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A CLASS OF BANACH SEQUENCE SPACES ANALOGOUS TO THE SPACE OF POPOV

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Abstract. Hagler and the first named author introduced a class of hereditarily l_1 Banach spaces which do not possess the Schur property. Then the first author extended these spaces to a class of hereditarily l_p Banach spaces for $1 \leqslant p < \infty$. Here we use these spaces to introduce a new class of hereditarily $l_p(c_0)$ Banach spaces analogous of the space of Popov. In particular, for $p = 1$ the spaces are further examples of hereditarily l_1 Banach spaces failing the Schur property.

Keywords: Banach spaces, Schur property, hereditarily l_p

MSC 2010: 46B20, 46E30

1. Introduction

A class of hereditarily l_1 Banach spaces was introduced by Hagler and the first named author. Among other interesting properties it does not possess the Schur property [2]. Then these spaces were extended to a new class of hereditarily l_p Banach spaces, $X_{\alpha,p}$ [1]. In 2005, Popov constructed a new class of hereditarily l_1 subspaces of L_1 without the Schur property [5] and generalized his result to a class of hereditarily l_p Banach spaces [6]. In this paper we use the spaces $X_{\alpha,p}$ [1] to introduce and study a new class of hereditarily l_p spaces, analogous of the space of Popov. In particular, we show that for $p = 1$ the spaces are further examples of hereditarily l_1 Banach spaces which do not possess the Schur property. This would be the fourth example of this type. The first was constructed by J. Bourgain [3], the second by Hagler and the first author, and the third by Popov.

Our construction shows that for the case $p = 0$ the spaces are hereditarily c_0.

Before we define these new spaces let us recall the definition of $X_{\alpha,p}$. Let (α_i) be a sequence of reals in $[0,1]$ (whose terms are used as the weighting factor in the definition of the norm) which has the following properties:

(1) $1 = \alpha_1 \geqslant \alpha_2 \geqslant \ldots,$

(2) $\lim\limits_i \alpha_i = 0,$

and

(3) $\sum\limits_{i=1}^{\infty} \alpha_i = \infty.$

By a block F we mean an interval (finite or infinite) of integers. For a block F and a sequence of scalars $x = (t_1, t_2, \ldots)$ such that $\sum\limits_j t_j$ converges, define $\langle x, F\rangle = \sum\limits_{j\in F} t_j$. A sequence $F_1, F_2, \ldots, F_n, \ldots$ where each F_i is a finite block is admissible if

$$\max F_i < \min F_{i+1} \text{ for } i = 1, 2, 3, \ldots$$

For a finitely nonzero sequence of scalars $x = (t_1, t_2, \ldots)$, define

$$\|x\| = \max \left(\sum_{i=1}^{n} \alpha_i |\langle x, F_i\rangle|^p \right)^{1/p},$$

where max is taken over all n, admissible sequences $F_1,\ F_2, \ldots, F_n$ and $1 \leqslant p < \infty$. Then $X_{\alpha,p}$ is the completion of the finitely nonzero sequences of scalars $x = (t_1, t_2, \ldots)$ in this norm. For a good information concerning these spaces, we refer to [1] and [2].

Now we go through the construction of the spaces X_p analogous to the space of Popov. Let α be a fixed sequence and $(X_{\alpha,p_n})_{n=1}^{\infty}$ a sequence of Banach spaces as above with $\infty > p_1 > p_2 > \ldots > 1$. The direct sum of these spaces in the sense of l_p is defined as the linear space

$$X_p = \left(\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n} \right)_p$$

with $p \in [1, \infty)$, which is the space of all sequences $x = (x^1, x^2, \ldots)$, $x^n \in X_{\alpha,p_n}$, $n = 1, 2, \ldots$ with

$$\|x\|_p = \left(\sum_{n=1}^{\infty} \|x^n\|_{\alpha,p_n}^p \right)^{1/p} < \infty.$$

