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# RESULTS ON $F$-CONTINUOUS GRAPHS

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*Abstract.* For any nontrivial connected graph $F$ and any graph $G$, the *$F$-degree* of a vertex $v$ in $G$ is the number of copies of $F$ in $G$ containing $v$. $G$ is called *$F$-continuous* if and only if the $F$-degrees of any two adjacent vertices in $G$ differ by at most 1; $G$ is *$F$-regular* if the $F$-degrees of all vertices in $G$ are the same. This paper classifies all $P_4$-continuous graphs with girth greater than 3. We show that for any nontrivial connected graph $F$ other than the star $K_{1,k}$, $k \geqslant 1$, there exists a regular graph that is not $F$-continuous. If $F$ is 2-connected, then there exists a regular $F$-continuous graph that is not $F$-regular.

*Keywords*: continuous, $F$-continuous, $F$-regular, regular graph

*MSC 2010*: 05C12, 05C78

## 1. INTRODUCTION

Chartrand et al. in [1] consider the general case of integer-valued functions $f$ defined on a metric space of objects associated with a particular graph $G$. Such a function is *continuous* if and only if $|f(x) - f(y)| \leqslant 1$ for every two adjacent elements $x$ and $y$ in the metric space. When the metric space is the vertex set of $G$, a continuous function defined on $V(G)$ is, in fact, a labeling of the vertices of $G$ with nonnegative integers such that the labels of any two vertices $v$ and $u$ connected with an edge differ by at most 1. Such a labeling is referred to as a *continuous labeling.* Degree-continuous graphs provide an example of graphs with a certain type of a continuous labeling. A graph $G$ is called *degree-continuous* if $|\deg(v) - \deg(v')| \leqslant 1$ for every pair $\{v, v'\}$ of adjacent vertices of $G$. For more information on degree-continuous graphs see [5].

This paper is concerned with graphs $G = (V, E)$ together with a different continuous labeling. Given any nontrivial connected graph $F$, and any vertex $v \in V(G)$, the *$F$-degree of $v$ in $G$*, denoted $F$-$\deg_G(v)$, is the number of copies (not necessarily induced) of $F$ in $G$ containing $v$. Thus, the degree of $v$, denoted $\deg_G(v)$, and the

$P_2$-degree of $v$ are the same where $P_n$ denotes the path on $n$ vertices. When no confusion is possible, we write $F$-deg$(v)$ instead of $F$-deg$_G(v)$, and deg$(v)$ instead of deg$_G(v)$. A graph $G$ is *$F$-continuous* (or *$F$-degree continuous*) if and only if the $F$-degrees of any two adjacent vertices in $V(G)$ differ by at most 1. If, in addition, $F$-deg$(v) = r$ for all $v \in V(G)$, then, $G$ is *$F$-regular* of degree $r$.

Without loss of generality we can assume that $G$, as well as $F$, is nontrivial and connected; we do not allow loops or multiple edges. If no copy of $F$ can be found in $G$, then $F$-deg$(v) = 0$ for all $v \in V(G)$, and trivially, $G$ is $F$-continuous and even $F$-regular. The *girth* $g(G)$ of a graph $G$ is the minimum among all cycle lengths taken over all cycles in $G$; the *circumference* $c(G)$ of $G$ is the length of the largest cycle appearing in $G$. If $G$ has no cycles, by default $g(G) = \infty$. The *distance* between any two vertices of $G$ is the length of the shortest path between them; the *diameter* $d(G)$ of $G$ is the largest over all distances between pairs of vertices in $G$.

The concept of $F$-degree was introduced by Chartrand et al. [2] in 1987; results on $F$-continuous graphs can be found in [3]. In addition to determining all $P_3$-continuous graphs, Chartrand, Jarrett et al. [3] show that if $G$ is $F$-continuous for all nontrivial connected graphs $F$, then, $G = P_n$ or $G$ is regular. However, there are nontrivial connected graphs $F$ such that there exists a regular graph $G$ that is not $F$-continuous. Certainly, if $F = K_{1,k}$, $k \geqslant 2$, and $G$ is an $r$-regular graph, then $K_{1,k}$-deg$(v) = (k+1)\binom{r}{k}$ for every $v \in V(G)$. Thus, there is no regular graph which is not $K_{1,k}$-continuous. In the case when $F$ is a 2-connected graph, however, Chartrand et al. construct a regular graph that is not $F$-continuous [3].

