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ON THE SCHRÖDER-BERNSTEIN PROBLEM FOR CARATHÉODORY VECTOR LATTICES

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Abstract. In this note we prove that there exists a Carathéodory vector lattice V such that $V \cong V^3$ and $V \ncong V^2$. This yields that V is a solution of the Schröder-Bernstein problem for Carathéodory vector lattices. We also show that no Carathéodory Banach lattice is a solution of the Schröder-Bernstein problem.

Keywords: vecrot lattice, Boolean algebra, internal direct factor

MSC 2010: 46A40, 06F20, 06F15

1. Introduction

We apply the standard terminology and notation for vector lattices; cf. e.g., [1]. Carathéodory vector lattices were investigated in several papers; we quote [8], [11] and [14]. If V is a Carathéodory vector lattice, then it is generated by a uniquely determined Boolean algebra B; in such a case we write $V = C(B)$.

Applying the results of [10], [20] and [14], we prove

Theorem 1.1. *There exists a Carathéodory vector lattice V such that $V \cong V^3$ and V fails to be isomorphic to V^2.*

Corollary 1.2. *There exist Carathéodory vector lattices A and B such that*

(i) *A is isomorphic to a direct factor of B and B is isomorphic to a direct factor of A;*

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(ii) *A is not isomorphic to B;*

(iii) $A^3 \cong A$ *and* $B^3 \cong B$.

Theorem 1.1′. *Let V be a Carathéodory vector lattice, $V = C(B)$. Assume that the Boolean algebra B is countable. Then V does not satisfy the conditions from Theorem* 1.1.

In accordance with the terminology applied in literature we can say that the validity of the conditions (i) and (ii) in Corollary 1.2 expresses the fact that the problem of Schröder-Bernstein is solvable for Carathéodory vector lattices.

We remark that if the condition (iii) in Corollary 1.2 is replaced by the condition

(iii_1) $A^2 \cong A$ and $B^2 \cong B$,

then the modified assertion fails to be valid.

A Carathéodory vector lattice V is defined to be a solution of the Schröder-Bernstein problem if there exist direct factors V_1 and V_2 of V such that $V_2 \subset V_1$, $V \cong V_2$ and V_1 fails to be isomorphic to V.

The relation between the conditions mentioned in this definition and the conditions (i), (ii) above are described in Section 2.

The notion of Carathéodory Banach space is defined in a natural way. We prove that if V is a Carathéodory Banach space, then it fails to be a solution of the Schröder-Bernstein problem.

Deep results on the Schröder-Bernstein problem for Banach spaces were proved in [6], [7], [9] and in the papers quoted therein.

The Schröder-Bernstein problem for abelian lattice ordered groups and for MV-algebras was studied in [15]. Some variations of this problem were investigated in [2] (for fields) and in [17] (for linearly ordered groups).

For some classes of algebraic structures, the Schröder-Bernstein problem has no solution. In such a case we say that the Cantor-Bernstein theorem is valid for the corresponding algebraic structure. Results of this type were proved in [4], [5], [12], [13], [18], [19].

2. Preliminaries

Assume that $V_1, V_2, \ldots, V_n$ are vector lattices; their direct product is denoted by $V_1 \times V_2 \times \ldots \times V_n$. We put

$$V_1 \times V_1 = V_1^2, \quad V_1 \times V_1 \times V_1 = V_1^3.$$

Let V be a vector lattice; an isomorphism $\varphi\colon V \to V_1 \times V_2 \times \ldots \times V_n$ is a *direct product decomposition* of V; V_i $(i = 1, 2, \ldots, n)$ are *direct factors* of V.

Lemma 2.1. *Assume that A and B are vector lattices such that*

(i) *A is isomorphic to a direct factor of B and B is isomorphic to a direct factor A;*

(ii) *A is not isomorphic to B.*

Then A is a solution of the Schröder-Bernstein problem for vector lattices.

