Lili Hu; Chunhui Lai; Ping Wang

On potentially $K_5 - H$-graphic sequences

Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 1, 173–182

Persistent URL: http://dml.cz/dmlcz/140471

Czechoslovak Mathematical Journal, 59 (134) (2009), 173–182

ON POTENTIALLY $K_5 - H$-GRAPHIC SEQUENCES

Lili Hu, Zhangzhou, Chunhui Lai, Zhangzhou,
Ping Wang, Antigonish

(Received March 16, 2007)

Abstract. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of H where H is a subgraph of K_m. In this paper, we characterize the potentially $K_5 - P_4$ and $K_5 - Y_4$-graphic sequences where Y_4 is a tree on 5 vertices and 3 leaves.

Keywords: graph, degree sequence, potentially $K_5 - H$-graphic sequence

MSC 2010: 05C07, 05C35

1. Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. An n-term non-increasing nonnegative integer sequence $\pi = (d_1, d_2, \ldots, d_n)$ is said to be graphic if it is the degree sequence of a simple graph G of order n; such a graph G is referred as a realization of π. Let C_k and P_k denote a cycle on k vertices and a path on $k+1$ vertices, respectively. We use the symbol E_4 to denote graphs on 5 vertices and 4 edges. Let $\sigma(\pi)$ be the sum of all the terms of π, and let $[x]$ be the largest integer less than or equal to x. Let Y_4 denote a tree on 5 vertices and 3 leaves. Z_4 is $K_4 - P_2$. A graphic sequence π is said to be potentially H-graphic if it has a realization G containing H as a subgraph. Let $G - H$ denote the graph obtained from G by removing the edges set $E(H)$ where H is a subgraph of G. In the degree sequence, r^t means r repeats t times, that is, in the realization of the sequence there are t vertices of degree r.

In 1907, Mantel first proposed the problem of determining the maximum number of edges in a graph without containing 3-cycles. In general, this problem can be

Research was supported by NNSF of China (10271105) and by NSF of Fujian (Z0511034), Fujian Provincial Training Foundation for "Bai-Quan-Wan Talents Engineering", Project of Fujian Education Department, Project of Zhangzhou Teachers College and by NSERC.

phrased as determining the maximum number of edges, denoted $\mathrm{ex}(n,H)$, of a graph with n vertices not containing H as a subgraph. This area of research is called extremal graph theory. In terms of graphic sequences, the number $2\,\mathrm{ex}(n,H)+2$ is the minimum even integer l such that every n-term graphic sequence π with $\sigma(\pi)\geqslant l$ is forcibly H-graphic. In 1991, Erdös, Jacobson and Lehel [2] showed $\sigma(K_k,n)\geqslant(k-2)(2n-k+1)+2$ and conjectured that the equality holds. In the same paper, they proved that the conjecture is true for the case $k=3$ and $n\geqslant 6$. The cases $k=4$ and 5 were proved separately in [3], [16] and [17]. Based on linear algebraic techniques, Li, Song and Luo [18] proved the conjecture true for $k\geqslant 6$ and $n\geqslant\binom{k}{2}+3$. Recently, Ferrara, Gould and Schmitt proved the conjecture [5] and they also determined in [6] $\sigma(F_k,n)$ where F_k denotes the graph of k triangles intersecting at exactly one common vertex.

In 1999, Gould, Jacobson and Lehel [3] considered the following generalized problem: determine the smallest even integer $\sigma(H,n)$ such that every n-term positive graphic sequence $\pi=(d_1,d_2,\ldots,d_n)$ with $\sigma(\pi)\geqslant\sigma(H,n)$ has a realization G containing H as a subgraph. They proved $\sigma(pK_2,n)=(p-1)(2n-p)+2$ for $p\geqslant 2$ and $\sigma(C_4,n)=2[\frac{1}{2}(3n-1)]$ for $n\geqslant 4$. Lai [10] determined $\sigma(K_4-e,n)$ for $n\geqslant 4$. Yin, Li, and Mao [24] determined $\sigma(K_{r+1}-e,n)$ for $r\geqslant 3$ and $r+1\leqslant n\leqslant 2r$ and $\sigma(K_5-e,n)$ for $n\geqslant 5$, and Yin and Li [23] further determined $\sigma(K_{r+1}-e,n)$ for $r\geqslant 2$ and $n\geqslant 3r^2-r-1$. Moreover, Yin and Li in [23] also gave two sufficient conditions for a sequence $\pi\varepsilon GS_n$ to be potentially $(K_{r+1}-e)$-graphic. Yin [26] determined $\sigma(K_{r+1}-K_3,n)$ for $r\geqslant 3$ and $n\geqslant 3r+5$. Lai [11]–[13] determined $\sigma(K_5-P_3,n)$, $\sigma(K_5-P_4,n)$, $\sigma(K_5-C_4,n)$ and $\sigma(K_5-K_3,n)$ for $n\geqslant 5$. Lai and Hu [14] determined $\sigma(K_{r+1}-H,n)$ for $n\geqslant 4r+10$, $r\geqslant 3$, $r+1\geqslant k\geqslant 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices and $\sigma(K_{r+1}-P_2,n)$ for $n\geqslant 4r+8$, $r\geqslant 3$. Lai [15] determined $\sigma(K_{r+1}-Z_4,n)$, $\sigma(K_{r+1}-(K_4-e),n)$, $\sigma(K_{r+1}-K_4,n)$ for $n\geqslant 5r+16$, $r\geqslant 4$ and $\sigma(K_{r+1}-Z,n)$ for $n\geqslant 5r+19$, $r+1\geqslant k\geqslant 5$, $j\geqslant 5$ where Z is a graph on k vertices and j edges which contains a graph Z_4 but does not contain a cycle on 4 vertices.

