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ON MINIMAL STRONGLY KC-SPACES

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Abstract. In this article we introduce the notion of strongly KC-spaces, that is, those spaces in which countably compact subsets are closed. We find they have good properties. We prove that a space (X,τ) is maximal countably compact if and only if it is minimal strongly KC, and apply this result to study some properties of minimal strongly KC-spaces, some of which are not possessed by minimal KC-spaces. We also give a positive answer to a question proposed by O. T. Alas and R. G. Wilson, who asked whether every countably compact KC-space of cardinality less than c has the FDS-property. Using this we obtain a characterization of Katětov strongly KC-spaces and finally, we generalize one result of Alas and Wilson on Katětov-KC spaces.

Keywords: KC-space, strongly KC-space, FDS-property, maximal (countably) compact

MSC 2010: 54A10, 54D25, 54D55

1. Introduction

The notion of KC-space was first introduced by A. Wilansky [12] in 1967. A topological space (X,τ) is called a KC-space if every compact subset is closed. One of the old questions on KC-spaces posed by R. Larson [9] is whether a space is maximal compact if and only if it is minimal KC. Many authors have investigated this problem, among them we might mention [1], [2], [10] and [11]. However, up to now, Larson's original question remains open and in the past few years, many new problems were formulated. For example, it is still an open problem whether a closed subspace of a minimal KC-space is minimal KC [1] and Alas [2] asked whether every countably compact KC-space of size less than c has the FDS-property. A related question to R. Larson's is whether every KC-space is Katětov-KC, that is, whether every KC topology contains a minimal KC topology. W. Fleissner [5] proved that

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this is not always true. Recently, for countable KC-spaces, a characterization of Katětov-KC spaces has been given [2].

In this article, we introduce the notion of strongly KC-spaces, that is, those spaces in which every countably compact subset is closed. We find minimal strongly KC-spaces have many nice properties, some of which are not possessed by minimal KC-spaces or remain uncertain for them. In the first section of this paper, we outline some known notions and preliminary results which will be used in the sequel. In the second section, we briefly discuss the relationship between strongly KC and KC-spaces. A natural question analogous to R. Larson's is whether a space (X,τ) is maximal countably compact if and only if it is minimal strongly KC. We give a positive answer to this question in Section 3. Applying this we show that minimal strongly KC-spaces are closed hereditarily and study some properties of them. We also answer affirmatively Question D of [2], and using this, for strongly KC-spaces of size less than c, we give a characterization of Katětov strongly KC-spaces. Finally we generalize Theorem 18 of [2] to hereditarily Lindelof spaces.

We first recall several definitions.

Definition 1.1 ([2]). If $\mathscr{P}$ is a topological property, then a space (X,τ) is said to be minimal $\mathscr{P}$ (respectively, maximal $\mathscr{P}$) if (X,τ) has property $\mathscr{P}$ but no topology on X which is strictly smaller (respectively, strictly larger) than τ has $\mathscr{P}$.

A space (X,τ) is said to be Katětov $\mathscr{P}$ if there is a topology $\sigma \subset \tau$ such that (X,σ) is minimal $\mathscr{P}$.

Specifically, we are interested here in minimal (strongly) KC-spaces, Katětov (strongly) KC-spaces and maximal (countably) compact spaces.

Definition 1.2 ([6]). A filter over a set X is a collection $\mathscr{F}$ of subsets of X such that

(i) $\emptyset \in \mathscr{F}$;
(ii) if $F_1 \in \mathscr{F}$ and $F_2 \in \mathscr{F}$ then $F_1 \cap F_2 \in \mathscr{F}$;
(iii) if $A, B \subset X$, $A \in \mathscr{F}$ and $B \subset A$ then $B \in \mathscr{F}$.

If a filter on X has the property that there is no filter on X which is strictly finer than $\mathscr{F}$, $\mathscr{F}$ is called an ultrafilter on X.

Following [11], for κ an infinite cardinal number, an ultrafilter $\mathscr{F}$ over κ is called uniform if $|F| = \kappa$ for all $F \in \mathscr{F}$.

Notice the following crucial property of the ultrafilter [6]:

If $\mathscr{F}$ is an ultrafilter in X and the union of two sets is a member of $\mathscr{F}$, then one of the two sets belongs to $\mathscr{F}$. In particular, if A is a subset of X, then either A or $X - A$ belongs to $\mathscr{F}$.