The direct sum of the spaces (X_{α,p_n}) in the sense of c_0 is the linear space

$$X_0 = \left(\sum_{n=1}^{\infty} \oplus X_{\alpha,p_n} \right)_0$$

of all sequences $x = (x^1, x^2, \ldots)$, $x^n \in X_{\alpha,p_n}$, $n = 1, 2, \ldots$ for which $\lim_n \|x^n\|_{\alpha,p_n} = 0$ with the norm

$$\|x\|_0 = \max_n \|x^n\|_{\alpha,p_n}.$$

A Banach space X is hereditarily l_p if every infinite dimensional subspace of X contains a subspace isomorphic to l_p. A Banach space X has the Schur property if the norm convergence and the weak convergence of sequences coincide. It is well known that l_1 has the Schur property.

We follow the same notation and terminology as in [4]. The construction and the idea of the proof follow [6] but the nature of these spaces is different, so for similar results we omit the details of proofs. In fact these spaces are a rich class of spaces which depend on the sequences (α_i) and (p_n) as above.

Fix a sequence (α_i) of reals which satisfies the above conditions, and a sequence (p_n) of reals with $\infty > p_1 > p_2 > \ldots > 1$. Consider the sequence of spaces X_p as above. For each $n \geqslant 1$, denote by $(\overline{e}_{i,n})_{i=1}^{\infty}$ the unit vector basis of X_{α,p_n} and by $(e_{i,n})_{i=1}^{\infty}$ its natural copy in X_p:

$$e_{i,n} = (\underbrace{0, \ldots 0}_{n-1}, \overline{e}_{i,n}, 0, \ldots) \in X_p.$$

Let $\delta_n > 0$ and $\Delta = (\delta_n)$ be such that $\sum\limits_{i=1}^{\infty} \delta_n^p = 1$ if $p \geqslant 1$, and $\lim_n \delta_n = 0$ and $\max_n \delta_n = 1$ if $p = 0$. For each $i \geqslant 1$ put $z_i = \sum\limits_{n=1}^{\infty} \delta_n e_{i,n}$. Then

$$\|z_i\|_p = \left(\sum_{n=1}^{\infty} \|\delta_n e_{i,n}\|_{\alpha,p_n}^p \right)^{1/p} = \left(\sum_{n=1}^{\infty} \delta_n^p \right)^{1/p} = 1.$$

Since $\|e_{i,n}\|_{\alpha,p} = 1$ and

$$\|z_i\|_0 = \max_n \|\delta_n e_{i,n}\|_{\alpha,p_n} = 1,$$

it is clear that for any sequence $(t_i)_{i=1}^m$ of scalars, we have

$$\left\| \sum_{i=1}^{m} t_i z_i \right\|_p^p = \sum_{n=1}^{\infty} \delta_n^p \left\| \sum_{i=1}^{m} t_i e_{i,n} \right\|_{\alpha,p_n}^p \quad \text{if } 1 \leqslant p < \infty$$

and

$$\left\| \sum_{i=1}^{m} t_i z_i \right\|_0 = \max \delta_n \left\| \sum_{i=1}^{m} t_i e_{i,n} \right\|_{\alpha,p_n} \quad \text{if } p = 0.$$

Let Z_p be the closed linear span of $(z_i)_{i=1}^{\infty}$. Here is the main result of this paper:

Theorem 1.1.
(i) *The Banach space Z_p is hereditarily l_p for $p > 1$.*
(ii) *For $p = 1$ the space Z_1 is hereditarily l_1 and does not possess the Schur property.*
(iii) *The space Z_0 is hereditarily c_0.*

2. The results

Before beginning our detailed analysis, we collect some basic facts about our spaces in the following lemmas. For $x \in X_{\alpha,p}$, put $s(x) = \max |\langle x, G\rangle|$ where the max is taken over all blocks G.