A harder question is to characterize the potentially H-graphic sequences without zero terms. That is, finding necessary and sufficient conditions for a sequence to be a potentially H-graphic sequence. Luo [20] characterized the potentially C_k-graphic sequences for each $k=3$, 4 and 5. Recently, in [21], Luo and Warner also characterized the potentially K_4-graphic sequences. Eschen and Niu [22] characterized the potentially K_4-e-graphic sequences. Hu and Lai [7]–[8] characterized the potentially K_5-C_4 and K_5-Z_4-graphic sequences. Yin and Chen [25] characterized the potentially $K_{r,s}$-graphic sequences for $r=2$, $s=3$ and $r=2$, $s=4$, where $K_{r,s}$ is an $r\times s$ complete bipartite graph. Gupta, Joshi and Tripathi [4] gave a necessary and sufficient condition for the existence of a tree of order n with a given degree set.

In attempt to completely characterize the potentially $K_5 - E_4$-graphic sequences, we will characterize the potentially $K_5 - P_4$ and $K_5 - Y_4$-graphic sequences in this paper. The problem of characterizing the potentially $K_5 - E_4$-graphic sequences has not been solved so far.

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing positive integer sequence. We write $m(\pi)$ and $h(\pi)$ to denote the largest positive terms of π and the smallest positive terms of π, respectively. $\pi'' = (d_1 - 1, d_2 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1})$ is the residual sequence obtained by laying off d_n from π. We denote $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$ where $d'_1 \geqslant d'_2 \geqslant \ldots \geqslant d'_{n-1}$ is a rearrangement of the $n-1$ terms in π''. We denote by π' the residual sequence obtained by laying off d_n from π and all the graphic sequences have no zero terms. We need the following results.

Theorem 1.1 ([3]). *If $\pi = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.*

Theorem 1.2 ([19]). *If $\pi = (d_1, d_2, \ldots, d_n)$ is a sequence of nonnegative integers with $1 \leqslant m(\pi) \leqslant 2$, $h(\pi) = 1$ and $\sigma(\pi)$ even, then π is graphic.*

Theorem 1.3 ([9]). *π is graphic if and only if π' is graphic.*

The following corollary is obvious.

Corollary 1.4. *Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.*

2. Main theorems

Theorem 2.1. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_5 - P_4$-graphic if and only if the following conditions hold:*

(1) $d_2 \geqslant 3$.

(2) $d_5 \geqslant 2$.

(3) $\pi \neq (n-1, k, 2^t, 1^{n-2-t})$ *where $n \geqslant 5$, $k, t = 3, 4, \ldots, n-2$, and, k and t have different parities.*

(4) *For $n \geqslant 5$, $\pi \neq (n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \ldots, n-2k$ and $k = 1, 2, \ldots, [\frac{1}{2}(n-1)] - 1$.*

(5) *If $n = 6, 7$, then $\pi \neq (3^2, 2^{n-2})$.*

Proof. First we show that the conditions (1)–(5) are necessary conditions for π to be potentially $K_5 - P_4$-graphic. Assume that π is potentially $K_5 - P_4$-graphic. (1), (2) and (5) are obvious. If $\pi = (n-1, k, 2^t, 1^{n-2-t})$ is potentially