Proof. In view of (i) there exist direct product decompositions

$$A = A_1 \times X, \quad B = B_1 \times Y$$

and isomorphisms

$$\varphi_1 \colon A \to B_1, \quad \varphi_2 \colon B \to A_1.$$

Put $A_2 = \varphi_2(B_1)$. Then $A_2 \cong A$ and A_2 is a direct factor of A_1. Hence A_2 is a direct factor of A. From (ii) we conclude that A_1 is not isomorphic to A, hence $A_2 \subset A_1$. Therefore A is a solution of the Schröder-Bernstein problem. $\square$

Further, from the definition of the solution of the Schröder-Bernstein problem we immediately obtain

Lemma 2.2. *Assume that V is a vector lattice such that V is a solution of the Schröder-Bernstein problem. Let V_1 and V_2 be as in Section* 1. *Put $V = A$ and $V_1 = B$. Then the pair (A, B) satisfies the conditions* (i) *and* (ii) *from Lemma* 2.1.

Let (iii) and (iii$_1$) be as in Section 1. Then the condition (iii) in Corollary 1.2 cannot be replaced by the condition (iii$_1$). In fact, assume that the conditions (i) (from Corollary 1.3) and (iii$_1$) are valid. Thus there are vector lattices X and Y with $A \cong B \times X$, $B \cong A \times Y$. We obtain

$$A \cong B \times X \cong B \times B \times X \cong B \times A \times Y \times X.$$

Similarly, $B \cong A \times B \times X \times Y$. Thus $A \cong B$. Therefore the condition (ii) from Corollary 1.2 fails to hold.

For the sake of completeness, we recall the basic definitions concerning Carathéodory functions which correspond to a Boolean algebra B (cf. [8]).

We denote by $C(B)$ the system consisting of all forms

$$f = a_1 b_1 + \ldots + a_n b_n,$$

where a_i are nonzero reals, b_i are elements of B such that $b_i > 0$ for each $i \in \{1, 2, \ldots, n\}$,

$$b_{i(1)} \wedge b_{i(2)} = 0 \quad \text{for distinct } i(1), i(2) \in \{1, 2, \ldots, n\},$$

and of the "empty form". If g is another such form,

$$g = a'_1 b'_1 + \ldots + a'_m b'_m,$$

then f and g are considered equal if

$$\bigvee_{i=1}^{n} b_i = \bigvee_{j=1}^{m} b'_j$$

and if $a_i = a'_j$ whenever $b_i \wedge b'_j > 0$.

For $b, b' \in B$ we denote by $b -_1 b'$ the relative complement of $b \wedge b'$ in the interval $[0, b]$.

If f and g are as above, then we put

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + a'_j)(b_i \wedge b'_j) + \sum_{i=1}^{n} a_i \left(b_i -_1 \bigvee_{j=1}^{m} b'_j \right) + \sum_{j=1}^{m} a'_j \left(b'_j -_1 \bigvee_{i=1}^{n} b_i \right),$$

where in the summations only those terms are taken into account in which $a_i + a'_j \neq 0$ and the elements

$$b_i \wedge b'_j, \quad b_i -_1 \bigvee_{j=1}^{m} b'_j, \quad b'_j -_1 \bigvee_{i=1}^{n} b_i$$

are nonzero. The empty form is considered to be the neutral element of $C(B)$ (with respect to the operation $+$) and is identified with the element 0 of B. Also, each element $0 \neq b \in B$ is identiefied with the element $1b$ of $C(B)$; hence $B \subseteq C(B)$.

We remark that we apply the same symbol for the zero element of $\mathbb{R}$, the least element of B and the neutral element of $C(B)$; the meaning of this symbol will be always clear from the context.

If $a_1 = 0 \in \mathbb{R}$ and $b_1 \in B$, or if $a_1 \in \mathbb{R}$ and $b_1 = 0 \in B$, then $a_1 b_1$ is identified with the neutral element of $C(B)$.