Definition 1.3 ([4]). A topological space is called a sequential space if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits.

Definition 1.4 ([2]). A space is said to have the finite derived set property (hereafter abbreviated as the FDS-property) if each infinite subset $A \subset X$ contains an infinite subset with only finitely many accumulation points in X.

Definition 1.5 ([8]). A topological space is called a US-space provided that each convergent sequence has a unique limit.

By definitions, clearly we have

$$\text{Hausdorff } \Rightarrow \text{KC} \Rightarrow US \Rightarrow \text{T}_1.$$

The following results are known and will be used in the next two sections.

Lemma 1.6 ([10]). *A maximal compact space is* KC*, and is minimal* KC.

Lemma 1.7 ([3]). *A topological space is maximal (countably) compact if and only if its (countably) compact subsets are precisely the closed sets.*

Lemma 1.8 ([1]). *A first countable* KC*-space is minimal* KC *if and only if it is compact Hausdorff.*

Lemma 1.9 ([2]). *A compact, countable* KC*-space is sequential.*

Lemma 1.10 ([11]). *Minimal* KC*-spaces are countably compact.*

Lemma 1.11 ([1]). *A compact, hereditarily Lindelof* KC*-space is sequential.*

Lemma 1.12 ([1]). *Every sequential* KC*-space is Katětov*-KC.

In this article, for $A \subset X$, the cardinality of A is denoted by $|A|$. The closure of a set A in a topological space (X, τ) is denoted by $\mathrm{cl}_\tau(A)$, or simply by $\mathrm{cl}(A)$ if no confusion is possible, and the set of accumulation points of A with respect to the topology τ is denoted by A^d_τ or simply A^d if no confusion arises. Denote the relative topology of the set A with respect to the topology τ by $\tau|A$. The symbols ω and ω_1 stand for the first infinite and the first uncountable ordinal number respectively and $c = 2^\omega$. All notation and terminology not defined here can be found in [4].

2. Relationship between strongly KC and KC-spaces

By definitions, it is clear that every strongly KC-space is KC, while Example 2.1 below shows that the inverse is not always true.

Example 2.1. Let $X = [0, \omega_1]$. Obviously X is a KC-space since X is Hausdorff. However, since $[0, \omega_1)$ is countably compact but not closed, it follows from the definition of strongly KC-space that X is not strongly KC.

Though strongly KC-spaces must be KC, minimal strongly KC and minimal KC-spaces do not imply each other. We will now illustrate this by examples. First we present two lemmas.

Lemma 2.2. *Let (X, τ) be a maximal countably compact space, then (X, τ) is minimal strongly* KC.

Proof. It follows from Lemma 1.7 and the definition of strongly KC-space that (X, τ) is strongly KC. Let $\sigma \subset \tau$ but $\sigma \neq \tau$ be a topology on X. Take any $U \in \tau \setminus \sigma$, then $X \setminus U$ is closed in (X, τ). Thus, $X \setminus U$ is countably compact in (X, τ) by Lemma 1.7 and also countably compact in (X, σ) since $\sigma \subset \tau$. Since $U \notin \sigma$, it follows that $X \setminus U$ is not closed in (X, σ) and therefore (X, σ) is not strongly KC. Hence (X, τ) is minimal strongly KC. $\square$

By Lemma 1.7 and the definition of (strongly) KC-spaces, we obtain easily the following lemma:

Lemma 2.3. *A space (X, τ) is maximal (countably) compact if and only if it is (countably) compact (strongly)* KC.

Example 2.4. Example of a minimal strongly KC but not minimal KC-space.

Let $X = [0, \omega_1)$. From [3] we know that X is maximal countably compact and hence it is minimal strongly KC by Lemma 2.2. It follows from Lemma 1.8 that X is not minimal KC, since X is first countable but not compact.

Example 2.5. Example of a minimal KC but not minimal strongly KC-space.

Let $X = \beta\omega$, the Stone-Cech compactification of natural numbers. From [3], we know that X is maximal compact and hence it is minimal KC by Lemma 1.6. Note in [7] the two facts about X: (i) There exists a countably compact subspace Y with $\omega \subset Y \subset \beta\omega$ and $|Y| \leqslant c$; (ii) Every infinite set in $\beta\omega$ has 2^c accumulation points. Thus we may assume that $A \subset X$ satisfies the condition (i), then by (ii), we have $\mathrm{cl}(A) \neq A$, so A is not closed in X and hence X is not strongly KC. Therefore, X is not minimal strongly KC.

