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DENJOY INTEGRAL AND HENSTOCK-KURZWEIL INTEGRAL IN VECTOR LATTICES, I

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Abstract. In this paper we define the derivative and the Denjoy integral of mappings from a vector lattice to a complete vector lattice and show the fundamental theorem of calculus.

Keywords: derivative, Denjoy integral, Henstock-Kurzweil integral, fundamental theorem of calculus, vector lattice, Riesz space

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1. Introduction

The purpose of our research is to consider some derivatives and some integrals of mappings in vector spaces and to study their relations, for instance, the fundamental theorem of calculus, inclusive relations between integrals and so on. To this aim we refer to the Fréchet derivative, the Denjoy integral of mappings from an abstract space to the real line in [4], [5], [17] and the Henstock-Kurzweil integral of mappings from the division space or the real line to a complete vector lattice in [15], [16], [2]. From the above theories to consider both derivatives and integrals of mappings in vector spaces a domain of mappings may be needed an interval structure and linearity and a range of mappings may be needed a convergence structure and linearity. Hereafter we consider that both a domain and a range of mappings are vector lattices.

In this paper we define the derivative and the Denjoy integral of mappings from a vector lattice to a complete vector lattice and show the fundamental theorem of calculus. In the next paper we will define the Henstock-Kurzweil integral of mappings from a vector lattice to a complete vector lattice and consider a relation between these two integrals.

Let X and Y be vector lattices. $e \in X$ is said to be unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathscr{K}_X$ be the class of units of X. Let $\mathscr{I}_X$ be the class of intervals of X and $\mathscr{I}\mathscr{K}_X$ the class of intervals $[a, b]$ with $b - a \in \mathscr{K}_X$. $x_1 \in X$ and $x_2 \in X$ are said to be orthogonal, denoted by $x_1 \perp x_2$, if $|x_1| \wedge |x_2| = 0$. Let $A^{\perp}$ be the class of $x_1 \in X$ satisfying $x_1 \perp x$ for any $x \in A \subset X$. Let $\mathscr{L}(X, Y)$ be the class of bounded linear mappings from X to Y. If Y is complete, then $\mathscr{L}(X, Y)$ is so, too [1], [3], [14], [18], [19].

2. Derivative

Definition 2.1. Let X be a vector lattice with unit.

$D \subset X$ is said to be open if for any $x \in D$ and for any $e \in \mathscr{K}_X$ there exists $\varepsilon \in \mathscr{K}_{\mathbb{R}}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset D$. Let $\mathcal{O}_X$ be the class of open subsets of X.

Definition 2.2. Let X be a vector lattice with unit and Y a vector lattice.

Let $\mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ be the class of $\{v_e \colon e \in \mathscr{K}_X\}$ satisfying the following conditions:

(U1) $v_e \in Y$ with $v_e > 0$.

$(\mathrm{U2})^d$ $v_{e_1} \geqslant v_{e_2}$ if $e_1 \geqslant e_2$.

$(\mathrm{U3})^s$ For any $e \in \mathscr{K}_X$ there exists $\theta(e) \in \mathscr{K}_{\mathbb{R}}$ such that $v_{\theta(e)e} \leqslant \frac{1}{2} v_e$.

Remark 2.1. It holds that $\{\alpha v_e + \beta v_{1,e}\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ for any $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$, for any $\{v_{1,e}\}$ satisfying $v_{1,e} \geqslant 0$ and $(\mathrm{U2})^d$ $(\mathrm{U3})^s$, for any $\alpha \in \mathbb{R}$ with $\alpha > 0$ and for any $\beta \in \mathbb{R}$ with $\beta \geqslant 0$.

Lemma 2.1. *Let X be a vector lattice with unit and Y an Archimedean vector lattice.*

Then $\bigwedge\limits_{\varepsilon \in \mathscr{K}_{\mathbb{R}}} v_{\varepsilon e} = 0$ for any $e \in \mathscr{K}_X$. In particular, $\bigwedge\limits_{e \in \mathscr{K}_X} v_e = 0$.

Proof. Let $\theta(e, n) = \underbrace{\theta(\theta(\ldots\theta(\theta}_{n}(e)e)\ldots e)e)$. Then by $(\mathrm{U3})^s$ it holds that $v_{\theta(e,n)e} \leqslant 2^{-n} v_e$ for any natural number n. Since Y is Archimedean, we have

$$\bigwedge_{e \in \mathscr{K}_X} v_e \leqslant \bigwedge_{\varepsilon \in \mathscr{K}_{\mathbb{R}}} v_{\varepsilon e} \leqslant \bigwedge_{n \in \mathbb{N}} v_{\theta(e,n)e} \leqslant \bigwedge_{n \in \mathbb{N}} 2^{-n} v_e = 0.$$

□

Definition 2.3. Let X be a vector lattice with unit, Y a complete vector lattice, $x \in D \in \mathcal{O}_X$ and $F \colon D \longrightarrow Y$.

F is said to be right differentiable at x if there exists $l \in \mathscr{L}(X, Y)$ satisfying the following condition:

(R) There exists $\{w^+_{x,e}\} \in \mathscr{U}^s_{\mathscr{L}(X,Y)}(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists $\delta^+_x \in \mathscr{K}_{\mathbb{R}}$ such that $|F(x+h) - F(x) - l(h)| \leqslant w^+_{x,e}(h)$ for any $h \in X$ with $0 < h \leqslant \delta^+_x e$.

Then we denote $o\text{-}D^+F(x) = l$. F is said to be left differentiable at x if there exists $l \in \mathscr{L}(X,Y)$ satisfying the following condition:

(L) There exists $\{w^-_{x,e}\} \in \mathscr{U}^s_{\mathscr{L}(X,Y)}(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists $\delta^-_x \in \mathscr{K}_{\mathbb{R}}$ such that $|F(x) - F(x-h) - l(h)| \leqslant w^-_{x,e}(h)$ for any $h \in X$ with $0 < h \leqslant \delta^-_x e$.

Then we denote $o\text{-}D^-F(x) = l$. F is said to be differentiable at x if $o\text{-}D^+F(x) = o\text{-}D^-F(x)$. Then $o\text{-}DF(x) = o\text{-}D^+F(x) = o\text{-}D^-F(x)$.

Let $A \subset D$ and let $F\colon\ D \longrightarrow Y$ be differentiable at every point of A.

F is said to be uniformly differentiable on A if there exists $\{w_e\} \in \mathscr{U}^s_{\mathscr{L}(X,Y)}(\mathscr{K}_X, \geqslant)$ such that for any $x \in A$ and for any $e \in \mathscr{K}_X$ there exists $\varrho^\pm(x,e) \in \mathscr{K}_X$ such that $w^\pm_{x,\varrho^\pm(x,e)} \leqslant w_e$.

Example 2.1. Let $X = \mathbb{R}^d$, let Y be a complete vector lattice with Archimedean unit, $D \in \mathcal{O}_X$ and let $F\colon\ D \longrightarrow Y$ be a differentiable at every point of D.

Then F is uniformly differentiable on D. Let $u_1, \ldots, u_d$ be Archimedean units of Y, $\alpha_e = e_1 \ldots e_d$ and $w_e = \alpha_e(u_1, \ldots, u_d)$ for any $e = (e_1, \ldots, e_d) \in \mathscr{K}_X$. Then $\{w_e\} \in \mathscr{U}^s_{\mathscr{L}(X,Y)}(\mathscr{K}_X, \geqslant)$. For any $\{w^\pm_{x,e}\} \in \mathscr{U}^s_{\mathscr{L}(X,Y)}(\mathscr{K}_X, \geqslant)$ there exists $\beta^\pm_{x,e} \in \mathscr{K}_{\mathbb{R}}$ such that $w^\pm_{x,e} \leqslant \beta^\pm_{x,e}(u_1, \ldots, u_d)$. Let $n^\pm(x,e)$ be a natural number with $2^{-n^\pm(x,e)} \leqslant \alpha_e/\beta^\pm_{x,e}$. Then by (U3)^s

$$w_e \geqslant \frac{\alpha_e}{\beta^\pm_{x,e}} w^\pm_{x,e} \geqslant 2^{-n^\pm(x,e)} w^\pm_{x,e} \geqslant w^\pm_{x,\theta(e,n^\pm(x,e))e},$$

where $\theta(e,n)$ is from the proof of Lemma 2.1.