$K_5 - P_4$-graphic, then according to Theorem 1.1, there exists a realization G of π containing $K_5 - P_4$ as a subgraph so that the vertices of $K_5 - P_4$ have the largest degrees of π. Therefore, the sequence $\pi^* = (n-4, k-3, 2^{t-3}, 1^{n-2-t})$ obtained from $G-(K_5-P_4)$ must be graphic. Since the edge between two vertices with degree $n-4$ and $k-3$ has been removed from the realization of π^*, thus, $\Delta(G-(K_5-P_4)) \leqslant n-5$, a contradiction. Hence, (3) holds. If $\pi = (n-k, k+i, 2^i, 1^{n-i-2})$ is potentially $K_5 - P_4$-graphic, then according to Theorem 1.1, there exists a realization G of π containing $K_5 - P_4$ as a subgraph so that the vertices of $K_5 - P_4$ have the largest degrees of π. Therefore, the sequence $\pi^* = (n-k-3, k+i-3, 2^{i-3}, 1^{n-i-2})$ obtained from $G - (K_5 - P_4)$ must be graphic and there is no edge between two vertices with degree $n-k-3$ and $k+i-3$ in the realization of π^*. Let G^* be a realization of π^*, and, $d_{G^*}(x) = n-k-3$ and $d_{G^*}(y) = k+i-3$. Consider a partition of G^* where $X = \{x, y\}$ and $Y = V(G^*) - \{x, y\}$. It follows that the number of edges between X and Y equals $(n-k-3) + (k+i-3) \leqslant 2(i-3) + (n-i-2)$, that is, $[(n-k-3) + (k+i-3)] - [2(i-3) + (n-i-2)] = 2 \leqslant 0$, a contradiction. Hence, (4) holds.

Now we show that the conditions (1)–(5) are sufficient conditions for π to be potentially $K_5 - P_4$-graphic. Suppose the graphic sequence π satisfies the conditions (1) to (5). Our proof is by induction on n. We first prove the base case where $n = 5$. Since $\pi \neq (4^2, 2^3)$, π must be one of the following sequences: (4^5), $(4^3, 3^2)$, $(4^2, 3^2, 2)$, $(4, 3^4)$, $(4, 3^2, 2^2)$, $(3^4, 2)$, $(3^2, 2^3)$. It is easy to check that all of these are potentially $K_5 - P_4$-graphic. Now we assume that the sufficiency holds for $n-1$ ($n \geqslant 6$). We will prove that π is potentially $K_5 - P_4$-graphic.

Case 1: $\pi' = (3^2, 2^4)$. Clearly, $n = 7$ and π must be one of the following sequences $(4^2, 2^5)$, $(4, 3^2, 2^4)$, $(3^4, 2^3)$, $(4, 3, 2^4, 1)$ or $(3^3, 2^3, 1)$. It is easy to check that all of these are potentially $K_5 - P_4$-graphic.

Case 2: $\pi' = (3^2, 2^5)$. Clearly, $n = 8$ and π must be one of the following sequences $(4^2, 2^6)$, $(4, 3^2, 2^5)$, $(3^4, 2^4)$, $(4, 3, 2^5, 1)$ or $(3^3, 2^4, 1)$. It is easy to check that all of these are potentially $K_5 - P_4$-graphic.

Case 3: $d_n \geqslant 3$. Clearly, π' satisfies the assumption, and thus, by the induction hypothesis, π' is potentially $K_5 - P_4$-graphic, and hence so is π. In the following, we only consider the cases $d_n = 1$ or $d_n = 2$.

Case 4: $\pi' = (n-2, k, 2^t, 1^{n-3-t})$ *where* $n-1 \geqslant 5$, $k, t = 3, 4, \ldots, n-3$, *and,* k *and* t *have different parities.*

If $d_n = 2$, then $\pi' = (n-2, k, 2^{n-3})$. If $k \geqslant 4$, then $\pi = (n-1, k+1, 2^{n-2})$ which contradicts condition (3). If $k = 3$, that is $\pi' = (n-2, 3, 2^{n-3})$, then $\pi = (n-1, 4, 2^{n-2})$ or $\pi = (n-1, 3^2, 2^{n-3})$. But $\pi = (n-1, 4, 2^{n-2})$ contradicts condition (3), thus $\pi = (n-1, 3^2, 2^{n-3})$ where n is odd. We will show that $\pi = (n-1, 3^2, 2^{n-3})$

is potentially $K_5 - P_4$-graphic. In other words, we would like to show that $\pi_1 = (n-4, 2^{n-5}, 1)$ is graphic. It suffices to show that $\pi_2 = (1^{n-5})$ where $n \geqslant 7$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 1.2, π_2 is graphic. Thus, $\pi = (n-1, 3^2, 2^{n-3})$ is potentially $K_5 - P_4$-graphic.

