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WEAKLY CONNECTED DOMINATION STABLE TREES

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Abstract. A dominating set $D \subseteq V(G)$ is a *weakly connected dominating set* in G if the subgraph $G[D]_w = (N_G[D], E_w)$ weakly induced by D is connected, where E_w is the set of all edges having at least one vertex in D. *Weakly connected domination number* $\gamma_w(G)$ of a graph G is the minimum cardinality among all weakly connected dominating sets in G. A graph G is said to be *weakly connected domination stable* or just *γ_w-stable* if $\gamma_w(G) = \gamma_w(G+e)$ for every edge e belonging to the complement $\overline{G}$ of G. We provide a constructive characterization of weakly connected domination stable trees.

Keywords: weakly connected domination number, tree, stable graphs

MSC 2010: 05C05, 05C69

1. Introduction

Let $G = (V, E)$ be a connected undirected simple graph. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v. For a set $X \subseteq V(G)$, the *neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the *closed neighbourhood* $N_G[X]$ is $N_G(X) \cup X$. The degree of a vertex v is $d_G(v) = |N_G(v)|$.

A subset D of $V(G)$ is *dominating* in G if every vertex of $V(G) - D$ has at least one neighbour in D. Let $\gamma(G)$ be the minimum cardinality among all dominating sets in G.

Subgraph weakly induced by a set $D \subseteq V(G)$ is the graph $G[D]_w = (N_G[D], E_w)$, where E_w is the set of all edges having at least one vertex in D. A dominating set $D \subseteq V(G)$ is a *weakly connected dominating set* in G if the subgraph weakly induced by D is connected. Dunbar et al. [2] have defined the *weakly connected domination number* $\gamma_w(G)$ of a graph G to be the minimum cardinality among all weakly connected dominating sets in G.

Let $n = n(G)$ be the order of a graph G and let $n_1 = n_1(G)$ denote the number of leaves of G, that is the number of vertices of degree one. A vertex v is called a *support vertex* if it is adjacent to a leaf.

It is easy to observe that for any graph G we have $\gamma(G) - 1 \leqslant \gamma(G+e) \leqslant \gamma(G)$ for every edge $e \in E(\overline{G})$. Sumner and Blitch [1] have defined domination critical graphs. A graph G is said to be *domination critical*, or just *γ-critical*, if $\gamma(G) = \gamma$ and $\gamma(G+e) = \gamma - 1$ for every edge e in the complement $\overline{G}$ of G.

A graph is said to be *domination stable*, or just *γ-stable*, if $\gamma(G) = \gamma(G+e)$ for every edge e in the complement $\overline{G}$ of G.

A subset D of $V(G)$ is *connected dominating* in G if D is dominating and a subgraph $G[D]$ induced by D is connected. Let $\gamma_c(G)$ be the minimum cardinality among all connected dominating sets in G.

In [4] X. Chen et al. defined the connected domination critical graphs. A graph G is said to be *connected domination critical* in the following sense: $\gamma_c(G+vu) < \gamma_c(G)$ for each $u, v \in V(G)$ with v not adjacent to u.

We define the graph G to be *weakly connected domination stable* (*γ_w-stable*) if $\gamma_w(G+vu) = \gamma_w(G)$ for each $u, v \in V(G)$ with v not adjacent to u.

In this paper we characterize all weakly connected domination stable trees.

2. Results

We begin with the following lemma.

Lemma 1. *If T is a tree and $D \subseteq V(T)$, then D is a weakly connected dominating set of T if and only if the set $V(T) - D$ is independent.*

P r o o f. Let D be a weakly connected dominating set of T and suppose there is an edge $uv \in E(T)$ such that $u, v \in V(T) - D$. Since D is dominating, $N_T(u) \cap D \neq \emptyset$, $N_T(v) \cap D \neq \emptyset$ and, since T is a tree, $N_T(u) \cap N_T(v) = \emptyset$. Let $u' \in N_T(u) \cap D$ and $v' \in N_T(v) \cap D$. Since D is weakly connected, there is an $(u' - v')$-path P such that $u, v \notin P$, what produces a cycle and gives contradiction.

Now let D be a subset of $V(T)$ such that $V(T) - D$ is independent. Suppose D is not weakly connected dominating set of T. If D is not weakly connected, then $T[D]_w$ is not connected and there is an edge uv such that $u, v \notin D$. Then $u, v \in V(T) - D$ and $V(T) - D$ is not independent. If D is not dominating, then there is a vertex $x \in V(T) - D$ which has no neighbour in D. Since G is connected, x has a neighbour in $V(T) - D$ and again $V(T) - D$ is not independent. $\square$

In [5] the following theorem was proved:

Theorem 2. *If G is a connected graph, then $\gamma_w(G) - 1 \leqslant \gamma_w(G+e) \leqslant \gamma_w(G)$ for every edge $e \in E(\overline{G})$.*

Corollary 3. *If G is γ_w-critical, then $\gamma_w(G) = \gamma_w(G+e) + 1$ for every edge $e \in E(\overline{G})$.*

We are now in position to constructively characterize all γ_w-stable trees. To this aim we define some operations and a family of trees, similarly to [3].