The derivative of mappings in vector lattices is introduced in the case of a domain with Archimedean unit in [6] and thereafter it is extended to the case of a domain with unit in [10]. The derivative in Definition 2.3 differs from both of them and is further extended.

Remark 2.2. By Definition 2.3 it is clear that $o\text{-}Dl(x) = o\text{-}D^+l(x) = o\text{-}D^-l(x) = l$ for any $l \in \mathscr{L}(X,Y)$ and for any $x \in X$.

The following is evident.

Theorem 2.1. *Let X be a vector lattice with unit, Y a complete vector lattice, $x \in D \in \mathcal{O}_X$, $F_1, F_2\colon D \longrightarrow Y$ and $\alpha, \beta \in \mathbb{R}$.*

(1) *If F_1 and F_2 are right differentiable, then $\alpha F_1 + \beta F_2$ is also so and*

$$o\text{-}D^+(\alpha F_1 + \beta F_2)(x) = \alpha o\text{-}D^+ F_1(x) + \beta o\text{-}D^+ F_2(x).$$

(2) *If F_1 and F_2 are left differentiable, then $\alpha F_1 + \beta F_2$ is also so and*

$$o\text{-}D^-(\alpha F_1 + \beta F_2)(x) = \alpha o\text{-}D^- F_1(x) + \beta o\text{-}D^- F_2(x).$$

3. Integral

3.1. Preliminary. All integrals have double-facedness of an inverse operation of the derivative and the limit of a certain sum. In the former setting a Newton integral in [7], [10] and a Lebesgue integral in [8] were given for mappings in vector lattices. In this paper a Denjoy integral is provided and in the next paper a Henstock-Kurzweil integral will be given in the latter setting.

First some concepts required in the subsequent arguments are defined.

Definition 3.1. Let X be a vector lattice with unit, $e \in \mathscr{K}_X$ and let $a, b \in D \subset X$ with $a \neq b$.

Let $\mathbf{CSIP}_e(a,b)$ be the class of $\varphi\colon [0,1] \longrightarrow D$ satisfying the conditions (P) (CP_e) (SI), $\mathbf{CSDP}_e(a,b)$ the class of $\varphi\colon [0,1] \longrightarrow D$ satisfying the conditions (P) (CP_e) (SD), and $\mathbf{CSMP}_e(a,b) = \mathbf{CSIP}_e(a,b) \cup \mathbf{CSDP}_e(a,b)$, where

(P) $\varphi(0) = a$ and $\varphi(1) = b$.

(CP_e) For any $t \in [0,1]$ and for any $\varepsilon \in \mathscr{K}_{\mathbb{R}}$ there exists $\delta \in \mathscr{K}_{\mathbb{R}}$ such that for any $s \in [0,1]$ if $|s-t| \leqslant \delta$, then $|\varphi(s) - \varphi(t)| \leqslant \varepsilon e$.

(SI) $\varphi(t_1) < \varphi(t_2)$ if $t_1 < t_2$.

(SD) $\varphi(t_1) > \varphi(t_2)$ if $t_1 < t_2$.

Remark 3.1. Let $\varphi^{\mathrm{rev}}(t) = \varphi(1-t)$. Then $\varphi \in \mathbf{CSIP}_e(a,b)$ is equivalent to $\varphi^{\mathrm{rev}} \in \mathbf{CSDP}_e(b,a)$ and $\varphi \in \mathbf{CSDP}_e(a,b)$ is equivalent to $\varphi^{\mathrm{rev}} \in \mathbf{CSIP}_e(b,a)$.

Definition 3.2. Let X be a vector lattice with unit.

Let $|\mathscr{K}_X|$ be the class of x satisfying $|x| \in \mathscr{K}_X$. For any $x \in |\mathscr{K}_X|$ let $x_+^\perp = \{0 \vee x\}^\perp$, $x_-^\perp = \{0 \vee (-x)\}^\perp$,

$$Q(x) = \{x_1\colon\ x_1 \in |\mathscr{K}_X|,\ (x_1)_+^\perp = x_+^\perp,\ (x_1)_-^\perp = x_-^\perp\}$$

and

$$\overline{Q}(x) = \Bigg(\bigcup_{x_1, x_2 \in Q(x)} [0 \wedge x_1, 0 \vee x_2]\Bigg) \setminus \{0\}.$$

Remark 3.2. The class of $Q(x)$'s is an equivalence class of $|\mathscr{K}_X|$. Therefore each $x \in |\mathscr{K}_X|$ belongs to unique $Q(x)$.

Lemma 3.1. *Let X be a vector lattice with unit satisfying the principal projection property and $x \in |\mathscr{K}_X|$. Then $x_+^\perp \oplus x_-^\perp = X$.*

Proof. Since X satisfies the principal projection property, it holds that $B(\{x_1\}) \oplus \{x_1\}^\perp = X$ for any $x_1 \in X$, where $B(A)$ is the smallest band containing $A \subset X$. Let $x_1 = 0 \vee (-x)$. Then $x_-^\perp = \{x_1\}^\perp$. Since $x_1 \wedge (0 \vee x) = 0$ and $x_+^\perp$ is a projection band, it holds that $B(\{x_1\}) \subset x_+^\perp$. Let $x_2 \in x_+^\perp \cap x_-^\perp$. Then

$$(0 \vee x) \wedge |x_2| = 0,\ (0 \vee (-x)) \wedge |x_2| = 0$$

and

$$\begin{aligned}|x| \wedge |x_2| &= ((0 \vee x) + (0 \vee (-x))) \wedge |x_2| \\ &\leqslant (0 \vee x) \wedge |x_2| + (0 \vee (-x)) \wedge |x_2| = 0\end{aligned}$$

proving that $x_2 = 0$. Therefore $x_+^\perp \oplus x_-^\perp = X$. $\square$

Lemma 3.2. *Let X be a vector lattice with unit satisfying the principal projection property and let $x \in |\mathscr{K}_X|$.*

If $(x_1)_+^\perp = x_+^\perp$ and $(x_1)_-^\perp = x_-^\perp$, then $x_1 \in |\mathscr{K}_X|$.

Proof. By Lemma 3.1 we have $(x_1)_+^\perp \oplus (x_1)_-^\perp = x_+^\perp \oplus x_-^\perp = X$. Therefore $(x_1)_+^\perp \cap (x_1)_-^\perp = \{0\}$. Assume that $x_1 \notin |\mathscr{K}_X|$. Then there exists $x_2 \in X$ with $x_2 > 0$ such that $|x_1| \wedge x_2 = 0$. Therefore

$$\begin{aligned}(0 \vee x_1) \wedge x_2 &\leqslant |x_1| \wedge x_2 = 0, \\ (0 \vee (-x_1)) \wedge x_2 &\leqslant |x_1| \wedge x_2 = 0\end{aligned}$$

implying that $x_2 \in (x_1)_+^\perp \cap (x_1)_-^\perp$. This is a contradiction. Therefore $x_1 \in |\mathscr{K}_X|$. $\square$

Remark 3.3. By Lemma 3.1 and Lemma 3.2 if X satisfies the principal projection property, then

$$\begin{aligned}Q(x) &= \{x_1 \colon\ x_1 \in |\mathscr{K}_X|,\ (x_1)_+^\perp = x_+^\perp\} \\ &= \{x_1 \colon\ x_1 \in |\mathscr{K}_X|,\ (x_1)_-^\perp = x_-^\perp\} \\ &= \{x_1 \colon\ (x_1)_+^\perp = x_+^\perp,\ (x_1)_-^\perp = x_-^\perp\}.\end{aligned}$$

Lemma 3.3. *Let X be a vector lattice with unit satisfying the principal projection property and let $x \in |\mathscr{K}_X|$.*

Then the mapping

$$\begin{array}{rccc} |\cdot|_{Q(x)}\colon & Q(x) & \longrightarrow & \mathscr{K}_X \\ & \cup & & \cup \\ & x_1 & \longmapsto & |x_1| \end{array}$$

is bijective.