In Section 3, we extend the above result from 2-connected graphs $F$ to all nontrivial connected graphs other than $K_{1,k}$, $k \geqslant 2$, confirming a conjecture in [3]. Furthermore, we show that for every 2-connected graph $F$, there exists a regular $F$-continuous graph that is not $F$-regular. We begin, in Section 2, by classifying all $P_4$-continuous graphs that contain no triangles.

## 2. $P_4$-continuous graphs

This section is entirely devoted to the case of $F = P_4$. All $P_2$-continuous graphs have been studied in [5], and all $P_3$-continuous graphs have been classified in [3]. We determine all $P_4$-continuous graphs with girth greater than 3.

Let $H$ and $K$ denote the graphs on five and four vertices, respectively, shown in Figure 1. Our main result is given below.

**Theorem 2.1.** *Let $G$ be a connected $P_4$-continuous graph with girth $g(G) > 3$ and minimum degree $\delta$. Then, $G$ is isomorphic to one of $H$, $P_n$, $K_{1,n}$, for some integer $n \geqslant 1$, or $G$ is $\delta$-regular.*

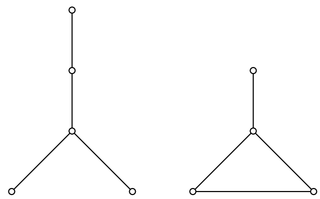

Figure 1. Two $P_4$-continuous graphs: acyclic graph $H$ on 5 vertices and graph $K$ on 4 vertices

Before we prove Theorem 2.1, we consider some special cases.

**Lemma 2.2.** *Let $G$ be a connected $P_4$-continuous graph, and let $C_3$ denote the cycle on 3 vertices.*

(i) *If $P_4$-deg$(v) = 0$ for some vertex $v$ of $G$, then $G \cong C_3$ or $K_{1,n}$ for some integer $n \geqslant 1$.*

(ii) *If $P_4$-deg$(v) = 1$ for some vertex $v$ of $G$, then $G \cong H$ or $P_n$ for some integer $n \geqslant 4$.*

(iii) *If deg$(v) = 1$ for some vertex $v$ of $G$ and $G$ contains a copy of $P_4$, then $G \cong H$, $K$ or $P_n$ for some integer $n \geqslant 4$.*

Proof. (i) The distance between any two vertices $x$ and $y$ of $G$ is less than or equal to the length of a path from $x$ to $y$ passing through $v$. Since $G$ is connected, such a path always exists; it must be that $d(G) \leqslant 2$ and the result follows.

(ii) Since $v$ is contained in a copy of $P_4$, there exists a vertex $u$ adjacent to $v$ with $\deg(u) > 1$; i.e. $\{u, w\} \in E(G)$ for some vertex $w$ other than $v$. For any vertex $x$ adjacent to $v$ other than $u$, $\langle w, u, v, x\rangle$ is a copy of $P_4$. Therefore, $\deg(v) = 1$ or $\deg(v) = 2$. If $\deg(v) = 2$, no new edges or vertices can be added without contradicting $P_4$-deg$(v) = 1$. It must be that $G \cong P_4$. Suppose that $\deg(v) = 1$ and let $\langle v, u, w, y_1\rangle$ be the copy of $P_4$ containing $v$. Now, no new edges adjacent to $w$ can be added; $u$ can only be adjacent to a new vertex $y_2$ in which case $G \cong H$ and no additional edges are present. Otherwise, it must be that $G \cong P_4$ or a new vertex $y_2$ is adjacent to $y_1$. Again, either $G \cong P_5$ or there is a new vertex $y_3$ that can only be adjacent to $y_2$. Repeating the same procedure we see that $G \cong P_n$ for some integer $n \geqslant 4$.