Lemma 2.1. *Let $p \geqslant 1$ and let (v_i) be a sequence in $X_{\alpha,p}$, (G_i) an admissible sequence of blocks such that $\{j\colon\ v_i(j) \neq 0\} \subset G_i$, and let*
1. $\|v_i\| \leqslant 2$,
2. $s(v_i) \to 0$.

Then

$$\Bigl\|\sum_{i=1}^{k} t_i v_i\Bigr\|^p \leqslant 2(3)^{p-1} \sum_{i=1}^{k} |t_i|^p.$$

Proof. Since $s(v_i) \to 0$ we have $\lim\limits_{i\to\infty} \langle v_i, \mathbb{N}\rangle = 0$. By passing to a subsequence of (v_i) (not renaming) we may assume that

$$\text{(A)} \qquad \sum_{i=1}^{\infty} |\langle v_i, \mathbb{N}\rangle|^q \leqslant 1.$$

By induction, we show that for any n, and admissible blocks $F_1, F_2, \ldots, F_m$ we have

$$\text{(B)} \qquad \sum_{j=1}^{m} \alpha_j \Bigl|\Bigl\langle \sum_{i=1}^{n} t_i v_i, F_j \Bigr\rangle\Bigr|^p \leqslant 2K \sum_{i=1}^{n-1} |t_i|^p + K|t_n|^p$$

for $K = 3^{p-1}$. Now we assume that (B) is true for all $k \leqslant n-1$, and note that it holds for $k = 1$. Let l be the largest integer for which

$$\operatorname{support}(v_{n-1}) \cap F_l \neq \varphi$$

and suppose that for $i = k, \ldots, n-1$

$$\operatorname{support}(v_i) \cap F_l \neq \varphi,$$

but

$$\operatorname{support}(v_{k-1}) \cap F_l = \varphi.$$

Thus $v_{k+1}, \ldots, v_{n-1}$ are entirely supported in F_l.

Next,

$$\text{(C)}\quad \sum_{j=1}^{m} \alpha_j \left| \left\langle \sum_{i=1}^{n} t_i v_i, F_j \right\rangle \right|^p = \sum_{j=1}^{l-1} \alpha_j \left| \left\langle \sum_{i=1}^{k} t_i v_i, F_j \right\rangle \right|^p + \alpha_l \left| \left\langle \sum_{i=k}^{n} t_i v_i, F_l \right\rangle \right|^p$$
$$+ \sum_{j=l+1}^{m} \alpha_j |\langle t_n v_n, F_j \rangle|^p = \sum_1 + \sum_2 + \sum_3 .$$

We will use the induction hypothesis on $\sum_1$, leave $\sum_3$ basically as it is, and estimate the middle term $\sum_2$.

$$\text{(D)}\qquad \sum_2 = \alpha_l \left| t_k \langle v_k, F_l \rangle + \sum_{i=k+1}^{n-1} \langle t_i v_i, F_l \rangle + t_n \langle v_n, F_l \rangle \right|^p$$
$$\leqslant \alpha_l 3^{p-1} \left[|t_k \langle v_k, F_l \rangle|^p + \left| \sum_{i=k+1}^{n-1} \langle t_i v_i, F_l \rangle \right|^p + |t_n \langle v_n, F_l \rangle|^p \right].$$

We estimate the middle term in (D) by

$$\left| \sum_{i=k+1}^{n-1} \langle t_i v_i, F_l \rangle \right|^p = \left| \sum_{i=k+1}^{n-1} t_i \langle v_i, F_l \rangle \right|^p \leqslant \left(\sum_{i=k+1}^{n-1} |t_i|^p \right) \left(\sum_{i=k+1}^{n-1} |\langle v_i, F_l \rangle|^q \right)^{p/q}$$
$$= \left(\sum_{i=k+1}^{n-1} |t_i|^p \right) \left(\sum_{i=k+1}^{n-1} |\langle v_i, \mathbb{N} \rangle|^q \right)^{p/q} \leqslant \sum_{i=k+1}^{n-1} |t_i|^p$$

