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A CLASS OF STRONG LIMIT THEOREMS FOR COUNTABLE NONHOMOGENEOUS MARKOV CHAINS ON THE GENERALIZED GAMBLING SYSTEM

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Abstract. In this paper, we study the limit properties of countable nonhomogeneous Markov chains in the generalized gambling system by means of constructing compatible distributions and martingales. By allowing random selection functions to take values in arbitrary intervals, the concept of random selection is generalized. As corollaries, some strong limit theorems and the asymptotic equipartition property (AEP) theorems for countable nonhomogeneous Markov chains in the generalized gambling system are established. Some results obtained are extended.

Keywords: local convergence theorem, stochastic adapted sequence, martingale

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1. Introduction

Consider a sequence of Bernoulli trials and suppose that at each trial the bettor has the free choice of whether or not to bet. A theorem on gambling system asserts that under any non-anticipative system the successive bets form a sequence of Bernoulli trials with unchanged probability for success. The importance of this statement was recognized by von Mises, who introduced the impossibility of a successful gambling system as a fundamental axiom (see [1], [2]). This topic was discussed still further by Kolmogrov (see [3]) and Liu and Wang (see [4] and [5]). Yang and Liu (see [14]) and Wang (see [15]) have studied the limit properties for Markov chains on the tree and on the random transform, respectively. Wang and Li (see[16-25]) have studied the strong limit theorems for nonhomogeneous Markov chains and Markov chains field on trees and gambling systems. On the basis of [3-7] and [15] we studied strong limit theorems for nonhomogeneous Markov chains on the generalized gambling system.

The purpose of this paper is to extend the discussion to the case of strong limit theorem for countable nonhomogeneous Markov chains by using the martingale method and constructing compatible distribution. By allowing random selection functions to take values in arbitrary intervals, the concept of random selection is generalized. As corollaries, the results of Liu and Yang (see [6]) are extended.

Let $\{X_n, n \geqslant 0\}$ be a stochastic sequence defined on a probability space $(\Omega, \mathscr{F}, P)$ which takes values in $S = \{s_1, s_2, \ldots\}$. The joint distribution is

$$P(X_0 = x_0, \ldots, X_n = x_n) = p(x_0, \ldots, x_n) > 0, \quad x_i \in S,\ 0 \leqslant i \leqslant n. \tag{1}$$

Let $\{X_n, n \geqslant 0\}$ be a nonhomogeneous Markov chain. The initial distribution and the transition matrix are respectively:

$$(p(s_1), p(s_2) \ldots), \ \ p(i) > 0,\ i \in S \tag{2}$$

and

$$P_n = (p_n(i,j)), p_n(i,j) > 0, \quad i,j \in S,\ n \geqslant 1, \tag{3}$$

where $p_n(i,j) = P(X_n = j | X_{n-1} = i)$ $(n \geqslant 1)$. Then

$$p(x_0, \ldots, x_n) = p(x_0) \prod_{k=1}^{n} p_k(x_{k-1}, x_k). \tag{4}$$

In order to extend the concept of random selection, which is the crucial part of the gambling system, we first give a set of real-valued functions $f_n(x_1, \ldots, x_n)$ defined on $S^n (n = 1, 2, \ldots)$, which will be called the A-valued selection function if they take values in a set A of real numbers. Then let

$$\begin{aligned} Y_1 &= y \quad (y \text{ is an arbitrary real number}), \\ Y_{n+1} &= f_n(X_1, \ldots, X_n), \quad n \geqslant 1, \end{aligned} \tag{5}$$

where $\{Y_n, n \geqslant 1\}$ will be called a generalized gambling system (the generalized random selection system). Let $\delta_i(j)$ be the Kronecker delta function on S, that is for $i, j \in S$

$$\delta_i(j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

In order to explain the real meaning of the extended notion of the random selection, we consider the following gambling model. Let $\{X_n, n \geqslant 0\}$ be a nonhomogeneous Markov chain with the initial distribution (2) and the transition matrix