(iii) Let $u$ be the only vertex adjacent to $v$. Denote by $G'$ the graph with vertex set $V(G) - v$ and edge set $E(G) - \{v, u\}$. A copy of $P_4$ in $G$ that contains $v$ must necessarily contain $u$ as well. Any copy of $P_4$ that contains $u$ but does not contain $v$ must lie entirely in $G'$. Therefore, $P_4$-deg$_{G'}(u) = 0$ or 1. If $P_4$-deg$_{G'}(u) = 0$, then, by $(i)$, either $G' \cong C_3$ and $G \cong K$, or $G' \cong K_{1,n}$ and $G$ is $P_4$-continuous and contains a copy of $P_4$ only if $G \cong P_4$ or $G \cong H$. If $P_4$-deg$_{G'}(u) = 1$, then, by (ii), $G' \cong H$ or

$G' \cong P_n$, $n \geqslant 4$. It is easy to see that the only way for $G$ to be $P_4$-continuous in this case is if $G \cong H$ or $G \cong P_n$, $n \geqslant 5$. $\square$

Proof of Theorem 2.1. By Lemma 2.2, we may assume that $\delta \geqslant 2$ and that $G$ contains a copy of $P_4$. Let $v$ be a vertex of $G$ of degree $\delta$, and let $u_1, \ldots, u_\delta \in V(G)$ denote the neighbours of $v$ where $\deg(u_i) := d_i + 1$. For each $i$, let $u_{i,1}, \ldots, u_{i,d_i} \in V(G)$ denote the $d_i$ neighbours of $u_i$ other than $v$ with $\deg(u_{i,j}) := d_{i,j} + 1$ for $j = 1, \ldots, d_i$. Certainly not all $u_{i,j}$ have to be distinct. Define

$$c_i := \sum_{j=1}^{d_i} d_{i,j}$$

and without loss of generality assume that $c_1 \geqslant c_i$ for $i = 2, \ldots, \delta$.

Since $G$ contains no triangles, the $P_4$-degree of a vertex in $G$ depends only on the degrees of all vertices of distance two or less from the given vertex. If $A$ denotes the number of copies of $P_4$ in $G$ that contain both $v$ and $u_1$, then

$$\begin{aligned} P_4\text{-deg}(v) &= A + \sum_{i=2}^{\delta} c_i + (\delta - 2) \sum_{i=2}^{\delta} d_i, \\ P_4\text{-deg}(u_1) &\geqslant A + c_1(d_1 - 1) + c_1(\delta - 1) = A + c_1(d_1 + \delta - 2) \end{aligned}$$

since each neighbour of $u_{1,j}$, $j = 1, \ldots, d_1$, must have degree at least $\delta$. For each $i = 2, \ldots, \delta$, $c_1 \geqslant c_i \geqslant d_i(\delta - 1)$, leading to

$$c_1 \geqslant \sum_{i=2}^{\delta} d_i.$$

It must be that

$$P_4\text{-deg}(v) \leqslant A + c_1(\delta - 1) + c_1(\delta - 2) = A + c_1(2\delta - 3)$$

and since $d_1 \geqslant \delta - 1$,

$$1 \geqslant P_4\text{-deg}(u_1) - P_4\text{-deg}(v) \geqslant c_1(d_1 - \delta + 1) \geqslant d_1(\delta - 1)(d_1 - \delta + 1).$$

The above inequality does not hold when $d_1 > \delta - 1$ since $\delta \geqslant 2$; we must have $d_1 = \delta - 1$ and $\deg(u_1) = \delta$. Then,

$$P_4\text{-deg}(v) \leqslant A + c_1(2\delta - 3) \leqslant P_4\text{-deg}(u_1). \tag{1}$$

If equality holds in the first part of (1), then $c_1 = c_i$ for all $i = 2, \ldots, \delta$, and by the same argument as above applied to $c_i$, $\deg(u_i) = \delta$. All neighbours of the arbitrary vertex $v$ of degree $\delta$ must also have degree $\delta$, showing that $G$ is $\delta$-regular.