Examples 2.4 and 2.5 show that minimal strongly KC and minimal KC-spaces are not the same. However it is easy to see that a minimal KC-space which is strongly KC is minimal strongly KC. What's more, under certain conditions, these two notions are equivalent.

Theorem 2.6. *If X is a hereditarily Lindelof or sequential space, then X is* KC *if and only if it is strongly* KC.

P r o o f. Sufficiency is trivial. It remains to prove the necessity.

If X is a hereditarily Lindelof space, since $A \subset X$ is countably compact if and only it is compact, we have that X is KC implies X is strongly KC.

If X is a sequential KC-space, suppose $A \subset X$ is countably compact. If A is not closed, then there exist $x \in \mathrm{cl}(A) \setminus A$ and $\{x_n \colon n \in \omega\} \subset A$ such that $x_n \to x$ ($n \to \infty$). Since $\{x_n \colon n \in \omega\} \cup \{x\}$ is compact and X is KC, we have $\{x_n \colon n \in \omega\} \cup \{x\}$ is closed and hence x is the unique accumulation point of $\{x_n \colon n \in \omega\}$. However, $\{x_n \colon n \in \omega\} \subset A$ and A is countably compact, thus $\{x_n \colon n \in \omega\}$ must have an accumulation point a in A. Clearly $a \neq x$, a contradiction. So X is strongly KC. □

3. Properties of minimal strongly KC-spaces

There has been some interesting work on R. Larson's question mentioned in the first section. In [2], it was shown that in the class of KC-spaces, each countable space has the FDS-property and this result was used to prove that every countable minimal KC-space is compact. In [1], the authors showed that in some fairly wide classes of KC-spaces, including all hereditarily Lindelof spaces, minimal KC implies compact. And T. Vidalis [11] proved that minimal KC-spaces are countably compact.

Although minimal strongly KC and minimal KC-spaces do not imply each other, it is interesting that minimal strongly KC- spaces are also countably compact. Now we are going to present a proof.

Theorem 3.1. *Minimal strongly* KC*-spaces are countably compact.*

P r o o f. Suppose by way of contradiction that (X, τ) is a minimal strongly KC-space which is not countably compact. Then there exists a set $\{x_n \colon n \in \omega\}$ which has no accumulation points in X, that is, $\{x_n \colon n \in \omega\}$ is a closed discrete set of X. Put $D = \{x_n \colon 0 < n < \omega\}$. Let $\mathscr{F}$ be a uniform ultrafilter on D, then by the definition, for any $F \in \mathscr{F}$ we have $|F| = \omega$. Define

$$\mu = \{U \in \tau \colon x_0 \notin U\} \cup \{U \in \tau \colon x_0 \in U \text{ and } U \cap D \in \mathscr{F}\}.$$

Then (X, μ) is a T_1 space and $\mu \subset \tau$. From the definition of μ, it is obvious that $U \subset X$ is an open neighborhood of x_0 in (X, μ) if and only if U is an open set in (X, τ) which contains x_0 and a member of $\mathscr{F}$. Thus, $x_0 \in \mathrm{cl}_\mu(D) \setminus D$. Since D is closed in (X, τ), it follows that $\mu \neq \tau$. For any $B \subset X$, it is easy to check that

$$(3.1)\quad \mathrm{cl}_\tau(B) \subset \mathrm{cl}_\mu(B),\ \mathrm{cl}_\mu(B) \subset \mathrm{cl}_\tau(B) \cup \{x_0\} \text{ and hence } \mathrm{cl}_\mu(B) \setminus \mathrm{cl}_\tau(B) \subset \{x_0\}.$$

Therefore, for any $B \subset X$, x_0 is the unique point which can be an accumulation point for B in (X, μ) while not being an accumulation point of it in (X, τ).