by virtue of (A). Returning to (C) we obtain

$$\sum_{j=1}^{m} \alpha_j \left| \left\langle \sum_{i=1}^{n} t_i v_i, F_j \right\rangle \right|^p \leqslant \left[2K \sum_{i=1}^{k-1} |t_i|^p + K |t_k|^p \right]$$
$$+ \left[K |t_k \langle v_k, F_l \rangle|^p + K \sum_{i=k+1}^{n-1} |t_i|^p + \alpha_l K |t_n \langle v_n, F_l \rangle|^p \right] + \sum_{j=l+1}^{m} \alpha_j |\langle t_n v_n, F_j \rangle|^p$$
$$\leqslant 2K \sum_{i=1}^{n-1} |t_i|^p + K \sum_{j=l}^{m} \alpha_j |\langle t_n v_n, F_j \rangle|^p \leqslant 2K \sum_{i=1}^{n-1} |t_i|^p + K |t_n|^p,$$

thus

$$\left\| \sum_{i=1}^{k} t_i v_i \right\|^p \leqslant 2(3)^{p-1} \sum_{i=1}^{k} |t_i|^p.$$

□

Let $1 < p_n < p_{n-1}$ and let (u_i) be a norm one sequence in X_{α,p_n}, (G_i) an admissible sequence of blocks such that $\{j\colon u_i(j) \neq 0\} \subset G_i$ and let $s(u_i) \to 0$. Then the norm of u_i in $X_{\alpha,p_{n-1}}$ is less than or equal to 1. Then using previous lemma with $p = p_{n-1}$ we obtain

Lemma 2.2. *Let (u_i) be a norm one sequence in X_{α,p_n}, (G_i) an admissible sequence of blocks such that $\{j\colon u_i(j) \neq 0\} \subset G_i$ and $s(u_i) \to 0$*

$$\Big\|\sum_{i=1}^{k} t_i v_i\Big\|^{p_{n-1}} \leqslant 2(3)^{p_{n-1}-1} \sum_{i=1}^{k} |t_i|^{p_{n-1}}.$$

We use the following lemma from [1].

Lemma 2.3. *Let (u_i) be a sequence of norm one vectors in X_{α,p_n} $(p_n \geqslant 1)$ and (G_i) an admissible sequence of blocks such that $\{j\colon u_i(j) \neq 0\} \subset G_i$. Then for a subsequence (v_k) of (u_k) and for a given sequence $t_1, t_2, \ldots, t_k$ of scalars we have*

$$\Big\|\sum_{i=1}^{k} t_i v_i\Big\|^{p_n} \geqslant \frac{1}{2} \sum_{i=1}^{k} |t_i|^{p_n}.$$

For each $I \subseteq \mathbb{N}$ the projection P_I denotes the natural projection of X onto $[e_{i,n}\colon i \in \mathbb{N}, n \in I]$.

Lemma 2.4. *Let E_0 be an infinite dimensional subspace of Z_p, $n, m, j \in \mathbb{N}$ $(n > 1)$ and $\varepsilon > 0$. Then there are $\{x_i\}_{i=1}^m \subset E_0$ and $\{u_i\}_{i=1}^m \subset Z_p$ such that the k'th component of u_i is of the form*

$$u_{i,k} = \delta_k \sum_{s=j_1+1}^{j_{i+1}} a_{i,s} v_s,$$

where $j = j_1 < j_2 < \ldots < j_{m+1}$. The v_i's are obtained from Lemmas 2.2 and 2.3 for $p = p_n$ such that

$$\sum_{s=j_i+1}^{j_{i+1}} |a_i|^{p_{n-1}} = 1 \quad \text{and} \quad \|u_i - x_i\| < \frac{\varepsilon}{m}\|u_i\|$$

for each $i = 1, \ldots, m$.