If f is as above and $a \in \mathbb{R}$, then we set

$$af = (aa_1)b_1 + \ldots + (aa_n)b_n.$$

Finally, we set $f > 0$ if $a_1 > 0, \ldots, a_n > 0$. In more detail: For f, g as above we have $f \leqslant g$ if $b_1 \vee \ldots \vee b_n \leqslant b'_1 \vee \ldots \vee b'_m$ and $a_i \leqslant a'_j$ whenever $b_i \wedge b'_j \neq 0$ (for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$). Then $C(B)$ turns out to be a vector lattice; its elements are *elementary Carathéodory functions* corresponding to the Boolean algebra B.

We remark that if f is as above, then without loss of generality we can assume that the elements $a_1, a_2, \ldots, a_n$ are mutually distinct. In fact, if we have, e.g., $a_1 = a_2$, then we can write

$$f = a_1 b_{10} + a_3 b_3 + \ldots + a_n b_n,$$

where $b_{10} = b_1 \vee b_2 = b_1 + b_2$. In view of this fact we can also assume that $a_1 > a_2 > \ldots > a_n$.

It is obvious that if B and B' are Boolean algebras such that $B \cong B'$, then $C(B) \cong C(B')$.

An element $b > 0$ of a vector lattice V is said to be *boolean* if the interval $[0, b]$ of $\ell(V)$ is a Boolean algebra ($\ell(V)$ is the underlying lattice of V). Let $\beta(V)$ be the set of all boolean elements of V. Then for each Boolean algebra B we have

$$\beta(C(B)) = B.$$

From this we obtain

Lemma 2.3. *Let B and B' be Boolean algebras such that $C(B) \cong C(B')$. Then $B \cong B'$.*

3. Internal direct product decompositions

Assume that

$$\varphi\colon\ V \to V_1 \times V_2 \times \ldots \times V_n \tag{1}$$

is a direct product decomposition of a vector lattice V.

If $i \in \{1, 2, \ldots, n\}$, $x \in V$ and $\varphi(x) = (x_1, x_2, \ldots, x_n)$, then we put $\varphi_i(x) = x_i$. Let V_{i0} be the set of all $y \in V$ such that

$$\varphi_j(y) = 0 \quad \text{for each } j \in \{1, 2, \ldots, n\} \text{ with } j \neq i.$$

Further, for $x \in V$ let $\varphi_{i0}(x)$ be the element x_{i0} of V_{i0} such that $\varphi_i(x_{i0}) = x_i$. We set

$$\varphi_0(x) = (x_{10}, x_{20}, \ldots, x_{n0}).$$

Then we have an isomorphism

$$\varphi_0\colon\ V \to V_{10} \times V_{20} \times \ldots \times V_{n0} \tag{2}$$

which is said to be an *internal direct product decomposition* of V; the subalgebras $V_{10}, \ldots, V_{n0}$ of V are *internal direct factors* of V. Thus to each direct product decomposition φ of V there corresponds an internal direct product decomposition φ_0 of V. Under the assumptions as above we write

$$V = (\mathrm{int})V_{10} \times V_{20} \times \ldots \times V_{n0}. \tag{3}$$

In the same way we define the notion of the internal direct product decomposition of a Boolean algebra. For the case of Boolean algebras we apply the analogous notation as above.

In view of [14, Proposition 5.8], we have

Lemma 3.1. *Let B be a Boolean algebra and suppose that*

(a) $B = (\mathrm{int})B_1 \times B_2 \times \ldots \times B_n$.

Then

(b) $C(B) = (\mathrm{int})C(B_1) \times C(B_2) \times \ldots \times C(B_n)$.

Conversely, if (b) *is satisfied, then* (a) *is valid.*

The following result was proved in [17] (cf. also [16]).

Proposition 3.2. *There exists a Boolean algebra B such that $B \cong B^3$ and B is not isomorphic to B^2.*

Proposition 3.3. *Let B be as in Proposition* 3.2. *Put $C(B) = V$. Then $V \cong V^3$ and V is not isomorphic to V^2.*

P r o o f. This is a consequence of Proposition 3.2, Lemma 3.2 and Lemma 2.3. □

In view of Proposition 3.3 we conclude that Theorem 1.1 is valid. Further, applying Lemma 2.2, we infer that Corollary 1.2 holds.