Proof. By Lemma 3.1 for any $e \in \mathscr{K}_X$ and for any $x \in |\mathscr{K}_X|$ there exist $x_1 \in x_+^{\perp}$ and $x_2 \in x_-^{\perp}$ such that $x_1 + x_2 = e$. Since $x_1 \perp x_2$, it holds that $|x_1 - x_2| = |x_1 + x_2|$. Therefore $|x_2 - x_1| = e$. Note that $x_2 \perp x_3$ for any $x_3 \in x_+^{\perp}$. Since

$$(0 \vee (x_2 - x_1)) \wedge |x_3| = (0 \vee (2x_2 - e)) \wedge |x_3| \leqslant (0 \vee (2x_2)) \wedge |x_3| = 0,$$

it holds that $x_3 \in (x_2 - x_1)_+^{\perp}$ proving that $x_+^{\perp} \subset (x_2 - x_1)_+^{\perp}$. Note that $x_1 \perp x_3$ for any $x_3 \in x_-^{\perp}$. Since

$$(0 \vee (x_1 - x_2)) \wedge |x_3| = (0 \vee (2x_1 - e)) \wedge |x_3| \leqslant (0 \vee (2x_1)) \wedge |x_3| = 0,$$

it holds that $x_3 \in (x_2 - x_1)_-^{\perp}$ proving that $x_-^{\perp} \subset (x_2 - x_1)_-^{\perp}$. Since $x_2 - x_1 \in |\mathscr{K}_X|$, by Lemma 3.1 it holds that $(x_2 - x_1)_+^{\perp} \oplus (x_2 - x_1)_-^{\perp} = X$, $(x_2 - x_1)_+^{\perp} = x_+^{\perp}$ and $(x_2 - x_1)_-^{\perp} = x_-^{\perp}$. Therefore $x_2 - x_1 \in Q(x)$ and $|\cdot|_{Q(x)}$ is surjective.

To prove that $|\cdot|_{Q(x)}$ is injective it should be proved that if $|x_1| = |x_2| = e$ and $x_1 \neq x_2$, then $Q(x_1) \neq Q(x_2)$. Note that $0 \vee (-x_1) \in (x_1)_+^{\perp}$ and $0 \vee (-x_2) \in (x_2)_+^{\perp}$. In general,

$$(0 \vee x_1) \wedge (0 \vee (-x_2)) + (0 \vee x_2) \wedge (0 \vee (-x_1)) = \frac{1}{2}(|x_1| + |x_2| - |x_1 + x_2|)$$

and $|x_1 + x_2| \wedge |x_1 - x_2| = ||x_1| - |x_2||$. Since $|x_1| = |x_2| = e$, it holds that $|x_1 + x_2| \notin \mathscr{K}_X$ and it does never hold that $|x_1| + |x_2| = |x_1 + x_2|$. Therefore

$$(0 \vee x_1) \wedge (0 \vee (-x_2)) + (0 \vee x_2) \wedge (0 \vee (-x_1)) > 0$$

and either $(0 \vee x_1) \wedge (0 \vee (-x_2)) > 0$ or $(0 \vee x_2) \wedge (0 \vee (-x_1)) > 0$, thus either $0 \vee (-x_2) \notin (x_1)_+^{\perp}$ or $0 \vee (-x_1) \notin (x_2)_+^{\perp}$. Therefore $(x_1)_+^{\perp} \neq (x_2)_+^{\perp}$ proving that $Q(x_1) \neq Q(x_2)$. $\square$

Lemma 3.4. *Let X be a vector lattice with unit satisfying the principal projection property and let $x \in |\mathscr{K}_X|$.*

If $x_1, x_2 \in Q(x)$, then $x_1 \wedge x_2, x_1 \vee x_2 \in Q(x)$.

P r o o f. Since

$$|x_1 \wedge x_2| = \frac{1}{2}|x_1 + x_2 - |x_1 - x_2|| \geqslant \frac{1}{2}||x_1 + x_2| - |x_1 - x_2|| = |x_1| \wedge |x_2|$$

and

$$|x_1 \vee x_2| = \frac{1}{2}|x_1 + x_2 + |x_1 - x_2|| \geqslant \frac{1}{2}||x_1 + x_2| - |x_1 - x_2|| = |x_1| \wedge |x_2|,$$

we have $x_1 \wedge x_2, x_1 \vee x_2 \in |\mathscr{K}_X|$. If $x_3 \in x_-^\perp = (x_1)_-^\perp = (x_2)_-^\perp$, then

$$\begin{aligned}(0 \vee (-(x_1 \wedge x_2))) \wedge |x_3| &\leqslant (0 \vee (-x_1) + 0 \vee (-x_2)) \wedge |x_3| \\ &\leqslant (0 \vee (-x_1)) \wedge |x_3| + (0 \vee (-x_2)) \wedge |x_3| = 0\end{aligned}$$

proving that $x_3 \in (x_1 \wedge x_2)_-^\perp$. Conversely, if $x_3 \in (x_1 \wedge x_2)_-^\perp$, then

$$(0 \vee (-x_1)) \wedge |x_3| \leqslant (0 \vee (-(x_1 \wedge x_2))) \wedge |x_3| = 0$$

proving that $x_3 \in (x_1)_-^\perp = x_-^\perp$. Therefore $(x_1 \wedge x_2)_-^\perp = x_-^\perp$. By Remark 3.3 it holds that $x_1 \wedge x_2 \in Q(x)$. The rest can be proved similarly. □

Lemma 3.5. *Let X be a vector lattice with unit and let $x \in |\mathscr{K}_X|$.*

If $x_1 \in \overline{Q}(x)$, $0 \wedge x_1 \leqslant x_2 \leqslant 0 \vee x_1$ and $x_2 \neq 0$, then $x_2 \in \overline{Q}(x)$.

P r o o f. Since $x_1 \in \overline{Q}(x)$, there exist $x_3, x_4 \in Q(x)$ such that $x_1 \in [0 \wedge x_3, 0 \vee x_4] \setminus \{0\}$. Since $0 \wedge x_1 \leqslant x_2 \leqslant 0 \vee x_1$ and $x_2 \neq 0$, it holds that $x_2 \in [0 \wedge x_3, 0 \vee x_4] \setminus \{0\}$. Therefore $x_2 \in \overline{Q}(x)$. □

Lemma 3.6. *Let X be a vector lattice with unit.*

(1) *Then $\alpha x_1 \in \overline{Q}(x)$ for any $x_1 \in \overline{Q}(x)$ and for any $\alpha \in \mathscr{K}_{\mathbb{R}}$.*

(2) *If X satisfies the principal projection property, then $x_1 + x_2 \in \overline{Q}(x)$ for any $x_1, x_2 \in \overline{Q}(x)$.*

P r o o f. (1) Since $x_1 \in \overline{Q}(x)$, there exist $x_3, x_4 \in Q(x)$ such that $x_1 \in [0 \wedge x_3, 0 \vee x_4] \setminus \{0\}$. Since $\alpha \in \mathscr{K}_{\mathbb{R}}$, it holds that $\alpha x_1 \in [0 \wedge (\alpha x_3), 0 \vee (\alpha x_4)] \setminus \{0\}$. Since

$$(0 \vee x) \wedge |\alpha x_3| \leqslant (1 \vee \alpha)((0 \vee x) \wedge |x_3|) = 0$$

and

$$(0 \vee x) \wedge |\alpha x_4| \leqslant (1 \vee \alpha)((0 \vee x) \wedge |x_4|) = 0,$$

it holds that $\alpha x_3, \alpha x_4 \in Q(x)$. Therefore $\alpha x_1 \in \overline{Q}(x)$.

(2) Since $x_1, x_2 \in \overline{Q}(x)$, there exists $x_3, x_4, x_5, x_6 \in Q(x)$ such that $x_1 \in [0 \wedge x_3, 0 \vee x_4] \setminus \{0\}$ and $x_2 \in [0 \wedge x_5, 0 \vee x_6] \setminus \{0\}$. Note that $0 \vee x_4, 0 \vee x_6 \in x_-^\perp$ and $0 \vee (-x_3), 0 \vee (-x_5) \in x_+^\perp$. Assume that $x_2 = -x_1$. Then

$$x_1 = x_1 \wedge (-x_2) \leqslant (0 \vee x_4) \wedge (0 \vee (-x_5)) = 0,$$
$$x_2 = x_2 \wedge (-x_1) \leqslant (0 \vee x_6) \wedge (0 \vee (-x_3)) = 0$$

proving that $x_1 = x_2 = 0$. This is a contradiction. Therefore $x_2 \neq -x_1$ and $x_1 + x_2 \in [0 \wedge 2(x_3 \wedge x_5), 0 \vee 2(x_4 \vee x_6)] \setminus \{0\}$. By Lemma 3.4 and the proof of (1) it holds that $2(x_3 \wedge x_5), 2(x_4 \vee x_6) \in Q(x)$. Therefore $x_1 + x_2 \in \overline{Q}(x)$. □

Definition 3.3. Let X be a vector lattice with unit and $a, b \in D \subset X$ with $a \neq b$.