Otherwise, equality must hold in the second part of (1). Assume that $c_k < c_1$ for some $k$, $2 \leqslant k \leqslant j$. Since $G$ is $P_4$-continuous, $P_4\text{-deg}(v) = A + c_1(2\delta - 3) - 1$, and then,

$$\begin{aligned} A + \sum_{i=2}^{\delta} c_i + (\delta - 2) \sum_{i=2}^{\delta} d_i &= A + c_1(2\delta - 3) - 1, \\ c_k + c_1(\delta - 2) + c_1(\delta - 2) &\geqslant c_1(2\delta - 3) - 1, \\ c_k &\geqslant c_1 - 1. \end{aligned}$$

Thus, $c_k = c_1 - 1$ and $c_i = c_1$ for all $i \neq k$. Our argument, then, applies to all $c_i$, $i \neq k$, and shows that $\deg(u_i) = \delta$ for all $i \neq k$. Then, $d_k \geqslant d_i$ for all $i = 1, \ldots, \delta$ since $d_i = \delta - 1$ is the smallest possible when $i \neq k$. We get,

$$\begin{aligned} A + c_1(2\delta - 3) - 1 &= A + c_1 - 1 + c_1(\delta - 2) + (\delta - 2) \sum_{i=2}^{\delta} d_i, \\ c_1 &= \sum_{i=2}^{\delta} d_i, \\ c_1 &\leqslant d_k(\delta - 1). \end{aligned}$$

But then, $c_k \geqslant d_k(\delta - 1) \geqslant c_1$ which contradicts the fact that $c_k = c_1 - 1$. Therefore, it must be that $c_1 = c_i$ for all $i = 2, \ldots, \delta$ and as before $G$ is $\delta$-regular. □

To complete the classification of $P_4$-continuous graphs of girth other than three, we conclude this section with a closer look at regular graphs.

**Lemma 2.3.** *Let $n \geqslant 4$ be a positive integer. Let $G$ be an $r$-regular connected graph with $g(G) \geqslant n - 1$. Then, for every $v \in V(G)$,*

$$P_n\text{-deg}(v) = \frac{nr(r-1)^{n-2}}{2} - n\, C_{n-1}\text{-deg}(v)$$

*where $C_{n-1}$ is the cycle on $n - 1$ vertices.*

Proof. Fix $v \in V(G)$. If $g(G) \geqslant n$, then $C_{n-1}\text{-deg}(v) = 0$, and for $v$ to be at position $i$ of the path, $1 \leqslant i \leqslant n$, we have $r$ choices for the first edge incident to $v$ and $r - 1$ choices for each additional edge of $P_n$. Finally, since there are $n$ possible positions for $v$ and since $P_n$ is symmetric, the result follows. If $g(G) = n - 1$, we are counting illegitimate copies of $P_n$ whenever $v$ lies on a copy of $C_{n-1}$. Moreover, every such false copy of $P_n$ is counted exactly $n$ times. □

**Corollary 2.4.** *Let $n \geqslant 4$ be a positive integer. A regular connected graph $G$ with $g(G) \geqslant n-1$ is $P_n$-continuous if and only if it is $C_{n-1}$-regular.*

**Corollary 2.5.** *An $r$-regular graph $G$ with girth $g(G) > 3$ is always $P_4$-continuous and, in fact, $P_4$-regular of degree $2r(r-1)^2$. An $r$-regular graph with girth equal to 3 is $P_4$-continuous if and only if it is $C_3$-regular. There does not exist a regular $P_4$-continuous graph that is not $P_4$-regular.*

**Open Problem 2.6.** Determine all $P_4$-continuous graphs with girth 3 and minimum degree at least 2.

## 3. $F$-continuous graphs and regular graphs

In this section we examine $F$-continuous and $F$-regular graphs for a general graph $F$. Using a counting argument similar to the one used in the proof of Lemma 2.3 we can consider the case when $F$ is any tree.