The following result was proved in [20] (applying a different terminology).

Theorem 3.4. *Let B be a countable Boolean algebra and let m, n be positive integers such that $n < m$ and $B^n \cong B^m$. Then $B^n \cong B^{n+1}$.*

P r o o f of Theorem 1.1′. Let $V = C(B)$ be a Carathéodory vector lattice such that the Boolean algebra B is countable. By way of contradiction, assume that V satisfies the conditions from Theorem 1.1; i.e., we have $V \cong V^3$ and $V \cong V^2$. Then in view of Lemma 3.1 we obtain $B \cong B^3$ and $B \ncong B^2$. Put $n = 1$ and $m = 3$. According to Theorem 3.4, we have arrived at a contradiction. □

4. Carathéodory Banach spaces

Assume that $V = C(B)$ is a Carathéodory vector lattice corresponding to the Boolean algebra B. To avoid the trivial case, let us suppose that B has more than one element.

We define a norm function on V as follows. For $0 \in V$ we put $\|0\| = 0 \in \mathbb{R}$. Let $f \in V$, $f \neq 0$; under the notation as in Section 2, let

$$f = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n.$$

Then we set

$$\|f\| = \max\{|a_1|, |a_2|, \ldots, |a_n|\}.$$

It is easy to verify that the norm function is correctly defined and that it satisfies the usual rules

$$|f| \leqslant |g| \Rightarrow \|f\| \leqslant \|g\|; \tag{1}$$

$$\|f\| \geqslant 0; \quad \text{moreover } \|f\| = 0 \text{ iff } f = 0; \tag{2}$$

$$\|f + g\| \leqslant \|f\| + \|g\|; \tag{3}$$

$$\|af\| = |a|\|f\| \quad \text{for each } a \in \mathbb{R}. \tag{4}$$

Let $f \in V$ and let (f_n) be a sequence of elements of V. We write

$$f_n \xrightarrow{b} f \quad \text{or} \quad f = (b)\lim(f_n)$$

if the relation

$$\lim_{n\to\infty} \|f_n - f\| = 0$$

is valid. In such a case we also say that the sequence (f_n) is convergent.

In the terminology of [16], V is a KB-lineal. Thus in view of [16, Chapter VI, Section 2.22] we have

Lemma 4.1. *Let (x_n) be a sequence in V and let $x, y \in V$ be such that $x_n \xrightarrow{b} x$. Then $x_n \vee y \xrightarrow{b} x \vee y$.*

Lemma 4.2. *Let* (x_n) *and* x *be as in Lemma* 4.1. *Assume that the sequence* (x_n) *is increasing. Then* $x = \sup\{x_n\}_{n\in\mathbb{N}}$.

P r o o f. By way of contradiction, suppose that the relation $x = \sup\{x_n\}_{n\in\mathbb{N}}$ fails to be valid. Then some of the following conditions is satisfied:

a) there is $x \in V$ such that $x_n \leqslant z < x$ for each $n \in \mathbb{N}$,

b) there is $m \in \mathbb{N}$ with $x_m \gneqq x$.

Let a) be valid. Then

$$\|x - x\| \geqslant \|x - z\| > 0 \quad \text{for each } n \in \mathbb{N},$$

hence the relation $x_n \xrightarrow{b} x$ cannot hold and we have arrived at a contradiction.

Further, let b) be valid. For each $n \in \mathbb{N}$ we put $x_{m+n} = y_n$, $y_n \vee x = z_n$. Then (y_n) is a subsequence of (x_n), hence $y_n \xrightarrow{b} x$. In view of Lemma 4.1 we get $z_n \xrightarrow{b} x \vee x = x$. The sequence (z_n) is increasing and $z_n > x$ for each $n \in \mathbb{N}$. Thus

$$\|z_n - x\| \geqslant \|z_1 - x\| > 0 \quad \text{for each } n \in \mathbb{N}.$$

This yields that (z_n) does not converge to the element x; again, we have arrived at a contradiction. $\square$

Let (x_n) be a sequence in V; (x_n) is a *Cauchy sequence* if for each real $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\|x_{n(1)} - x_{n(2)}\| < \varepsilon$ whenever $n(1), n(2) \in \mathbb{N}$ and $n(1) > m$, $n(2) > m$.