Let $\mathbf{CSSMP}(a, b)$ be the class of $\varphi\colon [0,1] \longrightarrow D$ satisfying the following conditions:

(CS1) There exist a natural number r_φ and $\{e_\varphi^i\colon e_\varphi^i \in \mathscr{K}_X \text{ for } i = 1, \dots, r_\varphi\}$ such that the mapping

$$\begin{array}{cccc} \varphi^i\colon & [0,1] & \longrightarrow & D \\ & \cup & & \cup \\ & s & \longmapsto & \varphi((s+i-1)/r_\varphi) \end{array}$$

belongs to $\mathbf{CSMP}_{e_\varphi^i}(\varphi((i-1)/r_\varphi), \varphi(i/r_\varphi))$.

(CS2) There exists $x \in |\mathscr{K}_X|$ such that $\varphi^i(1) - \varphi^i(0) \in \overline{Q}(x)$ for any $i = 1, \dots, r_\varphi$.

(CS3) $\varphi([0,1]) \subset [a \wedge b, a \vee b]$.

φ^i satisfies either (SI) or (SD). For convenience, φ^i is said to be **CSIP** if φ^i satiefies (SI) and φ^i is **CSDP** if φ^i satisfies (SD).

Remark 3.4. By Remark 3.1, $\varphi \in \mathbf{CSSMP}(a, b)$ is equivalent to $\varphi^{\mathrm{rev}} \in \mathbf{CSSMP}(b, a)$. Since $(\varphi^{\mathrm{rev}})^{\mathrm{rev}} = \varphi$, the mapping $\varphi \longmapsto \varphi^{\mathrm{rev}}$ is bijective.

Definition 3.4. Let X be a vector lattice with unit and $D \subset X$.

D is said to be connected if $\mathbf{CSSMP}(a, b) \neq \emptyset$ for any $a, b \in D$ with $a \neq b$. Let $\mathcal{CO}_X$ be the class of connected open subsets of X.

Definition 3.5. Let X be a vector lattice with unit and $a, b \in D \in \mathcal{CO}_X$.

The subset

$$\langle a|b\rangle = \begin{cases} \displaystyle\bigcup_{\varphi \in \mathbf{CSSMP}(a,b)} \varphi([0,1]) & \text{if } a \neq b, \\ \{a\} & \text{if } a = b \end{cases}$$

is called to be a stepwise interval from a to b.

Remark 3.5. By Remark 3.4, it holds that $\varphi([0,1]) = \varphi^{\mathrm{rev}}([0,1])$. Therefore $\langle a|b\rangle$ and $\langle b|a\rangle$ coincide as sets. But the former means an "interval from a to b", the letter means another "interval from b to a" and they are distinguished.

Remark 3.6. By (CS1) (CS3) we have that $\langle a|b\rangle \subset [a \wedge b, a \vee b] \cap D$.

Definition 3.6. Let X be a vector lattice with unit, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$.

$\langle c|d\rangle$ is said to be a subinterval of $\langle a|b\rangle$ if $c, d \in \langle a|b\rangle$ and there exists $x \in |\mathscr{K}_X|$ such that $c - a, d - c, b - d \in \overline{Q}(x)$.

Remark 3.7. By Lemma 3.6 and (CS2) if X satisfies the principal projection property, then $\langle c|d\rangle \subset \langle a|b\rangle$.

Definition 3.7. Let X be a vector lattice with unit, $e \in \mathscr{K}_X$ and $a, b \in X$ with $a \leqslant b$.

For an interval $[a, b]$ we consider the subset:

$$[a,b]^e = \{x\colon \text{ there exists some } \varepsilon \in \mathscr{K}_{\mathbb{R}} \text{ such that } x - a \geqslant \varepsilon e \text{ and } b - x \geqslant \varepsilon e\}.$$

Lemma 3.7. *Let X be a vector lattice with unit, $e \in \mathscr{K}_X$ and $a, b \in X$ with $a \leqslant b$.*

Then $[a,b]^e \neq \emptyset$ if and only if there exists $\varepsilon \in \mathscr{K}_{\mathbb{R}}$ such that $b - a \geqslant \varepsilon e$.

Proof. Suppose that $[a,b]^e \neq \emptyset$. Let $x \in [a,b]^e$. By Definition 3.7 there exists $\varepsilon \in \mathscr{K}_{\mathbb{R}}$ such that $x - a \geqslant \frac{1}{2}\varepsilon e$ and $b - x \geqslant \frac{1}{2}\varepsilon e$. Therefore $b - a \geqslant \varepsilon e$.

Conversely, suppose that there exists $\varepsilon \in \mathscr{K}_{\mathbb{R}}$ such that $b - a \geqslant \varepsilon e$. Let $x = \frac{1}{2}(a+b)$. Then $x - a = b - x = \frac{1}{2}(b-a) \geqslant \frac{1}{2}\varepsilon e$. Therefore $x \in [a,b]^e$. $\square$

Definition 3.8. Let X be a vector lattice with unit.

We consider the following condition:

(M) There exists an interval function $q\colon \mathscr{I}_X \longrightarrow [0,\infty)$ such that

(M1) $q(I_1) \leqslant q(I_2)$ if $I_1 \subset I_2$.

(M2) $q(I) > 0$ if $I \in \mathscr{I}\mathscr{K}_X$.

(M3) For any $x \in X$, for any $e \in \mathscr{K}_X$ and for any $\varepsilon \in \mathscr{K}_{\mathbb{R}}$ there exists $\delta \in \mathscr{K}_{\mathbb{R}}$ such that $q([x, x+\delta e]) \leqslant \varepsilon$ and $q([x-\delta e, x]) \leqslant \varepsilon$.

Let $A \subset D \subset X$.

Given a property $P(x)$ of $x \in D$ we say to be true for nearly every $x \in A$ if there exists a countable set $N \subset D$ independent of A such that $P(x)$ holds for any $x \in A \setminus N$. $N \subset D$ is said to be a null set if for any $e \in \mathscr{K}_X$ and for any $\varepsilon \in \mathscr{K}_{\mathbb{R}}$ there exists $\{I_k\colon I_k \in \mathscr{I}_X, k = 1, 2, \ldots\}$ such that it satisfies the following conditions:

(N1) $N \subset \bigcup\limits_{k=1}^{\infty} I_k^e$.

(N2) $\sum_{k=1}^{\infty} q(I_k) \leqslant \varepsilon$. Given a property $P(x)$ of $x \in D$ we say to be true for almost every $x \in A$ if there exists a null set $N \subset D$ independent of A such that $P(x)$ holds for any $x \in A \setminus N$.

Let $P(x)$ be a property of $x \in D \in \mathcal{O}_X$ and let $A \subset D$. For convenience, expressions such that $P(x)$ uniformly for every $x \in A$, for nearly every $x \in A$, for almost every $x \in A$ and so on are used. For instance, o-$DF(x) = f(x)$ uniformly for almost every $x \in A$ means that there exists a null set $N \subset D$ such that F is uniformly differentiable on $A \setminus N$ and o-$DF(x) = f(x)$ for every $x \in A \setminus N$.

Example 3.1. If X is a Banach lattice, then X satisfies (M). For any $x_1, x_2 \in X$ with $x_1 < x_2$ let $q([x_1, x_2]) = \|x_2 - x_1\|$. Then X endowed with q satisfies (M).

If $X = \mathbb{R}^d \times X_1$, where X_1 is any vector lattice, then X also satisfies (M). For $x_1 = ((x_{1,1}, \ldots, x_{1,d}), x_1')$, $x_2 = ((x_{2,1}, \ldots, x_{2,d}), x_2')$ with $x_1 \leqslant x_2$ let $q([x_1, x_2]) = \prod_{i=1}^{d} (x_{2,i} - x_{1,i})$. Then X endowed with q satisfies (M). Moreover, $N \subset X$ is a null set if and only if the Lebesgue measure of the projection on $\mathbb{R}^d$ of N is zero.

In general, many interval functions satisfying (M) in X can be considered. Hereafter in the case of $X = \mathbb{R}^d$ we always consider the Lebesgue measure as an interval function q.