(3), and $\{g_n(x,y), n \geqslant 1\}$ be a real-valued function sequence defined on S^2. Interpret X_n as the result of the nth trial, the type of which may change at each step. Let $\mu_n = Y_n g_n(X_{n-1}, X_n)$ denote the gain of the bettor at the nth trial, where Y_n represents the bet size, $g_n(X_{n-1}, X_n)$ is determined by the gambling rules, and $\{Y_n, n \geqslant 0\}$ is called a generalized gambling system or a generalized random selection system. The bettor's strategy is to determine $\{Y_n, n \geqslant 1\}$ by the results of the last trial. Let the entrance fee that the bettor pays at the nth trial be b_n. Also suppose that b_n depends on X_{n-1} as $n \geqslant 1$, and b_1 is a constant. Thus $\sum\limits_{k=1}^{n} Y_k g_k(X_{k-1}, X_k)$ represents the total gain in the first n trials, $\sum\limits_{k=1}^{n} b_k$ the accumulated entrance fees, and $\sum\limits_{k=1}^{n} [Y_k g_k(X_{k-1}, X_k) - b_k]$ the accumulated net gain. Motivated by the classical definition of "fairness" of game of chance (see [3]), we introduce the following definition:

Definition. The game is said to be fair, if for almost all $\omega \in \Big\{\omega\colon \sum\limits_{k=1}^{\infty} Y_k = \infty\Big\}$, the accumulated net gain in the first n trials is of smaller order of magnitude than the accumulated stake $\sum\limits_{k=1}^{n} Y_k$ as n tends to infinity, that is

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} Y_k} \sum_{k=1}^{n} [Y_k g_k(X_{k-1}, X_k) - b_k] = 0 \quad \text{a.s. on } \Big\{\omega\colon \sum_{k=1}^{\infty} Y_k = \infty\Big\}$$

We can obtain the following conclusion.

2. Main results and its proof

Theorem 1. *Let $\{X_n, n \geqslant 0\}$ be a nonhomogeneous Markov chain with the initial distribution* (2) *and the transition matrix* (3) *and $\{Y_n, n \geqslant 1\}$ be defined as before. Let $\{\sigma_n, n \geqslant 1\}$ be an arbitrary nonnegative stochastic aequence. Let $\{g_n(x,y), n \geqslant 1\}$ be a real-valued function sequence defined on S^2 and let $\alpha > 0$ be a constant. Let*

$$D = \Big\{\omega\colon \lim_{n\to\infty} \sigma_n = \infty, \quad \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E[\exp\{\alpha\, |Y_k g_k(X_{k-1}, X_k)|\} | X_{k-1}] < \infty\Big\}. \tag{6}$$

Then

$$\lim_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} Y_k \{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k) | X_{k-1})\} = 0 \quad \text{a.s. } \omega \in D \tag{7}$$

P r o o f. Let

$$D_{x_0\dots x_n} = \{\omega\colon\ X_k = x_k, 0 \leqslant k \leqslant n\}, \quad x_k \in S, \quad 1 \leqslant k \leqslant n. \tag{8}$$

Then

$$P(D_{x_0\dots x_n}) = p(x_0, \dots, x_n)$$

and

$$P(D_{x_0\dots x_n}) = p(x_0, \dots, x_n) = p(x_0)\prod_{k=1}^{n} p_k(x_{k-1}, x_k), \quad n \geqslant 1. \tag{9}$$

$D_{x_0\dots x_n}$ is called an nth-order elementary cylinder. Let N_n be the collection of nth-order elementary cylinders, N the collection consisting of $\emptyset$, Ω and all cylinder sets and let $|\lambda| \leqslant \alpha$. Define a set function μ on N as follows:

$$\begin{gathered} \mu(\emptyset) = 0, \qquad \mu(\Omega) = 1, \\ \mu(D_{x_0\dots x_n}) = p(x_0)\prod_{k=1}^{n} \frac{\exp\{\lambda y_k g(x_{k-1}, x_k)\} p_k(x_{k-1}, x_k)}{E[\exp\{\lambda y_k g(X_{k-1}, X_k)\} | X_{k-1} = x_{k-1}]}, \end{gathered} \tag{10}$$

where y_1 is an arbitrary real number, and

$$y_k = f_{k-1}(x_0, \dots, x_{k-1}), \quad k \geqslant 1. \tag{11}$$

We have by (10)