If T is a tree, then we define the status of a vertex $v \in V(T)$, denoted $\mathrm{sta}(v)$, to be A or B. Let $\mathscr{T}^*$ be a family of trees with a status coloring that can be obtained from a sequence $T_1, \ldots, T_j$ $(j > 1)$ of trees with a status coloring such that T_1 is a star $K_{1,s}$ for $s \geqslant 2$, where initially $\mathrm{sta}(v) = A$ for the central vertex v of T_1, $\mathrm{sta}(u) = B$ for every leaf u of T_1 and $T = T_j$, and, if $j \geqslant 2$, then T_{i+1} can be obtained from T_i by one of two operations $\mathscr{X}$ and $\mathscr{Y}$ listed below. Once a vertex is assigned a status, this status remains unchanged as the tree is recursively constructed.

Intuitively, if a vertex v has status A or B in a γ_w-stable tree with a status coloring, then using we construct a new γ_w-stable tree with a status coloring by adding certain stars using one of the operations $\mathscr{X}$ and $\mathscr{Y}$.

- **Operation** $\mathscr{X}$: The tree T_{i+1} is obtained from T_i by adding a star $K_{1,r}$ for $r \geqslant 2$ and an edge uv, where u is a vertex of T_i such that $\mathrm{sta}(u) = A$ and v is the center of $K_{1,r}$, and letting $\mathrm{sta}(v) = A$ and $\mathrm{sta}(x) = B$ for each leaf x from $K_{1,r}$.
- **Operation** $\mathscr{Y}$: The tree T_{i+1} is obtained from T_i by adding a star $K_{1,r}$ for $r \geqslant 1$ and an edge uv, where u is a vertex of T_i such that $\mathrm{sta}(u) = B$ and v is the center of $K_{1,r}$, and letting $\mathrm{sta}(v) = A$ and $\mathrm{sta}(x) = B$ for each leaf x from $K_{1,r}$. If $r = 1$, then we take one vertex of $K_{1,1}$ to be a center and the other one to be a leaf.

Let $\mathscr{T}$ be a family of all trees T for which there exists a status coloring of T such that T with this status coloring belongs to $\mathscr{T}^*$.

Lemma 4. *If T is a tree belonging to the family $\mathscr{T}$, then there is the unique minimum weakly connected dominating set of T.*

P r o o f. Let T be a tree belonging to the family $\mathscr{T}$. Then there exists a status coloring of T such that T with this status coloring belongs to $\mathscr{T}^*$. Assume there are k vertices with status A in T. Then $D = \{a_1, \ldots, a_k\}$, where $\mathrm{sta}(a_i) = A$ for $i = 1, \ldots, k$ is the unique minimum weakly connected dominating set of T. $\square$

Lemma 5. *If T is a tree with at least three vertices and D is the unique minimum γ_w-set in T, then D contains no leaves.*

P r o o f. Suppose there is a leaf $v \in D$, where D is the unique minimum weakly connected dominating set of T. Then $(D - \{v\}) \cup \{u\}$, where u is the only neighbour of v, is a weakly connected dominating set of T, a contradiction. □

Theorem 6. *If T is a tree with at least three vertices, then T belongs to the family $\mathscr{T}$ if and only if there is a unique minimum weakly connected dominating set in T.*

P r o o f. Denote by T^* the tree T with a status coloring such that $T^* \in \mathscr{T}^*$. If T belongs to the family $\mathscr{T}$, then the result follows from Lemma 1. Let T be a tree with at least three vertices and assume there is the unique minimum weakly connected dominating set in T. We use induction on $\gamma_w(T)$, the weakly connected domination number of T.

If $\gamma_w(T) = 1$, then T is a star with at least two leaves and of course $T \in \mathscr{T}$. Assume $\gamma_w(T) > 1$ and let $P = (v_0, \ldots, v_l)$ be a longest path in T. Since $\gamma_w(T) > 1$, we have $l \geqslant 3$. Let D be the minimum weakly connected dominating set in T. From Lemma 5 we have $v_0 \notin D$. Thus $v_1 \in D$.

We now consider three possibilities depending on the degree of v_2. Let T_1 be the tree obtained from T by removing v_1 and all of its neighbours except for v_2 and denote by T_1^* the tree T_1 with a status coloring such that $T_1^* \in \mathscr{T}^*$. It is possible to observe that there is a unique minimum weakly connected dominating set in T_1 and $\gamma_w(T_1) < \gamma_w(T)$. Thus by the induction hypothesis, T_1 belongs to the family $\mathscr{T}$.

Case 1. If v_2 is a support vertex, then $v_2 \in D$. Moreover, if $d_T(v_1) = 2$, then $D - \{v_1\} \cup \{v_0\}$ would be another $\gamma_w(T)$-set, which gives a contradiction. Hence $d_T(v_1) > 2$ and T^* may be obtained from T_1^* by Operation $\mathscr{X}$.

Case 2. If $d_T(v_2) > 2$ and v_2 is not a support vertex, then $\operatorname{sta}(v_2) = B$ and T^* may be obtained from T_1^* by Operation $\mathscr{Y}$.