P r o o f. Put $Z_1 = E_0 \cap [z_i]_{i=j+1}^{\infty}$. Since E_0 is infinite dimensional and $[z_i]_{i=j+1}^{\infty}$ has finite codimension in Z_p, Z_1 is infinite dimensional as well. Put $j_1 = j$ and choose any $\overline{x}_1 \in Z_1 \setminus \{0\}$ such that the k'th component of $\overline{x}_1$ has the form

$$\overline{x}_{1,k} = \delta_k \sum_{s=j_1+1}^{\infty} \overline{a}_{1,s} v_s.$$

Take $\overline{x}_1$ and use Lemma 2.2 of [6] to obtain x_1 and u_1 with the above properties and continue the procedure of that lemma to construct the desired sequence.

For $n \in \mathbb{N}$ denote $Q_n = P_{\{n,n+1,\dots\}}$. □

Lemma 2.5. *Let E_0 be an infinite dimensional subspace of Z_p, $j, n \in \mathbb{N}$ and $\varepsilon > 0$. There exist an $x \in E_0$, $x \neq 0$ and a $u \in Z_p$ such that*
(i) $\|Q_n u\| \geqslant (1-\varepsilon)\|u\|$,
(ii) $\|x - u\| < \varepsilon\|u\|$.

P r o o f. Choose m so that $2^{1/p_n}((2(3)^{p_{n-1}-1}))^{1/p_{n-1}}\delta_n^{-1} m^{1/p_{n-1}-1/p_n} < \varepsilon$.

Using Lemma 2.4, choose $\{x_i\}_{i=1}^{m} \subset E_0$ and $\{u_i\}_{i=1}^{m} \subset Z_p$ to satisfy the claims of the lemma and put

$$x = \sum_{i=1}^{m} x_i \qquad \text{and} \qquad u = \sum_{i=1}^{m} u_i.$$

First, we prove (ii). We know that $\|u_i\| \leqslant \|u\|$ for $i = 1, \dots, m$ and

$$\|x - u\| \leqslant \sum_{i=1}^{m} \|x_i - u_i\| < \sum_{i=1}^{m} \frac{\varepsilon\|u_i\|}{m} \leqslant \frac{\varepsilon\|u\|}{m} = \varepsilon\|u\|.$$

To prove (i), we first show that

$$\|u\| - \|Q_n u\| < \bigl(2(3)^{p_{n-1}-1}\bigr)\, m^{1/p_{n-1}}.$$

Indeed, $\|u\| - \|Q_n u\| \leqslant \|P_{\{1,\dots,n-1\}} u\|$. Hence, for $p \geqslant 1$ and by virtue of Lemma 2.2 we have

$$\begin{aligned}(\|u\| - \|Q_n u\|)^p &\leqslant \sum_{k=1}^{n-1} \delta_k^p \Bigl\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s \Bigr\|_{\alpha,p_k}^{p} \leqslant \sum_{k=1}^{n-1} \delta_k^p \Bigl\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s \Bigr\|_{\alpha,p_{n-1}}^{p} \\ &\leqslant \bigl(2(3)^{p_{n-1}-1}\bigr)^{p/p_{n-1}} \sum_{k=1}^{n-1} \delta_k^p \Bigl(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} \Bigr)^{p/p_{n-1}}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \bigl(2(3)^{p_{n-1}-1}\bigr)^{p/p_{n-1}} \sum_{k=1}^{n-1} \delta_k^p \biggl(\sum_{i=1}^{m} 1\biggr)^{p/p_{n-1}} \\
&= \bigl(2(3)^{p_{n-1}-1}\bigr)^{p/p_{n-1}} m^{p/p_{n-1}} \sum_{k=1}^{n-1} \delta_k^p \\
&< \bigl(2(3)^{p_{n-1}-1}\bigr)^{p/p_{n-1}} m^{p/p_{n-1}}.
\end{aligned}
$$