In view of the validity of relations (1)–(4) we obtain

Lemma 4.3. *Under the norm function defined as above,* V *is a Banach space iff each Cauchy sequence of elements of* V *is convergent.*

If V satisfies the conditions from Lemma 4.3, then it is said to be a Carathéodory Banach space.

A sequence (f_n) in V will be called *orthogonal* if $f_{n(1)} \wedge f_{n(2)} = 0$ whenever $n(1)$ and $n(2)$ are distinct positive integers. Analogously, a subset S of V is *orthogonal* if $0 \leqslant s \in S$ for each $s \in S$, and $s_1 \wedge s_2 = 0$ whenever s_1 and s_2 are distinct elements of S.

The vector lattice V is *orthogonally σ-complete* if for each orthogonal sequence (f_n) in V the element

$$\sup\{f_n\}_{n\in\mathbb{N}}$$

exists.

Since the direct product decompositions of a vector lattice are uniquely determined by the direct product decompositions of its underlying lattice ordered group, in view of [13] we obtain

Lemma 4.4. *Assume that V is an orthogonally σ-complete vector lattice. Then V fails to be a solution of the Schröder-Bernstein problem.*

Let (f_n) be an orthogonal sequence in a Carathéodory vector lattice $C(B)$. Suppose that $f_n > 0$ for each $n \in \mathbb{N}$. Each element of this sequence is a form (cf. Section 2)

$$f_n = a_{n1}b_{n1} + \ldots + a_{n,m(n)}b_{n,m(n)}.$$

We put $b_n = b_{n1}$ for each $n \in \mathbb{N}$; hence (b_n) is an orthogonal sequence in V, $0 < b_n \in B$ for each $n \in \mathbb{N}$.

We choose a strictly decreasing sequence (a_n) such that $a_n > 0$ for each $n \in \mathbb{N}$. Further, for each $n \in \mathbb{N}$ we set

$$g_n = a_1 b_1 + \ldots + a_n b_n.$$

Thus (g_n) is an increasing sequence in V and $0 < g_n$ for each $n \in \mathbb{N}$.

Let m, $n(1)$ and $n(2)$ be positive integers with $m < n(1) < m(2)$. Then $0 < g_{n(2)} - g_{n(1)} < g_{n(2)}$ and

$$\|g_{n(2)} - g_{n(1)}\| < a_m.$$

From this we conclude

Lemma 4.5. *(g_n) is a Cauchy sequence in V.*

Lemma 4.6. *Let us apply the assumptions as above. Then V fails to be a Carathéodory Banach space.*

Proof. By way of contradiction, suppose that V is a Carathéodory Banach space. Then in view of Lemma 4.5 we conclude that the sequence (g_n) is convergent. Hence there exists $g \in V$ with

$$g = (b) \lim_{n \to \infty} (g_n).$$

Since (g_n) is strictly increasing, in view of Lemma 4.2 we obtain

$$g = \bigvee_{n=1}^{\infty} g_n. \tag{5}$$

Then $0 < g$ and g can be written in the form

$$g = a'_1 b'_1 + \ldots + a'_m b'_m$$

(cf. Section 2). In view of Section 2 we can suppose, without loss of generality, that the relation $a'_1 > a'_2 > \ldots > a'_m$ is valid. For $i(1), i(2) \in \{1, 2, \ldots, m\}$ with $i(1) \neq i(2)$ we have $a'_{i(1)} b'_{i(1)} \wedge a'_{i(2)} b'_{i(2)} = 0$, hence