If $d_n = 1$, then $\pi = (n-1, k, 2^t, 1^{n-2-t})$ which contradicts condition (3).

Case 5: $\pi' = (n-1-k, k+i, 2^i, 1^{n-i-3})$ *where* $i = 3, 4, \ldots, n-1-2k$ *and* $k = 1, 2, \ldots, [n/2]-2$.

If $d_n = 2$, then $n-i-3 = 0$ and $\pi = (n-k, k+i+1, 2^{i+1})$ which contradicts condition (4).

If $d_n = 1$ and $n-1-k = k+i+1$, then $\pi = (n-k, k+i, 2^i, 1^{n-i-2})$ or $\pi = ((n-1-k)^2, 2^i, 1^{n-i-2})$, both of which contradict condition (4). If $d_n = 1$ and $n-1-k = k+i$ or $n-1-k \geqslant k+i+2$, then $\pi = (n-k, k+i, 2^i, 1^{n-i-2})$ which also contradicts condition (4).

Case 6: $d_n = 2$, $\pi' \neq (n-2, k, 2^{n-3})$, $\pi' \neq (n-1-k, n+k-3, 2^{n-3})$, $\pi' \neq (3^2, 2^4)$, *and* $\pi' \neq (3^2, 2^5)$.

Consider $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$. Since $d_2 \geqslant 3$, we have $d'_{n-1} \geqslant 2$. If $d'_2 \geqslant 3$, then π' satisfies the assumption. Thus, π' is potentially $K_5 - P_4$-graphic. Hence, we may assume $d'_2 = 2$, that is, $d_2 = 3$ and $d_3 = d_4 = \ldots = d_n = 2$. It follows that $\pi = (d_1, 3, 2^{n-2})$. Since $\sigma(\pi)$ is even, d_1 must be odd. If $d_1 = 3$, then $\pi = (3^2, 2^{n-2})$. Since $\pi \neq (3^2, 2^4)$ and $\pi \neq (3^2, 2^5)$, we have $n \geqslant 8$. We will show that π is potentially $K_5 - P_4$-graphic. It suffices to show $\pi_1 = (2^{n-5})$ is graphic. Clearly, C_{n-5} is a realization of π_1. If $d_1 \geqslant 5$, since $\pi \neq (n-1, 3, 2^{n-2})$, we have $d_1 \leqslant n-2$. We will prove that $\pi = (d_1, 3, 2^{n-2})$, where $d_1 \geqslant 5$ and $n \geqslant d_1 + 2$, is potentially $K_5 - P_4$-graphic. We would like to show that $\pi_1 = (d_1 - 3, 2^{n-5})$ is graphic. It suffices to show that $\pi_2 = (2^{n-d_1-2}, 1^{d_1-3})$ is graphic. Since $\sigma(\pi_2)$ is even, π_2 is graphic by Theorem 1.2. Thus, $\pi = (d_1, 3, 2^{n-2})$ is potentially $K_5 - P_4$-graphic.

Case 7: $d_n = 1$, $\pi' \neq (n-2, k, 2^t, 1^{n-3-t})$, $\pi' \neq (n-1-k, k+i, 2^i, 1^{n-i-3})$, $\pi' \neq (3^2, 2^4)$, *and* $\pi' \neq (3^2, 2^5)$.

Consider $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$. Since $d_2 \geqslant 3$ and $d_5 \geqslant 2$, we have $d'_1 \geqslant 3$ and $d'_5 \geqslant 2$. If $d'_2 \geqslant 3$, then π' satisfies the assumption. Thus, π' is potentially $(K_5 - P_4)$-graphic. Hence, we may assume $d'_2 = 2$, that is, $d_1 = d_2 = 3$ and $d_3 = d_4 = d_5 = 2$. Thus $\pi = (3^2, 2^t, 1^{n-2-t})$ where $t \geqslant 3$ and $n-2-t \geqslant 1$. Since $\sigma(\pi)$ is even, $n-2-t$ must be even. We will prove π is potentially $K_5 - P_4$-graphic. It suffices to show that $\pi_1 = (2^{t-3}, 1^{n-2-t})$ is graphic. Since $\sigma(\pi_1)$ is even, π_1 is graphic by Theorem 1.2 and, in turn, π is potentially $K_5 - P_4$-graphic.

This completes the proof. □

Theorem 2.2. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geqslant 5$. Then π is potentially $K_5 - Y_4$-graphic if and only if the following conditions hold:*
(1) $d_3 \geqslant 3$.
(2) $d_4 \geqslant 2$.
(3) $\pi \neq (3^6)$.