**Lemma 3.1.** *Let $T$ be a tree with diameter $d(T) = d \geqslant 3$ and let $G$ be an $r$-regular connected graph with $g(G) \geqslant d+1$. Then, $G$ is $T$-regular.*

Proof. Fix $v_0 \in V(G)$. When $v_0$ is contained in a copy $T'$ of $T$ in $G$, $v_0$ is identified with a vertex $t'$ of $T'$. Think of $T'$ as a rooted tree with root $t'$ and say that $v_0$ *lies in a copy of $T$ in position $T'$*. There exists a set of rooted trees $T_1, T_2, \ldots, T_a$ that satisfy

1. $T_i$ is isomorphic to $T$ as undirected graphs for $i = 1, 2, \ldots, a$, and
2. For any graph $H$, and any vertex $v \in V(H)$,

$$T\text{-deg}_H(v) = \sum_{i=1}^{a} n_i(H, v)$$

   where $n_i(H, v)$ denotes the number of times $v$ lies in a copy of $T$ in $H$ in position $T_i$.

The integer $a$ depends only on the structure of $T$. In the case of the tree $P_4$, for example, $a = 3$ and Figure 2 shows the set of three rooted trees.

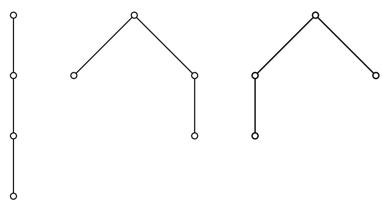

Figure 2. Set of three rooted trees for $P_4$

We want to show that $T\text{-deg}_G(v_0)$ is constant. For each $i = 1, 2, \ldots, a$,

$$n_i(G, v_0) = \binom{r}{\deg_{T_i}(t_i)} \prod_{u \in V(T_i),\ u \neq t_i} \binom{r-1}{\deg_{T_i}(u) - 1}$$

where $t_i \in V(T_i)$ denotes the root of $T_i$. The correctness of the counting argument is guaranteed by $g(G) \geqslant d+1$ which is large enough to never mistake a cyclic graph in $G$ for a copy of $T$. Therefore, $n_i(G, v_0)$, $i = 1, 2, \ldots, a$, is a function of $r$ and the structure of $T$ showing that $T\text{-deg}_G(v_0)$ will remain the same irrespective of the choice of vertex $v_0$ of $G$. □

We make use of the following result of Erdös and Sachs, the proof of which can be found in [4].

**Lemma 3.2** [4]**.** *For every two integers $r \geqslant 2$ and $g \geqslant 3$, there exists an $r$-regular graph $G$ with $g(G) = g$.*

The next theorem solves an open problem posed in [3].

**Theorem 3.3.** *Given any nontrivial connected graph $F$ other than the star $K_{1,k}$, $k \geqslant 1$, there exists a regular graph that is not $F$-continuous.*

Proof. Chartrand et al. in [3] have resolved the case of 2-connected graphs $F$. It will suffice, then, to construct a regular graph with the desired property for any other possible $F$, falling into two categories.

*Case 1*: $F$ is a tree.

Let $d(F) = d$, $d \geqslant 3$, and $|V(F)| = n$. Note that $d < 3$ implies that $F$ is a star graph. As hinted by Lemma 3.1, the idea is to construct a regular graph of girth $d$ which contains exactly one copy of $C_d$. We start with a copy of $F$ to avoid designing a regular graph that is trivially $F$-continuous because all of its $F$-degrees are zero. Pick two vertices $x$ and $y$ of $F$ distance $d$ apart and let the path $P$, passing through vertices $x, v_1, v_2, \ldots, v_{d-1}, y$ in that order, be a path of length $d$. Denote by $\Delta$ the highest degree of a vertex in $F$ and set $r = 4\Delta$. We will construct an $r$-regular graph that is not $F$-continuous.

Attach a single cycle $C_d$ to the vertex $x$ of a copy of $F$ by identifying $x$ with a vertex on the cycle. Each vertex of this new graph, that we will call $H$, has a degree less than or equal to $\Delta + 2 < r$. Collectively, for the vertices in the copy of $F$ we need additional

$$nr - 2 - \sum_{v \in V(F)} \deg_F(v)$$

edges, in order to make all of them have degree $r$ in the new graph we are creating. For the vertices in the cycle $C_d$, excluding $x$, we need $(d-1)(r-2)$ more edges. Note that

$$(d-1)(r-2) + nr - 2 - \sum_{v \in V(F)} \deg_F(v) = r(n+d-1) - 2d - \sum_{v \in V(F)} \deg_F(v) = 2q$$