Further, for $p = 0$,

$$
\begin{aligned}
\|u\| - \|Q_n u\| &\leqslant \max \delta_k \biggl\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s \biggr\|_{\alpha,p_k} \\
&\leqslant \max \delta_k \biggl\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s \biggr\|_{\alpha,p_{n-1}} \\
&\leqslant \bigl(2(3)^{p_{n-1}-1}\bigr)^{1/p_{n-1}} \max \delta_k \biggl(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}}\biggr)^{1/p_{n-1}} \\
&\leqslant \bigl(2(3)^{p_{n-1}-1}\bigr)^{1/p_{n-1}} \max \delta_k \biggl(\sum_{i=1}^{m} 1\biggr)^{1/p_{n-1}} \\
&= \bigl(2(3)^{p_{n-1}-1}\bigr)^{1/p_{n-1}} \max \delta_k m^{1/p_{n-1}} \\
&< \bigl(2(3)^{p_{n-1}-1}\bigr)^{1/p_{n-1}} m^{1/p_{n-1}},
\end{aligned}
$$

where max is taken over $1 \leqslant k < n$.

On the other hand, using Lemma 2.3 we obtain for $p \geqslant 1$

$$
\begin{aligned}
\|u\|^p &\geqslant \delta_n^p \biggl\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s \biggr\|_{\alpha,p_n}^p \\
&\geqslant \Bigl(\frac{1}{2}\Bigr)^{p/p_n} \delta_n^p \biggl(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_n}\biggr)^{p/p_n} \\
&\geqslant \Bigl(\frac{1}{2}\Bigr)^{p/p_n} \delta_n^p \biggl(\sum_{i=1}^{m} \biggl(\sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}}\biggr)^{p_n/p_{n-1}}\biggr)^{p/p_n} \\
&= \Bigl(\frac{1}{2}\Bigr)^{p/p_n} \delta_n^p \biggl(\sum_{i=1}^{m} 1\biggr)^{p/p_n} = \Bigl(\frac{1}{2}\Bigr)^{p/p_n} \delta_n^p m^{p/p_n}.
\end{aligned}
$$

Further, for $p = 0$,

$$\begin{aligned}
\|u\| &= \max \delta_k \Big\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s \Big\|_{\alpha,p_k} \geqslant \delta_n \Big\| \sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} a_{i,s} v_s \Big\|_{\alpha,p_n} \\
&\geqslant \Big(\frac{1}{2}\Big)^{1/p_n} \delta_n \Big(\sum_{i=1}^{m} \sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_n} \Big)^{1/p_n} \\
&\geqslant \Big(\frac{1}{2}\Big)^{1/p_n} \delta_n \Big(\sum_{i=1}^{m} \Big(\sum_{s=j_i+1}^{j_{i+1}} |a_{i,s}|^{p_{n-1}} \Big)^{p_n/p_{n-1}} \Big)^{1/p_n} \\
&= \Big(\frac{1}{2}\Big)^{1/p_n} \delta_n \Big(\sum_{i=1}^{m} 1 \Big)^{1/p_n} = \Big(\frac{1}{2}\Big)^{1/p_n} \delta_n m^{1/p_n},
\end{aligned}$$

where max is taken over $k \in \mathbb{N}$.

Thus, $\|u\| \geqslant (\frac{1}{2})^{1/p_n} \delta_n m^{1/p_n}$ and hence

$$1 - \frac{\|Q_n u\|}{\|u\|} \leqslant 2^{1/p_n} \left(2(3)^{p_{n-1}-1}\right)^{1/p_{n-1}} \frac{1}{\delta_n} m^{1/p_{n-1}-1/p_n} < \varepsilon$$

and $\|Q_n u\| \geqslant (1-\varepsilon)\|u\|$. □

To complete the proof of parts (i) and (iii) of Theorem 1.1 we will use the following two results of [6] (Lemma 2.4 and Theorem 2.5)