P r o o f. Assume that π is potentially $K_5 - Y_4$-graphic. In this case the necessary conditions (1) to (3) are obvious.

Now we prove the sufficient conditions. Suppose the graphic sequence π satisfies the conditions (1) to (3). Our proof is by induction on n. We first prove the base case where $n = 5$. In this case, π is one of the following sequences: (4^5), $(4^3, 3^2)$, $(4^2, 3^2, 2)$, $(4, 3^4)$, $(4, 3^3, 1)$, $(4, 3^2, 2^2)$, $(3^4, 2)$, or $(3^3, 2, 1)$. It is easy to check that all of these are potentially $K_5 - Y_4$-graphic. Now suppose the sufficiency holds for $n - 1$ ($n \geqslant 6$), and let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence which satisfies (1) to (3). We will prove π is potentially $K_5 - Y_4$-graphic.

Case 1: $\pi' = (3^6)$. We have $n = 7$ and π is one of the following sequences $(4^3, 3^4)$, $(4^2, 3^4, 2)$ or $(4, 3^5, 1)$. It is easy to check that all of these are potentially $K_5 - Y_4$-graphic.

Case 2: $d_n \geqslant 3$ and $\pi' \neq (3^6)$. Clearly, $d'_4 \geqslant 2$. If $d_3 \geqslant 4$, then $d'_3 \geqslant 3$. If $d_3 = \ldots = d_n = 3$ and $n \geqslant 6$, $d'_3 \geqslant 3$. It follows conditions (1) and (2) hold. Thus, by the induction hypothesis, π' is potentially $K_5 - Y_4$-graphic. Therefore, π is potentially $K_5 - Y_4$-graphic by Corollary 1.4. In the following, we only consider the cases where $d_n = 2$ or $d_n = 1$.

Case 3: $d_n = 2$ and $\pi' \neq (3^6)$. Consider $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$. Since $d_3 \geqslant 3$ and $d_n = 2$, we have $d'_1 \geqslant 3$ and $d'_{n-1} \geqslant 2$. If $d'_3 \geqslant 3$, then π' satisfies the assumption and it follows π' is potentially $K_5 - Y_4$-graphic. Therefore, π is potentially $K_5 - Y_4$-graphic by Corollary 1.4. Hence, we may assume $d'_3 = 2$. We will proceed with the following two cases $d_1 \geqslant 4$ and $d_1 = 3$.

Subcase 1: $d_1 \geqslant 4$. It suffices to consider the case where $d_2 = d_3 = 3$ and $d_4 = d_5 = \ldots = d_n = 2$. That is, $\pi = (d_1, 3^2, 2^{n-3})$. Since $\sigma(\pi)$ is even, d_1 must be even. We will prove π is potentially $K_5 - Y_4$-graphic. It is enough to show that $\pi_1 = (d_1 - 3, 2^{n-5}, 1)$ is graphic. If $d_1 = n - 1$, then $\pi_1 = (n - 4, 2^{n-5}, 1)$. It suffices to show that $\pi_2 = (1^{n-5})$ is graphic. Since $\sigma(\pi_2)$ is even, π_2 is graphic by Theorem 1.2. If $d_1 \leqslant n - 2$, it suffices to show that $\pi_2 = (2^{n-2-d_1}, 1^{d_1-2})$ (or $\pi_2 = (2^{n-1-d_1}, 1^{d_1-4})$) is graphic. Similarly, one can show π_2 is graphic. Thus, $\pi_1 = (d_1 - 3, 2^{n-5}, 1)$ is graphic and, in turn, π is potentially $K_5 - Y_4$-graphic.

Subcase 2: $d_1 = 3$. It suffices to consider the case where $d_1 = d_2 = d_3 = d_4 = 3$ and $d_5 = \ldots = d_n = 2$. That is, $\pi = (3^4, 2^{n-4})$. We will prove π is potentially $(K_5 - Y_4)$-

graphic. It is enough to show that $\pi_1 = (2^{n-5}, 1^2)$ is graphic. Since $\sigma(\pi_1)$ is even, π_1 is graphic by Theorem 1.2 and, in turn, π is potentially $K_5 - Y_4$-graphic.