is even since $r$ is even. By Lemma 3.2, there exists an $r$-regular graph $J$ with $g(J) = d+1$. Take $q$ distinct copies of $J$ and remove the same edge $\{s,t\}$ in each copy. Then, glue each of those graphs to $H$ by adding the edges $\{s,u\}$ and $\{t,w\}$, where $u$ and $w$ are vertices of $H$ in such a way that will complete the degree of each vertex to $r$. Denote the new $r$-regular graph by $G$. Certainly $G$ contains no cycles of length less than $d+1$ except the single cycle $C_d$ we started with. If $G'$ is any $r$-regular graph with $g(G') \geqslant d+1$, Lemma 3.1 will imply that $F\text{-}\deg_{G'}(v) = A$ for all $v \in V(G')$ and some positive constant $A$. Since we have the cycle $C_d$ in $G$, however, the $F$-degree of some vertices of $G$ will be less than A since $A$ would count some cyclic graphs as copies of $F$.

Consider the adjacent vertices $x$ and $v_1$ of $G$. Despite the edges we added to $H$, $v_1$ does not lie directly on the cycle $C_d$, and therefore, no double counting will occur and $F\text{-}\deg_G(v_1) = A$. However, the same counting procedure applied to $F\text{-}\deg_G(x)$ will consider the cycle $C_d$ as an acyclic path of length $d$ at least twice, once in either direction. Then, $F\text{-}\deg_G(x) \leqslant A-2$, making the $F$-degrees of $x$ and $v_1$ differ by more than 1; we have shown that $G$ is not $F$-continuous.

*Case 2*: $F$ is not a tree.

Let $c(F) = c$ and say that $F$ has $m$ cycles $C_c$. For each $v \in V(F)$, define the *proximity of $v$ in $F$*, denoted $\operatorname{prox}_F(v)$, to be the length of a shortest path from $v$ to a vertex on any of the $m$ cycles $C_c$ in $F$. If $v$ lies on one of the $m$ cycles, then $\operatorname{prox}_F(v) = 0$. Also, let

$$p = \max\{\operatorname{prox}_F(v) \colon\ v \in V(F)\}.$$

Identify two copies $F_1$ and $F_2$ of $F$ at the same vertex $x$, where $\operatorname{prox}_F(x) = p$. Add an additional vertex $y$ and the edge $\{x,y\}$, and denote the resulting graph by $H$. Let $r$ be the largest degree of a vertex in $H$. Using $H$, we will construct an $r$-regular graph $G$ that is not $F$-continuous. In particular, our goal is to make $F\text{-}\deg_G(y) = 0$ while $F\text{-}\deg_G(x) \geqslant 2$.

Using Lemma 3.2, there exists an $r$-regular graph $J$ of girth $g = \max\{c+1, p\}$. Note that

$$\sum_{u \in V(H), u \neq x,y} (r - \deg_H(u)) = 2 \sum_{u \in V(F), u \neq x} (r - \deg_F(u)) = 2q$$

is even. Let $J_1, J_2, \ldots, J_q$ be $q$ disjoint copies of $J$. Remove the same edge, say $\{s,t\}$ from each copy. Then, glue each copy $J_i - \{s,t\}$ to $H$ by adding the edges $\{s, v_1\}$ and $\{t, v_2\}$ where $v_1$ is a vertex of $F_1$, $v_2$ is the corresponding vertex of $F_2$, and $v_1, v_2 \neq x$. Continue to glue the copies of $J$ until all vertices of $H$, except possibly $x$ and $y$, have degree $r$.

Next, we deal with the vertices $x$ and $y$. Let $b = \deg_F(x)$. Note that $\deg_H(x) = 2b+1$ and $\deg_H(y) = 1$. Take $r - (2b+1)$ more copies of $J$, remove the same edge $\{s,t\}$, and attach each copy to $H$ by adding the edges $\{x,s\}$ and $\{y,t\}$. In the graph we have constructed so far, all vertices will have degree $r$, except possibly the vertex $y$ that will have degree $r - 2b$. So, finally, take $b$ copies of $J$, remove the same edge $\{s,t\}$, and glue each copy to our graph by the edges $\{y,s\}$ and $\{y,t\}$. Denote the final graph by $G$. Certainly $G$ is $r$-regular and the only cycles $C_c$ in $G$ are the $2m$ such cycles in $F_1$ and $F_2$. Furthermore, since $F_1$ and $F_2$ contain the vertex $x$, it is clear that $F\text{-}\deg_G(x) \geqslant 2$. We are left to show that $F\text{-}\deg_G(y) = 0$.