We will show that (X, μ) is a strongly KC-space and thus deduce a contradiction, since (X, τ) is minimal strongly KC. Let $K \subset X$ be countably compact in (X, μ). Then there are two possibilities:

(1) If $x_0 \notin K$, then $\mu|K = \tau|K$. So K is also a countably compact subset of (X, τ) and therefore K is closed in (X, τ). Since $\{x_n \colon n \in \omega\}$ has no accumulation points in (X, τ), it follows that $\{x_n \colon n \in \omega\} \cap K$ is finite. Thus we have $\{x_n \colon n \in \omega\} \cap K \notin \mathscr{F}$, since $\mathscr{F}$ is a uniform ultrafilter over D. Hence $D \setminus (\{x_n \colon n \in \omega\} \cap K) = D \setminus (D \cap K) = D \setminus K \in \mathscr{F}$. Since $D \setminus K \subset X \setminus K \in \tau$ and $x_0 \in X \setminus K$, it follows that $X \setminus K$ is an open neighborhood of x_0 in (X, μ) and therefore $x_0 \notin \mathrm{cl}_\mu(K)$. Then we have $\mathrm{cl}_\mu(K) = \mathrm{cl}_\tau(K) = K$ by (3.1) and hence K is closed in (X, μ).

(2) If $x_0 \in K$. Let $L = \mathrm{cl}_\tau(K) \cap D$. By (3.1), $\mathrm{cl}_\mu(K) = \mathrm{cl}_\tau(K)$, thus it remains to prove that K is closed in (X, τ).

If $L \notin \mathscr{F}$, then $F = D \setminus L \in \mathscr{F}$ and clearly $F \cap \mathrm{cl}_\tau(K) = \emptyset$. So for each $x \in F$, there is $V_x \in \tau$ such that $x \in V_x$ and $V_x \cap F = \emptyset$. Suppose that K is not countably compact in (X, τ), then there exists a set $S = \{s_n \colon n \in \omega\} \subset K$ without accumulation points in K with respect to the topology τ. We may assume that $s_n \neq x_0$ for any $n \in \omega$. Since x_0 is not an accumulation point of S in (X, τ), there is $V(x_0) \in \tau$ such that $x_0 \in V(x_0)$ and $V(x_0) \cap S = \emptyset$. Note that $V(x_0) \cup (\bigcup\{V_x \colon x \in F\})$ is an open neighborhood of x_0 in (X, μ), we know x_0 is not an accumulation point of S in (X, μ). Hence, by the comments following (3.1), S has no accumulation points in K with respect to the topology μ, contradicting the fact that K is countably compact in (X, μ). Consequently K is countably compact in (X, τ) and hence closed in (X, τ), since (X, τ) is strongly KC.

If, on the other hand, $L \in \mathscr{F}$, then there are two cases to consider:

a) If $K \cap D \in \mathscr{F}$, then $|K \cap D| = \omega$. Let $K \cap D = F_1 \cup F_2$ with $F_1 \cap F_2 = \emptyset$ and $|F_1| = |F_2| = \omega$. Then by the properties of ultrafilters, there is at least one of F_i $(i = 1, 2)$ belonging to $\mathscr{F}$; we may assume without loss of generality that $F_1 \in \mathscr{F}$. Since F_2 is closed in (X, τ), for each $x \in F_1$, there is $W_x \in \tau$ such that $x \in W_x$ and $W_x \cap F_2 = \emptyset$. Let $W(F_1) = \bigcup\{W_x \colon x \in F_1\}$. Then $W(F_1) \cap F_2 = \emptyset$ and so

$((X \setminus F_2) \cup W(F_1)) \cap F_2 = \emptyset$. Since $(X \setminus F_2) \cup W(F_1)$ is an open neighborhood of x_0 in (X, μ), we know that x_0 is not an accumulation point of F_2 in (X, μ). Since F_2 has no accumulation points in (X, τ), it follows that $F_2 \subset K$ has no accumulation points in (X, μ), a contradiction.

b) If $K \cap D \notin \mathscr{F}$, then $D \setminus (D \cap K) = D \setminus K \in \mathscr{F}$. Put $F_0 = (D \setminus K) \cap L$ and write $F_0 = \{x_{n_k}\colon k = 1, 2, \ldots\}$. Thus $F_0 \in \mathscr{F}$ and clearly $F_0 \subset \mathrm{cl}_\tau(K) \setminus K$. Thus K is not closed in (X, τ) and hence is not countably compact in (X, τ), since (X, τ) is strongly KC. Therefore, there is an infinite set $\{y_n\colon n \in \omega\} \subset K$ without accumulation points in K with respect to the topology τ, we may assume that $y_n \neq x_0$ for any $n \in \omega$ and since $x_0 \in K$, there exists an open neighborhood $U(x_0)$ of x_0 in (X, τ) with

$$U(x_0) \cap \{y_n\colon n \in \omega\} = \emptyset.$$

We claim that for every infinite subset $\{y_{n_k}\colon k \in \omega\}$ of $\{y_n\colon n \in \omega\}$ and for every $z \in F_0$ there is an open neighborhood $U(z)$ of z in (X, τ) such that $\{y_{n_k}\colon k \in \omega\} \setminus U(z)$ is infinite.