Case 4: $d_n = 1$ and $\pi' \neq (3^6)$. Consider $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$. Since $d_3 \geqslant 3$ and $d_4 \geqslant 2$, we have $d'_2 \geqslant 3$ and $d'_4 \geqslant 2$. If $d'_3 \geqslant 3$, then π' satisfies the assumptions and it follows π' is potentially $K_5 - Y_4$-graphic. Therefore, π is potentially $K_5 - Y_4$-graphic by Corollary 1.4. Hence, we may assume $d'_3 = 2$. It suffices to consider the case where $d_1 = d_2 = d_3 = 3$ and $d_4 = 2$. That is, $\pi = (3^3, 2^t, 1^{n-3-t})$ where $t \geqslant 1$ and $n - 3 - t \geqslant 1$. Since $\sigma(\pi)$ is even, $n - t$ must be even. We will prove π is potentially $K_5 - Y_4$-graphic. It is enough to show that $\pi_1 = (2^{t-2}, 1^{n-2-t})$ is graphic when $t \geqslant 2$. Since $\sigma(\pi_1)$ is even, π_1 is graphic by Theorem 1.2. If $t = 1$, then $\pi = (3^3, 2, 1^{n-4})$. Similarly we can show that $\pi_2 = (1^{n-5})$ is graphic and, in turn, π is potentially $K_5 - Y_4$-graphic.

This completes the proof. □

In the remainder of this section, we will use the above two theorems to find exact values of $\sigma(K_5 - P_4, n)$, $\sigma(K_5 - C_5, n)$, $\sigma(K_5 - Y_4, n)$, $\sigma(K_5 - (Y_4 + e), n)$ and $\sigma(K_5 - K_{2,3}, n)$. Note that the value of $\sigma(K_5 - P_4, n)$ was determined by Lai in [11] so a much simpler proof is given here.

Corollary 2.3 ([11]). *For $n \geqslant 5$, $\sigma(K_5 - P_4, n) = 4n - 4$.*

Proof. First we claim that $\sigma(K_5 - P_4, n) \geqslant 4n - 4$ for $n \geqslant 5$. We would like to show there exists π_1 with $\sigma(\pi_1) = 4n - 6$ such that π_1 is not potentially $K_5 - P_4$-graphic. Let $\pi_1 = ((n-1)^2, 2^{n-2})$. It is easy to see that $\sigma(\pi_1) = 4n - 6$ and π_1 is not potentially $K_5 - P_4$-graphic by Theorem 2.1.

Now we show if π is an n-term ($n \geqslant 5$) graphical sequence with $\sigma(\pi) \geqslant 4n - 4$, then there exists a realization of π containing a $K_5 - P_4$. If $d_5 = 1$, then $\sigma(\pi) = d_1 + d_2 + d_3 + d_4 + (n - 4)$. Let X be the four vertices of the largest degrees of G and $Y = V(G) - X$. Since there are at most six edges in X, $d_1 + d_2 + d_3 + d_4 \leqslant 12 + |E(X, Y)| \leqslant 12 + (n - 4) = n + 8$. This leads to $\sigma(\pi) \leqslant 2n + 4 < 4n - 4$, a contradiction. Thus, $d_5 \geqslant 2$. If $d_2 \leqslant 2$, then $\sigma(\pi) \leqslant d_1 + 2(n-1) \leqslant 3n - 3 < 4n - 4$, a contradiction. Thus, $d_2 \geqslant 3$. Since $\sigma(\pi) \geqslant 4n - 4$, then π is not one of the following: $(3^2, 2^4)$, $(3^2, 2^5)$, and $(n-1, k, 2^t, 1^{n-2-t})$ where $n \geqslant 6$ and $k, t = 3, 4, \ldots, n-2$, $(n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \ldots, n-2k$ and $k = 1, 2, \ldots, [\frac{1}{2}(n-1)] - 1$. Thus, π satisfies the conditions (1) to (5) in Theorem 2.1. Therefore, π is potentially $K_5 - P_4$-graphic by Theorem 2.1. □

Corollary 2.4 ([14]). *For $n \geqslant 5$, $\sigma(K_5 - C_5, n) = 4n - 4$.*

Proof. Obviously, for $n \geqslant 5$, $\sigma(K_5 - C_5, n) \leqslant \sigma(K_5 - P_4, n) = 4n - 4$. Now we claim that $\sigma(K_5 - C_5, n) \geqslant 4n - 4$ for $n \geqslant 5$. We would like to show there exists π_1 with $\sigma(\pi_1) = 4n - 6$, such that π_1 is not potentially $K_5 - C_5$-graphic. Let $\pi_1 = ((n-1)^2, 2^{n-2})$. It is easy to see that $\sigma(\pi_1) = 4n - 6$ and the only realization of π_1 does not contain $K_5 - C_5$. Thus, $\sigma(K_5 - C_5, n) = 4n - 4$. □

Corollary 2.5 ([14]). *For $n \geqslant 5$, $\sigma(K_5 - Y_4, n) = 4n - 4$.*

Proof. First we claim that $\sigma(K_5 - Y_4, n) \geqslant 4n - 4$ if $n \geqslant 5$. We would like to show there exists π_1 with $\sigma(\pi_1) = 4n - 6$, such that π_1 is not potentially $K_5 - Y_4$-graphic. Let $\pi_1 = ((n-1)^2, 2^{n-2})$. It is easy to see that $\sigma(\pi_1) = 4n - 6$ and π_1 is not potentially $K_5 - Y_4$-graphic by Theorem 2.2.