Lemma 2.6. *Suppose $\varepsilon > 0$ and ε_s for $s \in \mathbb{N}$ are such that $2\varepsilon_s \leqslant \varepsilon$ if $p = 1$, $\sum_{s=1}^{\infty} (2\varepsilon_s)^q \leqslant \varepsilon^q$ if $1 < p < \infty$ where $1/p + 1/q = 1$, $\sum_{s=1}^{\infty} 2\varepsilon_s \leqslant \varepsilon$ if $p = 0$.*

If, for given vectors $\{u_s\}_{s=1}^{\infty} \subset S(Z_p)$, there is a sequence of integers $1 \leqslant n_1 < n_2 < \ldots$ such that, for each $s \in \mathbb{N}$, one has
(i) $\|u_s - Q_{n_s} u_s\| \leqslant \varepsilon_s$,
(ii) $\|Q_{n_{s+1}} u_s\| \leqslant \varepsilon_s$
then $\{u_s\}_{s=1}^{\infty} \subset S(Z_p)$ is $(1+\varepsilon)(1-3\varepsilon)^{-1}$-equivalent to the unit vector basis of ℓ_p (as weall as, c_0).

Theorem 2.7. *The Banach space Z_p is hereditarily ℓ_p if $1 \leqslant p < \infty$ and is hereditarily c_0 if $p = 0$.*

The proof of 2.6 and 2.7 is based on the definition of Q_i and the norm on Z_p. In fact by the lemma conditions and for any sequence $(a_s)_{s=1}^{m}$ of scalars it follows that

$$(1-3\varepsilon)\Big(\sum_{s=1}^{m} |a_s|^p \Big)^{1/p} \leqslant \Big\| \sum_{s=1}^{m} a_s u_s \Big\| \leqslant (1+\varepsilon)\Big(\sum_{s=1}^{m} |a_s|^p \Big)^{1/p}$$

for $1 \leqslant p < \infty$, and

$$(1-3\varepsilon) \max_{1 \leqslant s \leqslant m} |a_s| \leqslant \Big\| \sum_{s=1}^{m} a_s u_s \Big\| \leqslant (1+\varepsilon) \max_{1 \leqslant s \leqslant m} |a_s|$$

for $p = 0$. Then by using the stability properties of the bases [4, P. 5] and Lemma 2.5 we conclude the proof.

The following lemma completes the proof of Theorem 1.1.

Lemma 2.8. *Z_1 does not possess the Schur property.*

Proof. Let $u_i = z_{2i-1} - z_{2i}$. Assume that u_i does not converge weakly to zero. Then there exist an $f \in Z_1^*$, $\|f\| = 1$, and a $\delta > 0$ such that (passing to a subsequence of (u_i) and not renaming) $f(u_i) > \delta$ for all i. Thus

$$\Big\| \frac{1}{N} \sum_{i=1}^{N} u_i \Big\|_1 > \delta \quad \text{for all } N.$$

Now, since $\alpha_i \to 0$ as $i \to \infty$, there exists N such that $N^{-1} \sum_{i=1}^{N} \alpha_i < \frac{1}{2}\delta$. Thus

$$\Big\| \frac{1}{N} \sum_{i=1}^{N} u_i \Big\|_1 = \frac{1}{N} \sum_{n=1}^{\infty} \delta_n \Big(\sum_{i=1}^{2N} \alpha_i \Big)^{1/p_n} \leqslant \sum_{n=1}^{\infty} \delta_n \frac{1}{N} \sum_{i=1}^{2N} \alpha_i < \frac{\delta}{2},$$

but this is a contradiction.

On the other hand, $\|u_i\|_1 = \sum_{n=1}^{\infty} \delta_n (1+\alpha_2)^{1/p_n} \geqslant \sum_{n=1}^{\infty} \delta_n = 1$. Hence, the sequence (u_i) is a weakly null sequence in Z_1 but not in norm. □

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