Assume to the contrary that there exist $\{y_{n_k}\colon k \in \omega\} \subset \{y_n\colon n \in \omega\}$ and some $z \in F_0$ such that, for any open neighborhood U of z in (X, τ), $\{y_{n_k}\colon k \in \omega\} \setminus U$ is a finite set. So $y_{n_k} \to z$ $(k \to \infty)$ in (X, τ) and therefore $\{y_{n_k}\colon k \in \omega\} \cup \{z\}$ is compact in (X, τ). Hence $\{y_{n_k}\colon k \in \omega\} \cup \{z\}$ is closed in (X, τ), since (X, τ) is a strongly KC-space. But, since $\mathscr{F}$ is the uniform ultrafilter on D, $\{z\} \notin \mathscr{F}$ and so $D \setminus \{z\} \in \mathscr{F}$. Let $F' = (D \setminus \{z\}) \cap F_0$, then by the definition of filter, $F' \in \mathscr{F}$. Clearly $z \notin F'$ and $F' \subset F_0$, and so $F' \cap K = \emptyset$. Since $\{y_{n_k}\colon k \in \omega\} \subset \{y_n\colon n \in \omega\} \subset K$, it follows that, for every $x \in F'$, $x \notin \{y_{n_k}\colon k \in \omega\} \cup \{z\}$, and so there is an open neighborhood U_x of x in (X, τ) such that $U_x \cap (\{y_{n_k}\colon k \in \omega\} \cup \{z\}) = \emptyset$. Let $U(F') = \bigcup\{U_x\colon x \in F'\}$, then $F' \subset U(F')$. So $U(F') \cup U(x_0)$ is an open neighborhood of x_0 in (X, μ) and $(U(F') \cup U(x_0)) \cap \{y_{n_k}\colon k \in \omega\} = \emptyset$. Consequently x_0 is not an accumulation point of $\{y_{n_k}\colon k \in \omega\}$ in (X, μ). Since $\{y_{n_k}\colon k \in \omega\} \subset \{y_n\colon n \in \omega\}$ has no accumulation points in K with respect to the topology τ, it follows that $\{y_{n_k}\colon k \in \omega\}$ has no accumulation points in K with respect to the topology μ, contradicting the fact that K is countably compact in (X, μ).

So, from the previous proof, it follows that for $x_{n_1} \in F_0$, there is an open neighborhood $U(x_{n_1})$ of x_{n_1} in (X, τ) such that $\{y_n\colon n \in \omega\} \setminus U(x_{n_1})$ is infinite. Choose $z_1 \in \{y_n\colon n \in \omega\} \setminus U(x_{n_1})$. Since (X, τ) is strongly KC, obviously it is T_1. Then for $x_{n_2} \in F_0$, there is an open neighborhood $U(x_{n_2})$ of x_{n_2} in (X, τ) such that $z_1 \notin U(x_{n_2})$ and $\{y_n\colon n \in \omega\} \setminus (U(x_{n_1}) \cup U(x_{n_2}))$ is infinite. Choose $z_2 \in \{y_n\colon n \in \omega\} \setminus (U(x_{n_1}) \cup U(x_{n_2}))$ with $z_2 \neq z_1$. Generally, suppose that we have chosen open neighborhoods $U(x_{n_1}), \ldots, U(x_{n_k})$ of $x_{n_1}, \ldots, x_{n_k}$ in (X, τ) and points $z_1, \ldots, z_k$ such that $z_i \notin U(x_{n_j})$ for each $i < j \leqslant k$, $z_k \notin \{z_1, \ldots, z_{k-1}\}$