Now we show if π is an n-term ($n \geqslant 5$) graphical sequence with $\sigma(\pi) \geqslant 4n - 4$, then there exists a realization of π containing a $K_5 - Y_4$. If $d_4 = 1$, then $\sigma(\pi) = d_1 + d_2 + d_3 + (n-3)$. Using a similar argument as in the above corollary, we have $d_1 + d_2 + d_3 \leqslant 6 + (n-3) = n + 3$. This leads to $\sigma(\pi) \leqslant 2n < 4n - 4$, a contradiction. Thus, $d_4 \geqslant 2$. Similarly, if $d_3 \leqslant 2$, then $\sigma(\pi) \leqslant d_1 + d_2 + 2(n-2) \leqslant 2(n-1) + 2(n-2) = 4n - 6 < 4n - 4$, a contradiction. Thus, $d_3 \geqslant 3$. Since $\sigma(\pi) \geqslant 4n - 4$, necessarily $\pi \neq (3^6)$. Thus, π satisfies the conditions (1) to (3) in Theorem 2.2. Therefore, π is potentially $K_5 - Y_4$-graphic by Theorem 2.2. □

Corollary 2.6 ([14]). *For $n \geqslant 5$, $\sigma(K_5 - (Y_4 + e), n) = 4n - 4$ where the two vertices of e are the leaves of Y_4 whose distance is 3.*

Proof. Obviously, for $n \geqslant 5$, $\sigma(K_5 - (Y_4 + e), n) \leqslant \sigma(K_5 - Y_4, n) = 4n - 4$. Now we claim that $\sigma(K_5 - (Y_4 + e), n) \geqslant 4n - 4$ for $n \geqslant 5$. We would like to show there exists π_1 with $\sigma(\pi_1) = 4n - 6$, such that π_1 is not potentially $K_5 - (Y_4 + e)$-graphic. Let $\pi_1 = ((n-1)^2, 2^{n-2})$. It is easy to see that $\sigma(\pi_1) = 4n - 6$ and the only realization of π_1 does not contain $K_5 - (Y_4 + e)$. Thus, $\sigma(K_5 - (Y_4 + e), n) = 4n - 4$. □

Corollary 2.7 ([14]). *For $n \geqslant 5$, $\sigma(K_5 - K_{2,3}, n) = 4n - 4$.*

Proof. Obviously, for $n \geqslant 5$, $\sigma(K_5 - K_{2,3}, n) \leqslant \sigma(K_5 - Y_4, n) = 4n - 4$. Now we claim $\sigma(K_5 - K_{2,3}, n) \geqslant 4n - 4$ for $n \geqslant 5$. We would like to show there exists π_1 with $\sigma(\pi_1) = 4n - 6$, such that π_1 is not potentially $K_5 - K_{2,3}$-graphic. Let $\pi_1 = ((n-1)^2, 2^{n-2})$. It is easy to see that $\sigma(\pi_1) = 4n - 6$ and the only realization of π_1 does not contain $K_5 - K_{2,3}$. Thus, $\sigma(K_5 - K_{2,3}, n) = 4n - 4$. □

Acknowledgements. The authors wish to thank R. J. Gould, Jiongsheng Li, Rong Luo, J. Schmitt, Amitabha Tripathi, Jianhua Yin and Mengxiao Yin for sending their papers to us.

References

[1] *J. A. Bondy, U. S. R. Murty*: Graph Theory with Applications. Macmillan Press, 1976.

[2] *P. Erdös, M. S. Jacobson, J. Lehel*: Graphs realizing the same degree sequences and their respective clique numbers. In: Graph Theory, Combinatorics and Application, Vol. 1 (Y. Alavi et al., eds.). John Wiley and Sons, New York, 1991, pp. 439–449.

[3] *R. J. Gould, M. S. Jacobson, J. Lehel*: Potentially G-graphic degree sequences. Combinatorics, Graph Theory and Algorithms, Vol. 2 (Y. Alavi et al., eds.). New Issues Press, Kalamazoo, 1999, pp. 451–460.