Definition 3.9. Let X be a vector lattice with unit, Y a vector lattice, $x_0 \in D \subset X$ and $F\colon D \longrightarrow Y$. Suppose that X satisfies (M).

F is said to be continuous at x_0 if it satisfies the following condition:

(C) There exists $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists $\delta \in \mathscr{K}_{\mathbb{R}}$ such that for any $x \in D$ if either $0 < x - x_0 \leqslant \delta e$ or $0 < x_0 - x \leqslant \delta e$, then $|F(x) - F(x_0)| \leqslant v_e$.

Let $\mathbf{C}(D, Y)$ be the class of mappings continuous at every point in D. F is said to be absolutely continuous if it satisfies the following condition:

(AC) There exists $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists $\delta \in \mathscr{K}_{\mathbb{R}}$ such that for any $x_{1,k}, x_{2,k} \in D$ with $x_{1,k} < x_{2,k}$ $(k = 1, \ldots, K)$

$$\text{if } \sum_{k=1}^{K} q([x_{1,k}, x_{2,k}]) \leqslant \delta, \quad \text{then } \sum_{k=1}^{K} |F(x_{2,k}) - F(x_{1,k})| \leqslant v_e.$$

Let $\mathbf{AC}(D, Y)$ be the class of absolutely continuous mappings. F is said to be restricted absolutely continuous if it satisfies the following condition:

(AC*) There exists $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists $\delta \in \mathscr{K}_\mathbb{R}$ such that for any $x_{1,k}, x_{2,k} \in D$ with $x_{1,k} < x_{2,k}$ $(k = 1, \ldots, K)$

$$\text{if } \sum_{k=1}^{K} q([x_{1,k}, x_{2,k}]) \leqslant \delta, \text{ then } \sum_{k=1}^{K} \omega(F, [x_{1,k}, x_{2,k}]) \leqslant v_e,$$

where

$$\omega(F, [u, v]) = \bigvee_{x_1, x_2 \in [u,v]} |F(x_2) - F(x_1)|$$

is the oscillation on $[u, v]$ of F.

Let $\mathbf{AC}^*(D, Y)$ be the class of restricted absolutely continuous mappings. F is said to be generalized absolutely continuous if it satisfies the following condition:

(ACG) There exists $\{E_p\colon E_p \subset D,\ p = 1, 2, \ldots\}$ with $\bigcup\limits_{p=1}^{\infty} E_p = D$ and $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any natural number p and for any $e \in \mathscr{K}_X$ there exists $\delta \in \mathscr{K}_\mathbb{R}$ such that for any $x_{1,k}, x_{2,k} \in D$ with $x_{1,k} < x_{2,k}$ and, $x_{1,k} \in E_p$ or $x_{2,k} \in E_p$ $(k = 1, \ldots, K)$

$$\text{if } \sum_{k=1}^{K} q([x_{1,k}, x_{2,k}]) \leqslant \delta, \text{ then } \sum_{k=1}^{K} |F(x_{2,k}) - F(x_{1,k})| \leqslant v_e.$$

Let $\mathbf{ACG}(D, Y)$ be the class of generalized absolutely continuous mappings. F is said to be restricted generalized absolutely continuous if it satisfies the following condition:

(ACG*) There exists $\{E_p\colon E_p \subset D,\ p = 1, 2, \ldots\}$ with $\bigcup\limits_{p=1}^{\infty} E_p = D$ and $\{v_e\} \in \mathscr{U}_Y^s(\mathscr{K}_X, \geqslant)$ such that for any natural number p and for any $e \in \mathscr{K}_X$ there exists $\delta \in \mathscr{K}_\mathbb{R}$ such that for any $x_{1,k}, x_{2,k} \in D$ with $x_{1,k} < x_{2,k}$ and, $x_{1,k} \in E_p$ or $x_{2,k} \in E_p$ $(k = 1, \ldots, K)$

$$\text{if } \sum_{k=1}^{K} q([x_{1,k}, x_{2,k}]) \leqslant \delta, \text{ then } \sum_{k=1}^{K} \omega(F, [x_{1,k}, x_{2,k}]) \leqslant v_e.$$

Let $\mathbf{ACG}^*(D, Y)$ be the class of generalized absolutely continuous mappings.

Remark 3.8. $\mathbf{C}(D, Y)$ is a vector lattice. First, it is clearly an ordered linear space. For $F_1, F_2 \in \mathbf{C}(D, Y)$ we have

$$\begin{aligned} |(F_1 \vee F_2)(x) - (F_1 \vee F_2)(x_0)| &= |F_1(x) \vee F_2(x) - F_1(x_0) \vee F_2(x_0)| \\ &\leqslant |F_1(x) - F_1(x_0)| + |F_2(x) - F_2(x_0)| \end{aligned}$$

and

$$|(F_1 \wedge F_2)(x) - (F_1 \wedge F_2)(x_0)| = |F_1(x) \wedge F_2(x) - F_1(x_0) \wedge F_2(x_0)| \leqslant |F_1(x) - F_1(x_0)| + |F_2(x) - F_2(x_0)|,$$

$\mathbf{C}(D, Y)$ is a lattice. Similarly it is proved that $\mathbf{AC}(D, Y)$, $\mathbf{AC}^*(D, Y)$, $\mathbf{ACG}(D, Y)$ and $\mathbf{ACG}^*(D, Y)$ are also vector lattices.

Lemma 3.8. *Let X be a vector lattice with unit, Y a complete vector lattice and $D \in \mathscr{O}_X$.*

If $F\colon D \longrightarrow Y$ is differentiable at $x_0 \in D$, then F is continuous at x_0. In particular, by Remark 2.2 any element of $\mathscr{L}(X, Y)$ is continuous.

Proof. By assumption there exists $\{w^{\pm}_{x_0,e}\} \in \mathscr{U}^s_{\mathscr{L}(X,Y)}(\mathscr{K}_X, \geqslant)$ such that for any $e \in \mathscr{K}_X$ there exists $\delta^{\pm}_{x_0} \in \mathscr{K}_{\mathbb{R}}$ such that $|F(x_0 \pm h) - F(x_0) \mp f(x_0)(h)| \leqslant w^{\pm}_{x_0,e}(h)$ for any $h \in X$ with $0 < h \leqslant \delta^{\pm}_{x_0} e$. Let $\{v_{1,e}\} \in \mathscr{U}^s_Y(\mathscr{K}_X, \geqslant)$ and $v_e = v_{1,e} + (|f(x_0)| + w^{+}_{x_0,e} + w^{-}_{x_0,e})(e)$. By Remark 2.1 it holds that $\{v_e\} \in \mathscr{U}^s_Y(\mathscr{K}_X, \geqslant)$. Let $\delta_{x_0} = \delta^{+}_{x_0} \wedge \delta^{-}_{x_0}$. Without loss of generality it may be assumed that $\delta_{x_0} \leqslant 1$. For any $x \in D$ if $0 < x - x_0 \leqslant \delta_{x_0} e$, then

$$|F(x) - F(x_0)| \leqslant |f(x_0)(x - x_0)| + w^{+}_{x_0,e}(x - x_0) \leqslant (|f(x_0)| + w^{+}_{x_0,e})(e),$$

and if $0 < x_0 - x \leqslant \delta_{x_0} e$, then

$$|F(x_0) - F(x)| \leqslant |f(x_0)(x_0 - x)| + w^{-}_{x_0,e}(x_0 - x) \leqslant (|f(x_0)| + w^{-}_{x_0,e})(e).$$

In either case we have $|F(x) - F(x_0)| \leqslant v_e$. Therefore F is continuous at x_0. □

3.2. Denjoy integral.

Definition 3.10. Let X be a vector lattice with unit, Y a complete vector lattice, $D \in \mathscr{CO}_X$ and $f\colon D \longrightarrow \mathscr{L}(X, Y)$. Suppose that X satisfies (M).