and $\{y_n\colon n \in \omega\} \setminus (U(x_{n_1}) \cup U(x_{n_2}) \cup \ldots \cup U(x_{n_k}))$ is infinite. Let $U(x_{n_{k+1}})$ be an open neighborhood of $x_{n_{k+1}}$ in (X, τ) satisfying $z_i \notin U(x_{n_{k+1}})$ for each $i \leqslant k$ and $\{y_n\colon n \in \omega\} \setminus (U(x_{n_1}) \cup U(x_{n_2}) \cup \ldots \cup U(x_{n_{k+1}}))$ is infinite. Take $z_{k+1} \in \{y_n\colon n \in \omega\} \setminus (U(x_{n_1}) \cup U(x_{n_2}) \cup \ldots \cup U(x_{n_{k+1}}))$ such that $z_{k+1} \notin \{z_1, \ldots, z_k\}$. Since $z_i \notin U(x_0) \cup U(x_{n_j})$ for each $i, j \in \omega$, it follows that

$$\{z_n\colon n \in \omega\} \cap \Big(U(x_0) \cup \bigcup\{U(x_{n_k})\colon k = 1, 2, \ldots\}\Big) = \emptyset.$$

Since $F_0 \subset \bigcup\{U(x_{n_k})\colon k = 1, 2, \ldots\}$, clearly $U(x_0) \cup (\bigcup\{U(x_{n_k})\colon k = 1, 2, \ldots\})$ is an open neighborhood of x_0 in (X, μ) and so x_0 is not an accumulation point of $\{z_n\colon n \in \omega\}$ with respect to the topology μ. However, $\{z_n\colon n \in \omega\} \subset \{y_n\colon n \in \omega\}$ has no accumulation points in K with respect to the topology τ, therefore $\{z_n\colon n \in \omega\} \subset K$ has no accumulation points in K with respect to the topology μ, which is impossible since K is countably compact in (X, μ).

Now we have shown that (X, μ) is strongly KC, which contradicts the fact that (X, τ) is minimal strongly KC. The theorem follows. $\square$

It is natural to ask whether every minimal strongly KC-space is compact. From Example 2.4 we deduce a negative answer to this question. However, it remains unknown whether every minimal KC-space is compact.

Below we will use Theorem 3.1 to establish two corollaries.

Corollary 3.2. *A closed subspace of a minimal strongly* KC*-space is minimal strongly* KC.

P r o o f. Let (X, τ) be a minimal strongly KC-space and $Y \subset X$ be closed. By Theorem 3.1, X is countably compact and hence Y is also countably compact. Clearly, Y is also strongly KC, and it follows from Lemmas 2.3 and 2.2 that Y is minimal strongly KC. $\square$

The next result is an immediate consequence of Lemma 2.2, Theorem 3.1 and Lemma 2.3:

Corollary 3.3. *A space (X, τ) is maximal countably compact if and only if it is minimal strongly* KC.

Theorem 3.4. *Let (X, τ) be a minimal strongly* KC*-space. Then X has the* FDS*-property if and only if it is a sequential space.*

P r o o f. For the necessity, suppose that $A \subset X$ is not closed. Since (X, τ) is strongly KC, A is not countably compact and hence we can find a countable discrete subset $D = \{x_n \colon n \in \omega\} \subset A$ which is closed in A, that is, $D^d \subset X \setminus A$. Since X has the FDS-property, there is some countably infinite set $E \subset D$ with only a finite number of accumulation points in X and $E^d \subset D^d \subset X \setminus A$. Thus $\mathrm{cl}(E)$ is a countable, strongly KC subspace and by Corollary 3.2 and Theorem 3.1, $\mathrm{cl}(E)$ is countably compact. Thus $E^d \neq \emptyset$ and $\mathrm{cl}(E)$ is compact. It follows from Lemma 1.9 that $\mathrm{cl}(E)$ is sequential, thus there is a sequence in E converging out of E and hence out of A. Consequently, X is a sequential space.

The sufficiency is trivial since we observe in [2] that each sequential KC-space has the FDS-property. □

However, the following example shows that a minimal KC-space with the FDS-property need not be sequential.

Example 3.5. Let $X = [0, \omega_1]$. Obviously X is compact and KC. By Lemmas 2.3 and 1.6, we know that X is minimal KC. Since $[0, \omega_1)$ is a sequential KC-space, by the proof of the preceding theorem, $[0, \omega_1)$ has the FDS-property and so does X. Clearly X is not a sequential space, since $[0, \omega_1)$ is not closed in X but ω_1 is not the limit point of any sequence of points in $[0, \omega_1)$.