$$\begin{aligned} \sum_{x_n\in S} \mu(D_{x_0\dots x_n}) &= \sum_{x_n\in S} \mu(D_{x_0\dots x_{n-1}}) \frac{\exp\{\lambda y_n g(x_{n-1}, x_n)\} p_n(x_{n-1}, x_n)}{E[\exp\{\lambda y_n g(X_{n-1}, X_n)\} | X_{n-1} = x_{n-1}]} \\ &= \mu(D_{x_0\dots x_{n-1}}) \frac{\sum_{x_n\in S}\exp\{\lambda y_n g(x_{n-1}, x_n)\} p_n(x_{n-1}, x_n)}{E[\exp\{\lambda y_n g(X_{n-1}, X_n)\} | X_{n-1} = x_{n-1}]} \\ &= \mu(D_{x_0\dots x_{n-1}}) \frac{E[\exp\{\lambda y_n g(X_{n-1}, X_n)\} | X_{n-1} = x_{n-1}]}{E[\exp\{\lambda y_n g(X_{n-1}, X_n)\} | X_{n-1} = x_{n-1}]} \\ &= \mu(D_{x_0\dots x_{n-1}}). \end{aligned} \tag{12}$$

It follows from (10)–(12) that μ is a measure on N. Since N is semialgebra, μ has a unique extension to the σ-field $\sigma(N)$. Let

$$T_n(\lambda, \omega) = \sum_{D\in N_n} \frac{\mu(D_{x_0\dots x_n})}{P(D_{x_0\dots x_n})}\, I_{D_{x_0\dots x_n}} \tag{13}$$

where $I_{D_{x_0\ldots x_n}}$ denotes the indicator function of $D_{x_0\ldots x_n}$, that is

$$T_n(\lambda,\omega)=\frac{\mu(D_{X_0(\omega)\ldots X_n(\omega)})}{P(D_{X_0(\omega)\ldots X_n(\omega)})}. \tag{14}$$

It is easy to see that $\{N_n, n\geqslant 0\}$ is a net relative to (Ω, A, P), where A denotes the σ-algebra of events on which P is defined. By the differentiation on a net of Hewitt and Stromberg (see [7], p. 373), there exists $A(\lambda)\in\sigma(N)$ with $P(A(\lambda))=1$ such that

$$\lim_n T_n(\lambda,\omega)=T_\infty(\lambda,\omega)<\infty \qquad \omega\in A(\lambda)$$

that is

$$\lim_n T_n(\lambda,\omega)=T_\infty(\lambda,\omega)<\infty \qquad \text{a.s.} \tag{15}$$

By (6) and (15) we have

$$\limsup_{n\to\infty}\frac{1}{\sigma_n}\log T_n(\lambda,\omega)\leqslant 0 \quad \text{a.s.} \quad \omega\in D. \tag{16}$$

By (9), (10), (13) and (14), we have

$$\log T_n(\lambda;\omega)=\sum_{k=1}^{n}\lambda Y_k g_k(X_{k-1},X_k)-\sum_{k=1}^{n}\log E[\exp\{\lambda Y_k g_k(X_{k-1},X_k)|X_{k-1}]. \tag{17}$$

By (16) and (17) we have

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{\sigma_n}\Bigg\{&\sum_{k=1}^{n}\lambda Y_k g_k(X_{k-1},X_k)\\ &-\sum_{k=1}^{n}\log E[\exp\{\lambda Y_k g_k(X_{k-1},X_k)|X_{k-1}]\Bigg\}\leqslant 0 \quad \text{a.s. } \omega\in D.\end{aligned} \tag{18}$$

By virtue of the property of limeas superior:

$$\limsup_{n\to\infty}(a_n-b_n)\leqslant d\Rightarrow \limsup_{n\to\infty}(a_n-c_n)\leqslant\limsup_{n\to\infty}(b_n-c_n)+d,$$

and the inequalities $\log x\leqslant x-1$ $(x>0)$, $\mathrm{e}^x-1-x\leqslant\frac{1}{2}x^2\mathrm{e}^{|x|}$, noticing

$$\max\{x^2\mathrm{e}^{-hx}, x\geqslant 0\}=4\mathrm{e}^{-2}/h^2 \quad (h>0),$$

and letting $0 \leqslant |\lambda| < \alpha$, we have by (18)