For $a, b \in D$ f is said to be Denjoy integrable on $\langle a|b\rangle$ and F is the Denjoy primitive of f on $\langle a|b\rangle$ if there exists $F \in \mathbf{ACG}^*(D, Y) \cap \mathbf{C}(D, Y)$ such that o-$DF(x) = f(x)$ uniformly for almost every $x \in \langle a|b\rangle$. If for any $a, b \in D$, f is Denjoy integrable on $\langle a|b\rangle$, then f is said to be Denjoy integrable on D and F is a Denjoy primitive of f, denoted by

$$F(x) = o\text{-}(D^*) \int f(x)\,\mathrm{d}x.$$

The value

$$F(b) - F(a) = o\text{-}(D^*) \int_a^b f(x)\,\mathrm{d}x$$

is said to be the Denjoy integral of f on $\langle a|b\rangle$. Let $(\mathbf{D}^*)(\langle a|b\rangle, Y)$ and $(\mathbf{D}^*)(D, Y)$ be the class of Denjoy integrable mappings on $\langle a|b\rangle$ and D, respectively.

We must show that Definition 3.10 is well-defined, that is, if the difference of constant values is disregarded, then for any Denjoy integrable mapping f its Denjoy primitive F is uniquely determined on $\langle a|b\rangle$.

Lemma 3.9. *Let X be a vector lattice with unit, $a, b \in D \in \mathcal{CO}_X$ with $a \neq b$ and $\varphi \in \mathbf{CSSMP}(a, b)$.*

If $\varphi^i([0,1]) \subset \bigcup\limits_{\lambda\in\Lambda} [c_\lambda, d_\lambda]^{e^i_\varphi}$ for $c_\lambda, d_\lambda \in D$ $(\lambda \in \Lambda)$ with $\varphi^i([0,1]) \cap [c_\lambda, d_\lambda]^{e^i_\varphi} \neq \emptyset$, then

(1) *For any $\lambda \in \Lambda$ there exists $I_\lambda = [0,1], (\alpha_\lambda, 1], [0, \beta_\lambda)$ or $(\alpha_\lambda, \beta_\lambda)$ with $0 \leqslant \alpha_\lambda < \beta_\lambda \leqslant 1$ such that $\varphi^i([0,1]) \cap [c_\lambda, d_\lambda]^{e^i_\varphi} = \varphi^i(I_\lambda)$.*

(2) *It is possible to select a finite subset $\{I_{\lambda_k}\colon k = 1, \ldots, K\}$ in $\{I_\lambda\colon \lambda \in \Lambda\}$ such that $[0,1] = \bigcup\limits_{k=1}^{K} I_{\lambda_k}$.*

Proof. We prove the case where φ^i is **CSIP**. When φ^i is **CSDP**, it can be proved similarly. We consider the following four cases.

(Case I) $\varphi^i(0), \varphi^i(1) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$:

Clearly (1) is satisfied for $I_\lambda = [0,1]$.

(Case II) $\varphi^i(0) \notin [c_\lambda, d_\lambda]^{e^i_\varphi}$ and $\varphi^i(1) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$:

Let $\alpha_\lambda = \inf\limits_{\varphi^i(s)\in[c_\lambda,d_\lambda]^{e^i_\varphi}} s$. Then $\varphi^i(s) \notin [c_\lambda, d_\lambda]^{e^i_\varphi}$ if $s < \alpha_\lambda$ and $\varphi^i(s) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$ if $\alpha_\lambda < s$. Assume that $\varphi^i(\alpha_\lambda) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$. Then there exists $\varepsilon \in \mathscr{K}_\mathbb{R}$ such that $\varphi^i(\alpha_\lambda) - c_\lambda \geqslant \varepsilon e^i_\varphi$ and $d_\lambda - \varphi^i(\alpha_\lambda) \geqslant \varepsilon e^i_\varphi$. There exists $\delta \in \mathscr{K}_\mathbb{R}$ such that for any $s \in [0,1]$ if $|s - \alpha_\lambda| \leqslant \delta$, then $|\varphi^i(s) - \varphi^i(\alpha_\lambda)| \leqslant \frac{1}{2}\varepsilon e^i_\varphi$. Since

$$\varphi^i(\alpha_\lambda - \delta) - c_\lambda \geqslant \varphi^i(\alpha_\lambda) - \frac{1}{2}\varepsilon e^i_\varphi - c_\lambda \geqslant \frac{1}{2}\varepsilon e^i_\varphi$$

and

$$d_\lambda - \varphi^i(\alpha_\lambda - \delta) \geqslant d_\lambda - \varphi^i(\alpha_\lambda) \geqslant \varepsilon e^i_\varphi,$$

it holds that $\varphi^i(\alpha_\lambda - \delta) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$. It is a contradiction. Therefore $\varphi^i(\alpha_\lambda) \notin [c_\lambda, d_\lambda]^{e^i_\varphi}$ proving that (1) is satisfied for $I_\lambda = (\alpha_\lambda, 1]$.

(Case III) $\varphi^i(0) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$ and $\varphi^i(1) \notin [c_\lambda, d_\lambda]^{e^i_\varphi}$:

Let $\beta_\lambda = \sup\limits_{\varphi^i(s)\in[c_\lambda,d_\lambda]^{e^i_\varphi}} s$. Then $\varphi^i(s) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$ if $s < \beta_\lambda$ and $\varphi^i(s) \notin [c_\lambda, d_\lambda]^{e^i_\varphi}$ if $\beta_\lambda < s$. Assume that $\varphi^i(\beta_\lambda) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$. Then there exists $\varepsilon \in \mathscr{K}_\mathbb{R}$ such that $\varphi^i(\beta_\lambda) - c_\lambda \geqslant \varepsilon e^i_\varphi$ and $d_\lambda - \varphi^i(\beta_\lambda) \geqslant \varepsilon e^i_\varphi$. There exists $\delta \in \mathscr{K}_\mathbb{R}$ such that for any $s \in [0,1]$ if $|s - \beta_\lambda| \leqslant \delta$, then $|\varphi^i(s) - \varphi^i(\beta_\lambda)| \leqslant \frac{1}{2}\varepsilon e^i_\varphi$. Since

$$\varphi^i(\beta_\lambda + \delta) - c_\lambda \geqslant \varphi^i(\beta_\lambda) - c_\lambda \geqslant \varepsilon e^i_\varphi$$

and

$$d_\lambda - \varphi^i(\beta_\lambda + \delta) \geqslant d_\lambda - \varphi^i(\beta_\lambda) - \frac{1}{2}\varepsilon e^i_\varphi \geqslant \frac{1}{2}\varepsilon e^i_\varphi,$$

it holds that $\varphi^i(\beta_\lambda + \delta) \in [c_\lambda, d_\lambda]^{e^i_\varphi}$. It is a contradiction. Therefore $\varphi^i(\beta_\lambda) \notin [c_\lambda, d_\lambda]^{e^i_\varphi}$ proving that (1) is satisfied for $I_\lambda = [0, \beta_\lambda)$.

(Case IV) $\varphi^i(0), \varphi^i(1) \notin [c_\lambda, d_\lambda]^{e^i_\varphi}$:

Let $\alpha_\lambda = \inf\limits_{\varphi^i(s)\in[c_\lambda,d_\lambda]^{e^i_\varphi}} s$ and $\beta_\lambda = \sup\limits_{\varphi^i(s)\in[c_\lambda,d_\lambda]^{e^i_\varphi}} s$. Then similarly (1) is satisfied for $I_\lambda = (\alpha_\lambda, \beta_\lambda)$.

Next we show (2). Since $\varphi^i([0,1]) = \bigcup\limits_{\lambda\in\Lambda} \varphi^i(I_\lambda)$ and φ^i is injective, it holds that $[0,1] = \bigcup\limits_{\lambda\in\Lambda} I_\lambda$. Since $[0,1]$ is compact, (2) is satisfied. $\square$

Lemma 3.10. *Let X be a vector lattice with unit, Y a complete vector lattice, $a, b \in D \in \mathcal{CO}_X$ with $a \neq b$ and $\varphi \in$ **CSSMP**(a,b). Suppose that X satisfies* (M) *and let $N \subset D$ be a null set.*

*If $F \in$ **ACG**$^*(D,Y) \cap$ **C**(D,Y) and o-$DF(x) \geqslant 0$ uniformly for every $x \in \varphi^i([0,1]) \setminus N$, then $F(\varphi^i(0)) \leqslant F(\varphi^i(1))$ when φ^i is* **CSIP** *and $F(\varphi^i(0)) \geqslant F(\varphi^i(1))$ when φ^i is* **CSDP**.