$$g = a'_1 b'_1 \vee \ldots \vee a'_m b'_m. \tag{6}$$

Similarly, for each $n \in N$ we get

$$g_n = a_1 b_1 \vee \ldots \vee a_n b_n.$$

Thus in view of (5) we obtain

$$g = \bigvee_{n=1}^{\infty} a_n b_n. \tag{7}$$

Let $n(1) \in \mathbb{N}$. In view of (6) and (7),

$$\begin{aligned} a_{n(1)} b_{n(1)} &= a_{n(1)} b_{n(1)} \wedge g = a_{n(1)} b_{n(1)} \wedge (a'_1 b'_1 \vee \ldots \vee a'_m b'_m) \\ &= ((a_{n(1)} b_{n(1)}) \wedge (a'_1 b'_1)) \vee \ldots \vee ((a_{n(1)} b_{n(1)} \wedge (a'_m b'_m). \end{aligned}$$

Hence there exists $j(1) \in \{1, 2, \ldots, m\}$ such that

$$a_{n(1)} b_{n(1)} \wedge a'_{j(1)} b'_{j(1)} > 0.$$

Then we also have

$$b_{n(1)} \wedge b'_{j(1)} > 0. \tag{8}$$

We put $b_{n(1)} \wedge b'_{j(1)} = b_0$. Thus

$$\begin{aligned} b_0 \wedge b_n = 0 \quad &\text{for each} \quad n \in \mathbb{N},\ n \neq n(1), \\ b_0 \wedge b'_j = 0 \quad &\text{for each} \quad j \in \{1, 2, \ldots, m\},\ j \neq j(1). \end{aligned}$$

Hence for such n and j we obtain

$$a'_{j(1)} b_0 \wedge a_n b_n = 0, \tag{9}$$

$$a_{n(1)} b_0 \wedge a'_j b'_j = 0. \tag{10}$$

Since

$$a'_{j(1)} b_0 \leqslant a'_{j(1)} b'_{j(1)} \leqslant g,$$

in view of (9) we get

$$\begin{aligned}a'_{j(1)}b_0 &= a'_{j(1)}b_0 \wedge g = a'_{j(1)}b_0 \wedge \bigvee_{n=1}^{\infty} a_n b_n \\ &= \bigvee_{n=1}^{\infty} (a'_{j(1)}b_0 \wedge a_n b_n) = a'_{j(1)}b_0 \wedge a_{n(1)}b_{n(1)}.\end{aligned}$$

Thus $a'_{j(1)}b_0 \leqslant a_{n(1)}b_{n(1)}$. Therefore $a'_{j(1)} \leqslant a_{n(1)}$.

Similarly, $a_{n(1)}b_0 \leqslant a_{n(1)}b_{n(1)} \leqslant g$, thus in view of (10) we obtain

$$\begin{aligned}a_{n(1)}b_0 &= a_{n(1)}b_0 \wedge g = a_{n(1)}b_0 \wedge (a'_1 b'_1 \vee \ldots \vee a'_m b'_m) \\ &= (a_{n1}b_0 \wedge a'_1 b'_1) \vee \ldots \vee (a_{n(1)}b_0 \wedge a'_m b'_m) \\ &= a_{n(1)}b_0 \wedge a'_{j(1)}b'_{j(1)}.\end{aligned}$$

Hence $a_{n(1)} \leqslant a'_{j(1)}$. Summarizing, $a_{n(1)} = a'_{j(1)}$.

For each $n(1) \in \mathbb{N}$ we put $\varphi(n(1)) = j(1)$, where $j(1)$ is as above. Since the set $\mathbb{N}$ is infinite and the set $\{1, 2, \ldots, m\}$ is finite, we have arrived at a contradiction. $\square$

Corollary 4.7. *Let V be a Carathéodory Banach space. Then each orthogonal subset of V is finite.*

Theorem 4.8. *Let V be a Carathéodory Banach space. Then V fails to be a solution of the Schröder-Bernstein problem.*

Proof. From Corollary 4.7 we conclude that V is orthogonally σ-complete. Thus in view of Lemma 4.4, V is not a solution of the Schröder-Bernstein problem. $\square$

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