Theorem 3.6. *An infinite minimal strongly* KC*-space possesses a non-trivial convergent sequence.*

P r o o f. Suppose X is such a space. Then by Theorem 3.1, X is countably compact. Let $p \in X$ be non-isolated. Then $X \setminus \{p\}$ is not closed, hence not countably compact, since X is strongly KC. So there is a countably infinite subset $A \subset X \setminus \{p\}$ which has no accumulation points in $X \setminus \{p\}$ and therefore, for every $x \in A$, there is an open neighborhood U_x of x such that $U_x \cap A = \{x\}$. It is clear that p is the unique accumulation point of A in X and so $A \cup \{p\}$ is closed in X. Thus $A \cup \{p\}$ is countably compact and hence compact in X. Let $A = \{x_n \colon n \in \omega\}$. It is obvious that, for any open neighborhood V of p, the collection $\{U_{x_n} \colon n \in \omega\} \cup \{V\}$ is an open cover of $A \cup \{p\}$ and hence it has a finite subcover, say, $U_{x_{n_1}} \cup U_{x_{n_2}} \cup \ldots \cup U_{x_{n_k}} \cup V$. Since $U_{x_{n_i}} \cap A = \{x_{n_i}\}$ for $i = 1, 2, \ldots, k$, it follows that $x_n \in V$ whenever $n > n_k$ and this implies $x_n \to p$ $(n \to \infty)$. □

But, Theorem 3.6 does not hold for minimal KC-spaces.

Example 3.7. Let $X = \beta\omega$. It follows from Example 2.5 that X is minimal KC. But from [7] we know that there are no non-trivial convergent sequences in X.

In [2], the authors raised the following question:

Question D. Does every countably compact KC-space of size less than c have the FDS-property?

Now we will give a positive answer to the above question.

Theorem 3.8. *A countably compact* KC*-space of cardinality less than c has the* FDS*-property.*

P r o o f. Suppose X satisfies the hypothesis. According to [7], every countably compact space of cardinality less than c is sequentially compact, so X is sequentially compact. Thus for any infinite subset $A \subset X$, we may assume without loss of generality that $A = \{x_n \colon n \in \omega\}$, then A must have a subsequence $\{x_{n_k} \colon k = 1, 2, \ldots\}$ such that $x_{n_k} \to x$ $(k \to \infty)$. Thus $\{x_{n_k} \colon k = 1, 2, \ldots\} \cup \{x\}$ is compact and hence closed in X, so x is the unique accumulation point of $\{x_{n_k} \colon k = 1, 2, \ldots\}$. Therefore X has the FDS-property. $\square$

In [2], for countable KC-spaces, a characterization of Katětov-KC spaces is given. Below we will give a characterization of Katětov strongly KC-spaces of cardinality less than c and at last extend the previous result of [2] to hereditarily Lindelof spaces.

Theorem 3.9. *Let (X, τ) be a strongly* KC*-space and $|X| < c$. Then (X, τ) is Katětov strongly* KC *if and only if there is a weaker sequential strongly* KC *topology $\sigma \subset \tau$.*

P r o o f. If (X, τ) is a Katětov strongly KC-space and $|X| < c$, then by the definition, there is a topology $\sigma \subset \tau$ such that (X, σ) is a minimal strongly KC-space. From Theorem 3.1, it follows that (X, σ) is countably compact and so it has the FDS-property by Theorem 3.8. Therefore, by Theorem 3.4, (X, σ) is sequential.

For the sufficiency, suppose that (X, τ) is a strongly KC-space with $|X| < c$ and $\sigma \subset \tau$ is a sequential strongly KC topology. If (X, σ) is countably compact, then by Lemmas 2.3 and 2.2, it is minimal strongly KC and hence (X, τ) is Katětov strongly KC. So we assume that (X, σ) is not countably compact. Fix $p \in X$ and define a new topology μ on X as follows:

$$\mu = \{U \in \sigma \colon p \notin U\} \cup \{U \in \sigma \colon p \in U \text{ and } X \setminus U \text{ is countably compact in } (X, \sigma)\}.$$

Clearly, (X, μ) is a countably compact T_1-space and $\mu \subset \sigma$. To complete the proof, we need to show that (X, μ) is a minimal strongly KC-space. By Lemmas 2.3 and 2.2, we need only to show that (X, μ) is a strongly KC-space. To this end, suppose that $K \subset X$ is a countably compact subset of (X, μ). It is clear that

$$\text{(3.2)} \quad \mathrm{cl}_\sigma(K) \subset \mathrm{cl}_\mu(K),\ \mathrm{cl}_\mu(K) \subset \mathrm{cl}_\sigma(K) \cup \{p\} \text{ and hence } \mathrm{cl}_\mu(K) \setminus \mathrm{cl}_\sigma(K) \subset \{p\}.$$

There are two possibilities:

(1) If $p \notin K$, then $\sigma|K = \mu|K$, K is countably compact in (X, σ), and hence closed in (X, σ). So $X \setminus K$ is an open neighborhood of p in (X, μ). Thus, $p \notin \mathrm{cl}_\mu(K)$ and hence we have $\mathrm{cl}_\mu(K) = \mathrm{cl}_\sigma(K) = K$ by (3.2). So, K is closed in (X, μ).