P r o o f. We prove the case where φ^i is **CSIP**. When φ^i is **CSDP**, it can be proved similarly. Let f be the derivative of F. Since $F \in$ **ACG**$^*(D,Y)$, there exists $\{E_p\colon E_p \subset D,\ p = 1, 2, \ldots\}$ with $\bigcup\limits_{p=1}^{\infty} E_p = D$ and $\{v_e\} \in \mathscr{U}^s_Y(\mathscr{K}_X, \geqslant)$ such that for any natural number p and for any $\varepsilon \in \mathscr{K}_X$ there exists $\delta_p \in \mathscr{K}_\mathbb{R}$ such that for any $x_{1,k}, x_{2,k} \in D$ with $x_{1,k} < x_{2,k}$ and $x_{1,k} \in E_p$ or $x_{2,k} \in E_p$ $(k = 1, \ldots, K)$

$$\text{if } \sum_{k=1}^{K} q([x_{1,k}, x_{2,k}]) \leqslant \delta_p, \text{ then } \sum_{k=1}^{K} \omega(F, [x_{1,k}, x_{2,k}]) \leqslant v_{\theta(\varepsilon e^i_\varphi, p)\varepsilon e^i_\varphi} \leqslant 2^{-p} v_{\varepsilon e^i_\varphi}.$$

Since $N_p = N \cap E_p$ is a null set, there exists $\{[a_{p,j}, b_{p,j}]\colon j = 1, 2, \ldots\}$ such that

$$N_p \subset \bigcup_{j=1}^{\infty} [a_{p,j}, b_{p,j}]^{e^i_\varphi} \quad \text{and} \quad \sum_{j=1}^{\infty} q([a_{p,j}, b_{p,j}]) \leqslant \delta_p.$$

Since $\bigcup\limits_{p=1}^{\infty} N_p = N$, it holds that $N \subset \bigcup\limits_{p=1}^{\infty} \bigcup\limits_{j=1}^{\infty} [a_{p,j}, b_{p,j}]^{e^i_\varphi}$. Since F is uniformly differentiable on $x \in \varphi^i([0,1]) \setminus N$, there exists $\{w_e\} \in \mathscr{U}^s_{\mathscr{L}(X,Y)}(\mathscr{K}_X, \geqslant)$ such that for any $x \in \varphi^i([0,1]) \setminus N$ and for any $\varepsilon \in \mathscr{K}_X$ there exists $\delta^\pm_x \in \mathscr{K}_\mathbb{R}$ such that $|F(x \pm h) - F(x) \mp f(x)(h)| \leqslant w_{\varepsilon e^i_\varphi}(h)$ for any $h \in X$ with $0 < h \leqslant \delta^\pm_x \varepsilon e^i_\varphi$. Moreover,

$$\varphi^i([0,1]) \setminus N \subset \bigcup_{x\in\varphi^i([0,1])\setminus N} [x - \delta^-_x \varepsilon e^i_\varphi, x + \delta^+_x \varepsilon e^i_\varphi]^{e^i_\varphi}.$$

By Lemma 3.9 there exist $I_k \subset [0,1]$ $(k=1,\ldots,K)$, $x_k \in \varphi^i(I_k)$ $(k=1,\ldots,K_1)$, $p_{K_1+1} < \ldots < p_K$ and $j_{K_1+1} < \ldots < j_K$ such that

$$\begin{aligned}\varphi^i([0,1]) \cap [x_k - \delta^-_{x_k}\varepsilon e^i_\varphi, x_k + \delta^+_{x_k}\varepsilon e^i_\varphi]^{e^i_\varphi} &= \varphi^i(I_k) \ (k=1,\ldots,K_1),\\ \varphi^i([0,1]) \cap [a_{p_k,j_k}, b_{p_k,j_k}]^{e^i_\varphi} &= \varphi^i(I_k) \ (k=K_1+1,\ldots,K),\\ [0,1] &= \bigcup_{k=1}^{K} I_k.\end{aligned}$$

Let α_k be the left end of I_k and β_k the right end of I_k. Order I_k according to increasing α_k and denote them by I_k's again. Without loss of generality it may be assumed that an I_k is not covered by the union of other I_k's because the above formulae are true even if I_k covered the union of other I_k's is excepted. Then

$$\begin{aligned}&0 = \alpha_1 < \alpha_2,\\ &\alpha_k < \beta_{k-1} < \alpha_{k+1} < \beta_k \ (k=2,\ldots,K-1),\\ &\beta_{K-1} < \beta_K = 1.\end{aligned}$$

Let

$$\begin{aligned}&\gamma_0 = \alpha_1 = 0,\\ &\alpha_k < \gamma_{k-1} < \beta_{k-1}, \quad \text{where}\\ &\qquad x_{k-1} < \varphi^i(\gamma_{k-1}) < x_k \ \text{ if } \ x_{k-1} < x_k\\ &\qquad \text{and } \ x_{k-1} > \varphi^i(\gamma_{k-1}) > x_k \ \text{ if } \ x_{k-1} > x_k\\ &\qquad\qquad (k=2,\ldots,K),\\ &\gamma_K = \beta_K = 1.\end{aligned}$$

When $\varphi^i([0,1]) \cap [a_{p_k,j_k}, b_{p_k,j_k}]^{e^i_\varphi} = \varphi^i(I_k)$, let x_k satisfy $a_{p_k,j_k} < x_k < b_{p_k,j_k}$, for instance, $x_k = \frac{1}{2}(a_{p_k,j_k} + b_{p_k,j_k})$. Since F is absolutely continuous on E_p, we have

$$\sum_k (F(\varphi^i(\gamma_k)) - F(\varphi^i(\gamma_{k-1}))) \geqslant -\sum_{p=1}^{\infty} 2^{-p} v_{\varepsilon e^i_\varphi} = -v_{\varepsilon e^i_\varphi}.$$

When $\varphi^i([0,1]) \cap [x_k - \delta^-_{x_k}\varepsilon e^i_\varphi, x_k + \delta^+_{x_k}\varepsilon e^i_\varphi]^{e^i_\varphi} = \varphi^i(I_k)$, we consider the following cases.

(Case I) $K_1 = 1$:

Since $\varphi^i(\gamma_0) \leqslant x_1 \leqslant \varphi^i(\gamma_1)$, it holds that $0 \leqslant \varphi^i(\gamma_1) - x_1 \leqslant \delta^+_{x_1}\varepsilon e^i_\varphi$ and $0 \leqslant x_1 - \varphi^i(\gamma_0) \leqslant \delta^-_{x_1}\varepsilon e^i_\varphi$. Therefore

$$\begin{aligned}&|F(\varphi^i(\gamma_1)) - F(x_1) - f(x_1)(\varphi^i(\gamma_1) - x_1)| \leqslant w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_1) - x_1),\\ &|F(x_1) - F(\varphi^i(\gamma_0)) - f(x_1)(x_1 - \varphi^i(\gamma_0))| \leqslant w_{\varepsilon e^i_\varphi}(x_1 - \varphi^i(\gamma_0)).\end{aligned}$$

Since $f(x_1) \geqslant 0$, it holds that

$$F(\varphi^i(\gamma_1)) - F(\varphi^i(\gamma_0)) \geqslant -w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_1) - \varphi^i(\gamma_0)) \geqslant -w_{\varepsilon e^i_\varphi}(\varphi^i(1) - \varphi^i(0)).$$

(Case II) $K_1 \geqslant 2$:

(Case II-1) $\varphi^i(\gamma_{k-1}) \leqslant x_k \leqslant \varphi^i(\gamma_k)$:

Since $0 \leqslant \varphi^i(\gamma_k) - x_k \leqslant \delta^+_{x_k}\varepsilon e^i_\varphi$ and $0 \leqslant x_k - \varphi^i(\gamma_{k-1}) \leqslant \delta^-_{x_k}\varepsilon e^i_\varphi$, it holds that

$$\begin{aligned}|F(\varphi^i(\gamma_k)) - F(x_k) - f(x_k)(\varphi^i(\gamma_k) - x_k)| &\leqslant w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_k) - x_k),\\ |F(x_k) - F(\varphi^i(\gamma_{k-1})) - f(x_k)(x_k - \varphi^i(\gamma_{k-1}))| &\leqslant w_{\varepsilon e^i_\varphi}(x_k - \varphi^i(\gamma_{k-1})).\end{aligned}$$

Since $f(x_k) \geqslant 0$, it holds that

$$F(\varphi^i(\gamma_k)) - F(\varphi^i(\gamma_{k-1})) \geqslant -w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_k) - \varphi^i(\gamma_{k-1})).$$