Case 3. If $d_T(v_2) = 2$, then v_2 is a leaf of T_1 and T^* may be obtained from T_1^* by Operation $\mathscr{Y}$. □

Theorem 7. *A tree T is γ_w-stable if and only if there is a unique minimum weakly connected dominating set in T.*

P r o o f. Let T be a tree. Suppose there is a unique minimum weakly connected dominating set in T and T is not γ_w-stable. Then there is an edge $uv \in E(\overline{T})$ such that $\gamma_w(T') < \gamma_w(T)$, where $T' = T + uv$. Observe that by Corollary 1 $\gamma_w(T') + 1 = \gamma_w(T)$. Let D' be a minimum weakly connected dominating set of T'. We consider three cases.

Case 1. If $u, v \notin D'$, then D' is a weakly connected dominating set in T and $\gamma_w(T) \leqslant |D'| = \gamma_w(T')$, which gives a contradiction.

Case 2. If $u, v \in D'$, then if D' is weakly connected in T, then similarly to Case 1 we obtain a contradiction. If D' is not weakly connected in T, then there is exactly one $(u - v)$-path in $T'[D']_w$. Hence there exists an edge xy in T' such that neither of the vertices x, y belongs to D' and x, y belong to the unique $(u - v)$-path in T.

Thus $D_1 = D' \cup \{x\}$ and $D_2 = D' \cup \{y\}$ are weakly connected dominating sets in T and $|D_1| = |D_2| = \gamma_w(T)$, which gives a contradiction with the fact that there exists exactly one minimum weakly connected dominating set in T.

Case 3. Exactly one of the vertices of $\{u, v\}$ does not belong to D', assume $u \in D', v \notin D'$. If D' is a weakly connected dominating set of T, then similarly to *Case* 1 we obtain a contradiction. If D' is dominating, but not weakly connected in T, we again obtain a contradiction, similarly to *Case* 2. Thus assume that D' is not dominating in T. Then u is the unique neighbour of v in T' belonging to D'. Since T is a tree, T' is a unicyclic graph and for this reason at most one edge of T' is not incident with a vertex of D'. In this way we conclude that v is a leaf of T and D' is a weakly connected set in T. Hence $D' \cup \{v\}$ and $D' \cup \{z\}$, where z is the neighbour of v in T, are two distinct weakly connected dominating sets of cardinality $\gamma_w(T)$ in T, a contradiction.

Now we show that if T is γ_w-stable, then there exists exactly one minimum weakly connected dominating set in T. Suppose to the contrary that there are at least two $\gamma_w(T)$-sets, say D_1 and D_2. Then $|D_1 \oplus D_2| \geqslant 2$, where $D_1 \oplus D_2 = (D_1 - D_2) \cup (D_2 - D_1)$.

Claim 1. *Every vertex belonging to $D_1 - D_2$ has a neighbour in $D_2 - D_1$ and every vertex belonging to $D_2 - D_1$ has a neighbour in $D_1 - D_2$.*

Suppose this is not true, let $u \in D_1 - D_2$ and $N_T(u) \cap (D_2 - D_1) = \emptyset$. Then of course $u \notin D_2$, but from Observation 1, every neighbour of u belongs to D_2. Since $N_T(u) \cap (D_2 - D_1) = \emptyset$, we have $N_T(u) \subseteq D_1$. But then $D_1 - \{u\}$ is a smaller weakly connected dominating set of T, which gives a contradiction. □

Since T is a tree, Claim 1 implies that $T[D_1 \oplus D_2]$ is a non-trivial forest. Let u be a leaf of $T[D_1 \oplus D_2]$. Without loss of generality let $u \in D_1 - D_2$ and let v be the neighbour of u such that $v \in D_2 - D_1$. Let us choose v such that v is not a leaf of T (if v is a leaf of T, then we can take u instead of v and v instead of u).

Let x be a neighbour of v such that $x \neq u$. Since D_1 is weakly connected, $x \in D_1$ and since T is a tree, $ux \notin E(T)$. For this reason, $D = D_1 - \{u\}$ is a weakly connected dominating set of $T + ux$ and $\gamma_w(T + ux) \leqslant |D| < \gamma_w(T)$, which contradicts the fact that T is γ_w-stable.

Lemma 8. *If there is the unique maximum independent set in T, then also there is the exactly one minimum weakly connected dominating set in T.*

P r o o f. Let $D \subseteq V(T)$ such that $V(T)-D$ is the unique maximum independent set of T. Since $V(T)-D$ is independent, from Lemma 1 D is weakly connected. Since $V(T)-D$ is maximal, D is a minimum weakly connected dominating set of T. If D is not the unique minimum weakly connected dominating set of T, $V(T)-D$ is not the unique maximum independent set of T, what gives a contradiction. Hence D is exactly one minimum weakly connected dominating set in T. $\square$

Corollary 9. *Let T be a tree of order at least three. Then the following conditions are equivalent:*

- *T belongs to the family $\mathscr{T}$;*
- *T is γ_w-stable;*
- *there is exactly one minimum weakly connected dominating set in T;*
- *there is a unique maximum independent set in T.*

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