[4] *G. Gupta, P. Joshi, A. Tripathi*: Graphic sequences of trees and a problem of Frobenius. Czech. Math. J. *57* (2007), 49–52.

[5] *M. Ferrara, R. Gould, J. Schmitt*: Potentially K_s^t-graphic degree sequences. Submitted.

[6] *M. Ferrara, R. Gould, J. Schmitt*: Graphic sequences with a realization containing a friendship graph. Ars Comb. Accepted.

[7] *Lili Hu, Chunhui Lai*: On potentially $K_5 - C_4$-graphic sequences. Ars Comb. Accepted.

[8] *Lili Hu, Chunhui Lai*: On potentially $K_5 - Z_4$-graphic sequences. Submitted.

[9] *D. J. Kleitman, D. L. Wang*: Algorithm for constructing graphs and digraphs with given valences and factors. Discrete Math. *6* (1973), 79–88.

[10] *Chunhui Lai*: A note on potentially $K_4 - e$ graphical sequences. Australas J. Comb. *24* (2001), 123–127.

[11] *Chunhui Lai*: An extremal problem on potentially $K_m - P_k$-graphic sequences. Int. J. Pure Appl. Math. Accepted.

[12] *Chunhui Lai*: An extremal problem on potentially $K_m - C_4$-graphic sequences. J. Comb. Math. Comb. Comput. *61* (2007), 59–63.

[13] *Chunhui Lai*: An extremal problem on potentially $K_{p,1,1}$-graphic sequences. Discret. Math. Theor. Comput. Sci. *7* (2005), 75–80.

[14] *Chunhui Lai, Lili Hu*: An extremal problem on potentially $K_{r+1} - H$-graphic sequences. Ars Comb. Accepted.

[15] *Chunhui Lai*: The smallest degree sum that yields potentially $K_{r+1} - Z$-graphical sequences. Ars Comb. Accepted.

[16] *Jiong-Sheng Li, Zi-Xia Song*: An extremal problem on the potentially P_k-graphic sequences. In: Proc. International Symposium on Combinatorics and Applications, June 28–30, 1996 (W. Y. C. Chen et. al., eds.). Nankai University, Tianjin, 1996, pp. 269–276.

[17] *Jiong-Sheng Li, Zi-Xia Song*: The smallest degree sum that yields potentially P_k-graphical sequences. J. Graph Theory *29* (1998), 63–72.

[18] *Jiong-sheng Li, Zi-Xia Song, Rong Luo*: The Erdös-Jacobson-Lehel conjecture on potentially P_k-graphic sequence is true. Sci. China (Ser. A) *41* (1998), 510–520.

[19] *Jiong-sheng Li, Jianhua Yin*: A variation of an extremal theorem due to Woodall. Southeast Asian Bull. Math. *25* (2001), 427–434.

[20] *R. Luo*: On potentially C_k-graphic sequences. Ars Comb. *64* (2002), 301–318.

[21] *R. Luo, M. Warner*: On potentially K_k-graphic sequences. Ars Combin. *75* (2005), 233–239.

[22] *E. M. Eschen, J. Niu*: On potentially $K_4 - e$-graphic sequences. Australas J. Comb. *29* (2004), 59–65.

[23] *J.-H. Yin, J. S. Li*: Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size. Discrete Math. *301* (2005), 218–227.

[24] *J.-H. Yin, J.-S. Li, R. Mao*: An extremal problem on the potentially $K_{r+1} - e$-graphic sequences. Ars Comb. *74* (2005), 151–159.

[25] *J.-H. Yin, G. Chen*: On potentially $K_{r_1,r_2,\dots,r_m}$-graphic sequences. Util. Math. *72* (2007), 149–161.

[26] *M. Yin*: The smallest degree sum that yields potentially $K_{r+1} - K_3$-graphic sequences. Acta Math. Appl. Sin., Engl. Ser. *22* (2006), 451–456.

Authors' addresses: L. Hu, Department of Mathematics, Zhangzhou Teachers College, Zhangzhou 363000, P. R. China, e-mail: jackey2591924@163.com; Ch. Lai (corresponding author), Department of Mathematics, Zhangzhou Teachers College, Zhangzhou 363000, P. R. China, e-mail: zjlaichu@public.zzptt.fj.cn; P. Wang, Dept. of Math., Stats. and Computer Science, St. Francis Xavier University, Antigonish, NS, Canada, B2G 2W5, e-mail: pwang@stfx.ca.