(Case II-2) $\varphi^i(\gamma_{k-1}) < \varphi^i(\gamma_k) < x_k$:

Note that this case occurs in the case of $k < K$. Since $0 < x_k - \varphi^i(\gamma_k) \leqslant \delta^-_{x_k}\varepsilon e^i_\varphi$ and $0 < x_k - \varphi^i(\gamma_{k-1}) \leqslant \delta^-_{x_k}\varepsilon e^i_\varphi$, it holds that

$$\begin{aligned}|F(x_k) - F(\varphi^i(\gamma_k)) - f(x_k)(x_k - \varphi^i(\gamma_k))| &\leqslant w_{\varepsilon e^i_\varphi}(x_k - \varphi^i(\gamma_k)),\\ |F(x_k) - F(\varphi^i(\gamma_{k-1})) - f(x_k)(x_k - \varphi^i(\gamma_{k-1}))| &\leqslant w_{\varepsilon e^i_\varphi}(x_k - \varphi^i(\gamma_{k-1})).\end{aligned}$$

Since $f(x_k) \geqslant 0$, it holds that

$$\begin{aligned}F(\varphi^i(\gamma_k)) - F(\varphi^i(\gamma_{k-1})) &\geqslant -w_{\varepsilon e^i_\varphi}(2x_k - \varphi^i(\gamma_k) - \varphi^i(\gamma_{k-1}))\\ &\geqslant -2w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_{k+1}) - \varphi^i(\gamma_{k-1})).\end{aligned}$$

(Case II-3) $x_k < \varphi^i(\gamma_{k-1}) < \varphi^i(\gamma_k)$:

Note that this case occurs in the case of $k > 1$. Since $0 < \varphi^i(\gamma_k) - x_k \leqslant \delta^+_{x_k}\varepsilon e^i_\varphi$ and $0 < \varphi^i(\gamma_{k-1}) - x_k \leqslant \delta^+_{x_k}\varepsilon e^i_\varphi$, it holds that

$$\begin{aligned}|F(\varphi^i(\gamma_k)) - F(x_k) - f(x_k)(\varphi^i(\gamma_k) - x_k)| &\leqslant w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_k) - x_k),\\ |F(\varphi^i(\gamma_{k-1})) - F(x_k) - f(x_k)(\varphi^i(\gamma_{k-1}) - x_k)| &\leqslant w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_{k-1}) - x_k).\end{aligned}$$

Since $f(x_k) \geqslant 0$, it holds that

$$\begin{aligned}F(\varphi^i(\gamma_k)) - F(\varphi^i(\gamma_{k-1})) &\geqslant -w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_k) + \varphi^i(\gamma_{k-1}) - 2x_k)\\ &\geqslant -2w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_k) - \varphi^i(\gamma_{k-2})).\end{aligned}$$

In any (Case II-1), (Case II-2) or (Case II-3) we have

$$
\begin{aligned}
&F(\varphi^i(\gamma_k)) - F(\varphi^i(\gamma_{k-1}))\\
&\qquad \geqslant \begin{cases} -2w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_{k+1}) - \varphi^i(\gamma_{k-1})) & \text{if } \ k=1,\\ -2w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_{k+1}) - \varphi^i(\gamma_{k-2})) & \text{if } \ 2 \leqslant k \leqslant l-1,\\ -2w_{\varepsilon e^i_\varphi}(\varphi^i(\gamma_{k}) - \varphi^i(\gamma_{k-2})) & \text{if } \ k=K. \end{cases}
\end{aligned}
$$

Summing up for k,

$$\sum_k (F(\varphi^i(\gamma_k)) - F(\varphi^i(\gamma_{k-1}))) \geqslant -6w_{\varepsilon e^i_\varphi}(\varphi^i(1) - \varphi^i(0)).$$

Therefore in either (Case I) or (Case II) we have that $F(\varphi^i(1)) - F(\varphi^i(0)) \geqslant -6w_{\varepsilon e^i_\varphi}(\varphi^i(1) - \varphi^i(0)) - v_{\varepsilon e^i_\varphi}$. Since ε is arbitrary, by Lemma 2.1 we have $F(\varphi^i(1)) - F(\varphi^i(0)) \geqslant 0$. □

By Lemma 3.10 it can be proved that the Denjoy integral is well-defined.

Theorem 3.1. *Let X be a vector lattice with unit, Y a complete vector lattice, $a, b \in D \in \mathcal{CO}_X$ and let $f\colon D \longrightarrow \mathscr{L}(X,Y)$ be Denjoy integrable on $\langle a|b\rangle$. Suppose that X satisfies* (M).

Then the Denjoy primitive of f is uniquely determined on $\langle a|b\rangle$.

P r o o f. Let $F, G \in \mathbf{ACG}^*(D,Y) \cap \mathbf{C}(D,Y)$ be two Denjoy primitives of f. We shall show that $(F-G)(a) = (F-G)(c) = (F-G)(b)$ for any $c \in \varphi([0,1])$, where $\varphi \in \mathbf{CSSMP}(a,b)$. Without loss of generality it may be assumed that there exists a natural number i such that $c = \varphi(i/r_\varphi)$. Then there exist null sets N_F, N_G such that $o\text{-}DF(\varphi(t)) = f(\varphi(t))$ for any $t \in [0,1] \setminus \varphi^{-1}(N_F)$ and $o\text{-}DG(\varphi(t)) = f(\varphi(t))$ for any $t \in [0,1] \setminus \varphi^{-1}(N_G)$. By Theorem 2.1 for any $t \in [0,1] \setminus \varphi^{-1}(N_F \cup N_G)$ we have

$$o\text{-}D(F-G)(\varphi(t)) = o\text{-}DF(\varphi(t)) - o\text{-}DG(\varphi(t)) = f(\varphi(t)) - f(\varphi(t)) = 0.$$

Similarly $o\text{-}D(G-F)(\varphi(t)) = 0$. By Lemma 3.10

$$
\begin{aligned}
(F-G)(\varphi^i(0)) &\leqslant (F-G)(\varphi^i(1)),\\
(G-F)(\varphi^i(0)) &\leqslant (G-F)(\varphi^i(1)).
\end{aligned}
$$

Thus

$$(F-G)\Big(\varphi\Big(\frac{i-1}{r_\varphi}\Big) = (F-G)\Big(\varphi\Big(\frac{i}{r_\varphi}\Big)\Big)\Big).$$

Therefore $(F-G)(a) = (F-G)(c) = (F-G)(b)$. □

In general, integrals should satisfy the following conditions:

(1) Linearity of integrand, that is, the space consisting of integrable mappings is linear and for any integrable mappings f, g and for any $\alpha, \beta \in \mathbb{R}$

$$\int (\alpha f + \beta g)(x)\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x + \beta \int g(x)\,\mathrm{d}x.$$

(2) Additivity of interval, that is, for any $a, b, c \in D$ if f is integrable from a to b and from b to c, then it is integrable from a to c and

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x.$$

(3) Integrability on subinterval, that is, if f is integrable on an interval, then it is also integrable on any subinterval of the interval.

For the Denjoy integral (1) is clear by Theorem 2.1 and Definition 3.10. (3) is true by Remark 3.7 if X satisfies the principal projection property. (2) is not true generally. Nonetheless, if $f \in (\mathbf{D}^*)(\langle a|b\rangle, Y) \cap (\mathbf{D}^*)(\langle b|c\rangle, Y) \cap (\mathbf{D}^*)(\langle c|a\rangle, Y)$, then (2) is true.

4. Fundamental theorem of calculus

The following fundamental theorem of calculus is clear by Definition 3.10.

Theorem 4.1. *Let X be a vector lattice with unit, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$. Suppose that X satisfies* (M).

If o-$DF(x) = f(x)$ for $F \in \mathbf{ACG}^*(D, Y) \cap \mathbf{C}(D, Y)$ and for almost every $x \in \langle a|b\rangle$, then f is Denjoy integrable on $\langle a|b\rangle$ and for any $x \in \langle a|b\rangle$

$$F(x) = o\text{-}(D^*) \int f(x)\,\mathrm{d}x.$$

Conversely, if $F\colon \langle a|b\rangle \longrightarrow Y$ is a Denjoy primitive of f, then it is differentiable and o-$DF(x) = f(x)$ for almost every $x \in \langle a|b\rangle$.

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