$$
\begin{aligned}
(19)\quad & \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \lambda Y_k \{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k)|X_{k-1})\} \\
& \leqslant \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \{\log E[\exp\{\lambda Y_k g_k(X_{k-1}, X_k)\}|X_{k-1}] \\
& \qquad\qquad - \lambda Y_k E(g_k(X_{k-1}, X_k)|X_{k-1})\} \\
& \leqslant \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \{E[\exp\{\lambda Y_k g_k(X_{k-1}, X_k)\}|X_{k-1}] - 1 \\
& \qquad\qquad - \lambda Y_k E(g_k(X_{k-1}, X_k)|X_{k-1})\} \\
& \leqslant \frac{1}{2} \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} (\lambda Y_k)^2 E\{[g_k(X_{k-1}, X_k)]^2 \mathrm{e}^{|\lambda Y_k g_k(X_{k-1}, X_k)|}|X_{k-1}\} \\
& \leqslant \frac{1}{2} (\lambda Y_k)^2 \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E\{[g_k(X_{k-1}, X_k)]^2 \mathrm{e}^{|\lambda Y_k|\cdot|g_k(X_{k-1}, X_k)|}|X_{k-1}\} \\
& = \frac{1}{2} (\lambda Y_k)^2 \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E\{g_k(X_{k-1}, X_k)^2 \mathrm{e}^{(|\lambda|-\alpha)|Y_k g_k(X_{k-1}, X_k)|} \\
& \qquad\qquad \times \mathrm{e}^{\alpha|Y_k g_k(X_{k-1}, X_k)|}|X_{k-1}\} \\
& \leqslant \frac{1}{2} (\lambda Y_k)^2 \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E\{\mathrm{e}^{\alpha|Y_k g_k(X_{k-1}, X_k)|} \\
& \qquad\qquad \times 4\mathrm{e}^{-2}/(Y_k \cdot (|\lambda| - \alpha))^2 |X_{k-1}\} \text{ a.s. } \omega \in D.
\end{aligned}
$$

When $0 < \lambda < \alpha$, we have by (6) and (19)

$$
\begin{aligned}
(20)\quad & \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} Y_k \{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k)|X_{k-1})\} \\
& \leqslant \frac{1}{2} \lambda \cdot \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E\{\mathrm{e}^{\alpha|Y_k g(X_{k-1}, X_k)|} 4\mathrm{e}^{-2}/(\lambda - \alpha)^2 |X_{k-1}\} \quad \text{a.s. } \omega \in D.
\end{aligned}
$$

Choose $0 < \lambda_i < \alpha$, $i = 1, 2, \ldots$ such that $\lambda_i \to 0$ (as $i \to \infty$). Therefore, for all i, we have by (20)

$$
(21)\quad \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} Y_k \{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k)|X_{k-1})\} \leqslant 0 \quad \text{a.s. } \omega \in D
$$

When $-\alpha < \lambda < 0$, we have by (6) and (19)

$$\text{(22)} \qquad \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} Y_k\{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k)|X_{k-1})\}$$
$$\geqslant \frac{1}{2}\lambda \cdot \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E\{\mathrm{e}^{\alpha|Y_k g(X_{k-1},X_k)|} 4\mathrm{e}^{-2}/(\lambda+\alpha)^2 |X_{k-1}\} \quad \text{a.s. } \omega \in D.$$

Choose $-\alpha < \lambda_i < 0$, $i = 1, 2, \ldots$ such that $\lambda_i \to 0$ (as $i \to \infty$). Therefore, for all i, we have by (22)

$$\text{(23)} \quad \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} Y_k\{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k)|X_{k-1})\} \geqslant 0 \quad \text{a.s. } \omega \in D.$$

Therefore (7) follows from (21) and (23). □

Corollary 1 (see [2]). *Let $\{X_n, n \geqslant 0\}$ be a nonhomogeneous Markov chain with the initial distribution* (2) *and the transition matrix* (3) *and let $\{Y_n, n \geqslant 1\}$ and $\{g_n(x, y), n \geqslant 1\}$ be defined as before. Let*

$$\text{(24)} \qquad D_0 = \left\{\omega \colon \lim_{n\to\infty} \sum_{k=1}^{n} E[\exp\{\alpha|Y_k g_k(X_{k-1}, X_k)|\}|X_{k-1}] = \infty\right\}.$$

where $\alpha > 0$ is a constant. Then

$$\text{(25)} \qquad \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} Y_k\{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k)|X_{k-1})\}}{\sum\limits_{k=1}^{n} E[\exp\{\alpha|Y_k g_k(X_{k-1}, X_k)|\}|X_{k-1}]} = 0 \quad \textit{a.s. } \omega \in D_0.$$

P r o o f. Let

$$\sigma_n = \sum_{k=1}^{n} E[\exp\{\alpha|Y_k g_k(X_{k-1}, X_k)|\}|X_{k-1}], \quad n \geqslant 1.$$

Then

$$\text{(26)} \qquad \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E[\exp\{\alpha|Y_k g_k(X_{k-1}, X_k)|\}|X_{k-1}] \leqslant 1.$$

It can be known that $D = D_0$ from (24) and (26). Therefore (25) follows from (7).