Assume on the contrary that $y$ is contained in a copy $F'$ of $F$, where $F'$ is a subgraph of $G$. Then, $F'$ must contain $m$ of the $2m$ cycles $C_c$ in $G$. By our definition of $p$, $\operatorname{prox}_{F'}(y) \leqslant p$.

However, if we remove the vertex $x$ from $G$, $G$ is no longer connected, and all of the cycles of type $C_c$ will lie in a different component than the vertex $y$. Also, since $g(J) \geqslant p$, the shortest distance from $x$ to a cycle $C_c$ in $G$ remains $p$. That is, any shortest path from $y$ to a cycle $C_c$ must start with the edge $\{y,x\}$ and continue with a path from $x$ to a cycle $C_c$. Thus,

$$\operatorname{prox}_G(y) \geqslant 1 + p$$

which is impossible because $\operatorname{prox}_{F'}(y) \geqslant \operatorname{prox}_G(y)$. Therefore, $x$ and $y$ are adjacent vertices of $G$ whose $F$-degrees differ by more than 1; $G$ is not $F$-continuous. □

Chartrand et al. in [3] pose yet another open problem concerning regular graphs. They question whether for every nontrivial connected graph $F$, $F \neq K_{1,k}$ for $k \geqslant 1$, there exists a regular $F$-continuous graph which is not $F$-regular. In [3] they answer this question in the affirmative if $F$ is any nontrivial complete graph $K_n$. Here, we show that the answer is still affirmative if $F$ is any 2-connected graph - a graph which remains connected after removing any two of its vertices and their adjacent edges.

**Theorem 3.4.** *For every nontrivial 2-connected graph $F$, there exists a regular $F$-continuous graph that is not $F$-regular.*

P r o o f. Let $c(F) = c$ and take two disjoint copies $F_1$ and $F_2$ of $F$. Add a new vertex $y$ and two new edges $\{y, x_1\}$ and $\{y, x_2\}$, where $x_1$ is a vertex in $F_1$ and $x_2$ is the corresponding vertex in $F_2$. Denote the graph constructed so far by $H$. If $\Delta(H)$ is the largest degree of a vertex in $H$, let $r = 4\Delta(H)$. We will add edges and vertices to $H$ to convert it to an $r$-regular graph. Observe that

$$r - 2 + 2\Big(\sum_{v\in V(F)} r - \deg_F(v)\Big) - 2 = 2q$$

is even since $r$ is even. Using $q$ disjoint copies of an $r$-regular graph $J$ with $g(J) = c+1$ we can transform $H$ into an $r$-regular graph $G$ with girth $c$ using the same approach as in the proof

of Theorem 3.3. The only cycles of length $c$ in $G$ would be the ones in $F_1$ and $F_2$. This and the fact that $F$ is 2-connected guarantees that $F_1$ and $F_2$ are the only copies of $F$ in $G$. Then, $F$-$\deg_G(y) = 0$ while $F$-$\deg_G(x_1) = F$-$\deg_G(x_2) = 1$ and there is no vertex in $G$ that is contained in both $F_1$ and $F_2$. Therefore, $G$ is not $F$-regular but it is $F$-continuous. □

When $F$ is not 2-connected, however, the same result does not necessarily hold. In particular, when $F = P_4$ there does not exist a regular $P_4$-continuous graph that is not $P_4$-regular as seen in Corollary 2.5.

**Open Problem 3.5.** For every integer $n \geqslant 5$, does there exist a regular $P_n$-continuous graph that is not $P_n$-regular?

**Open Problem 3.6.** Given any nontrivial connected graph $F$ that is not 2-connected, does there exist a regular $F$-continuous graph that is not $F$-regular?

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