow P \circ S$ in $\mathcal{AM}^r(E,H)$.

Moreover,

$$P_n \circ P \circ S_n \leqslant P_n \circ P \circ S \text{ for each } n \in \mathbb{N}^*.$$

We claim that we have equality. In fact, if not, there exists some $n_1 \in \mathbb{N}^*$ such that

$$P_{n_1} \circ P \circ S_{n_1} \lneqq P_{n_1} \circ P \circ S,$$

and in this case we have

$$S_n \leqslant P \circ S - (P_{n_1} \circ P \circ S - P_{n_1} \circ P \circ S_{n_1}) \lneqq P \circ S \text{ for each } n \in \mathbb{N}^*.$$

Since

$$P \circ S - (P_{n_1} \circ P \circ S - P_{n_1} \circ P \circ S_{n_1}) \in \mathcal{AM}^r(E,H),$$

the above inequality gives a contradiction because $P \circ S$ is a supremum of the sequence (S_n). Hence, for each $n \in \mathbb{N}^*$, we have

$$P_n \circ P \circ S_n = P_n \circ P \circ S.$$

On the other hand, since $P \circ S \in \mathcal{AM}^r(E,H)$, the sequence $(P \circ S(y_k))_{k \geqslant 0}$ admits a convergent subsequence, and then it is Cauchy in H. But for each $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$ such that $4 \leqslant k < n$ we have

$$\frac{\alpha}{2} \leqslant \alpha - \frac{\alpha}{n} - \frac{\alpha}{4} \leqslant |P_n \circ P \circ S(y_n - y_k)| \leqslant |P \circ S(y_n) - P \circ S(y_k)|.$$

This proves that our subsequence cannot be Cauchy. We obtain a contradiction.

If in Theorem 2.2 we assume F is order complete, by an analogous proof, we obtain the following result:

Theorem 2.4. *For Banach lattices E and F the following assertions are equivalent:*

1. *One of the following two conditions holds:*
   a) *F has an order continuous norm.*
   b) *The topological dual E' is discrete and F is order complete.*
2. *The vector lattice F is order complete and if $0 \leqslant S \leqslant T$ with $T \in \mathcal{AM}^r(E,F)$, then $S \in \mathcal{AM}^r(E,F)$.*
3. *$\mathcal{AM}^r(E,F)$ is an order complete vector lattice.*
4. *Any increasing order bounded generalized sequence in $\mathcal{AM}^r(E,F)$ has a supremum.*

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Authors' addresses: Belmesnaoui Aqzzouz, Université Mohammed V-Souissi, Faculté des Sciences Economiques, Juridiques et Sociales, Département d'Economie, B. P. 5295, Sala Eljadida, Morocco, e-mail: `baqzzouz@hotmail.com`; Redouane Nouira, Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, B.P. 133, Kénitra, Morocco.