(2) If $p \in K$, then by (3.2), $\mathrm{cl}_\mu(K) = cl_\sigma(K)$. So if K is not closed in (X, μ), then it is not closed in (X, σ) either. Since (X, σ) is sequential, there is some $x \in \mathrm{cl}_\sigma(K) \setminus K$ and a sequence $\{x_n\}_{n\in\omega}$ in K convergent to x with respect to the topology σ. Since $x \neq p$, we may assume that $x_n \neq p$ for all $n \in \omega$. Then $C = \{x_n \colon n \in \omega\} \cup \{x\}$ is compact in (X, σ) and thus closed in (X, σ), since (X, σ) is strongly KC. Therefore, x is the unique accumulation point of $\{x_n \colon n \in \omega\}$ in (X, σ). Since $\mu \subset \sigma$, C is also compact in (X, μ) and therefore countably compact in (X, μ). Clearly, $p \notin C$ and hence $X \setminus C$ is an open neighborhood of p in (X, μ). Thus p is not an accumulation point of $\{x_n \colon n \in \omega\}$ with respect to the topology μ, since $\{x_n \colon n \in \omega\}$ has no accumulation points in K with respect to the topology σ, we conclude that $\{x_n \colon n \in \omega\} \subset K$ has no accumulation points in K with respect to the topology μ either, contradicting the fact that K is countably compact in (X, μ). So K is closed in (X, μ). The theorem follows. $\square$

In fact, Theorem 3.9 can be improved. We need the following lemma.

Lemma 3.10. *If (X, τ) is a sequential US-space, then X is strongly* KC.

P r o o f. Let A be a countably compact subset of X. If A is not closed, since (X, τ) is a sequential space, there is some $x \in \mathrm{cl}(A) \setminus A$ and a sequence $\{x_n\}_{n\in\omega} \subset A$ convergent to x. Since A is countably compact, $\{x_n \colon n \in \omega\}$ must have an accumulation point y in A and so $\{x_n \colon n \in \omega\} \cup \{x\}$ is not closed in X. Again since X is sequential, it follows that there is some sequence $\{x_{n_k} \colon k \in \omega\} \subset \{x_n \colon n \in \omega\}$ which converges to x' and $x' \notin \{x_n\} \cup \{x\}$. Then $\{x_{n_k} \colon k \in \omega\}$ must also converge to x, contradicting the definition of US-space. Therefore, X is strongly KC. $\square$

After the above arguments, the next statement becomes obvious:

Corollary 3.11. *Let (X, τ) be a strongly* KC*-space and $|X| < c$. Then (X, τ) is Katětov strongly* KC *if and only if there is a weaker sequential US topology $\sigma \subset \tau$.*

The next result generalizes Theorem 18 in [2], stating that a countable KC-space (X, τ) is Katětov-KC if and only if there is a weaker sequential KC topology $\sigma \subset \tau$.

Theorem 3.12. *A hereditarily Lindelof* KC*-space* (X, τ) *is Katětov-*KC *if and only if there is a weaker sequential* KC *topology* $\sigma \subset \tau$.

P r o o f. If (X, τ) is Katětov-KC, then by the definition, there is a weaker topology $\sigma \subset \tau$ such that (X, σ) is minimal KC. By Lemma 1.10, (X, σ) is countably compact. Since (X, τ) is hereditarily Lindelof, it follows that (X, σ) is also hereditarily Lindelof and hence compact. So, (X, σ) is sequential by Lemma 1.11.

The sufficiency follows easily from Lemma 1.12. □

The next statement is obvious and further generalizes Theorem 18 in [2].

Corollary 3.13. *A hereditarily Lindelof* KC*-space* (X, τ) *is Katětov-*KC *if and only if there is a weaker sequential* US *topology* $\sigma \subset \tau$.

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