Corollary 2. *Let $\{X_n, n \geqslant 0\}$ be a nonhomogeneous Markov chain with the initial distribution (2) and the transition matrix (3) and let $\{\sigma_n, n \geqslant 1\}$ and $\{g_n(x,y), n \geqslant 1\}$ be defined as before, where $\alpha > 0$ is a constant. Let*

$$(27) \qquad D_2 = \Big\{\omega\colon \lim_{n\to\infty} \sigma_n = \infty, \\ \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E[\exp\{\alpha|g_k(X_{k-1}, X_k)|\}|X_{k-1}] < \infty\Big\}.$$

Then

$$(28) \qquad \lim_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k)|X_{k-1})\} = 0 \quad \text{a.s. } \omega \in D_2.$$

P r o o f. Let $Y_k \equiv 1$ in Theorem 1. It is easy to see that $D_2 = D$ then. Thus (28) can follow from (7). $\square$

Remark. It can be seen that the condition (27) weakens the condition of Theorem 1 in the paper of Liu and Yang (see [6]):

$$\limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E[g_k^2(X_{k-1}, X_k) \mathrm{e}^{\alpha|g_k(X_{k-1}, X_k)|}|X_{k-1}] < \infty.$$

Correspondingly the conclusion is strengthened.

3. The generalization for AEP theorems for nonhomogeneous Markov chains in the generalized gambling system

Let $\{X_n, n \geqslant 0\}$ be an arbitrary stochastic sequence defined on the probability space $(\Omega, \mathscr{F}, P)$ which takes values in $S_0 = \{s_1, s_2, \ldots, s_N\}$. The joint distribution is defined as in (1). Let

$$(29) \qquad f_n(\omega) = -\frac{1}{n} \log p(X_0, \ldots, X_n).$$

$f_n(\omega)$ is called as the relative entropy density of $\{X_n, n \geqslant 0\}$. If $\{X_n, n \geqslant 0\}$ is the nonhomogeneous Markov chain with the initial distribution (2) and the transition matrix (3), then by virtue of (4) and (29) we have

$$(30) \qquad f_n(\omega) = -\frac{1}{n}\Big[\log p(X_0) + \sum_{k=1}^{n} \log p_k(X_{k-1}, X_k)\Big].$$

The limit property of the relative entropy density is an important problem in information theory. Shannon (see [8]) first showed that for stationary ergodic Markov chains $f_n(\omega)$ converges in probability to a constant. McMillan (see [9]) and Breiman (see [10]) proved, respectively, that if $\{X_n, n \geqslant 0\}$ is stationary and ergodic, then $f_n(\omega)$ converges in L_1 and almost everywhere to a constant. This is the famous Shannon-McMillan theorem. The extension of the Shannon-McMillan theorem to the general stochastic process can be found, for example, in Barron (see [11]), Chung (see [12]) and FeinStein (see [13]). In this paper we mainly study the some limit properties and asymptotic equipartition property (AEP) theorems for countable nonhomogeneous Markov chains in the generalized gambling system.

Corollary 3 (see [3]). *Let $\{X_n, n \geqslant 0\}$ be a nonhomogeneous Markov chain with the initial distribution* (2) *and the transition matrix* (3), *and let $f_n(\omega)$ be the relative entropy density defined as in* (30). *Then*

$$\lim_{n\to\infty}\left\{f_n(\omega) - \frac{1}{n}\sum_{k=1}^{n}\sum_{j\in S_0} p_k(X_{k-1},j)\log p_k(X_{k-1},j)\right\} = 0 \quad \text{a.s.} \tag{31}$$

P r o o f. Let $g_k(i,j) = -\log p_k(i,j)$, $Y_k \equiv 1$, $\sigma_n = n$ and $\alpha = 1$ in Theorem 1. Then

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{n} E[\exp\{\alpha|Y_k g_k(X_{k-1},X_k)|\}|X_{k-1}] \\
&\leqslant \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} E[\exp\{|-\log p_k(X_{k-1},X_k)|\}|X_{k-1}] \\
&= \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{j\in S_0}\frac{p_k(X_{k-1},j)}{p_k(X_{k-1},j)} \leqslant N.
\end{aligned} \tag{32}$$

It can be shown that $D = \Omega$ from $\sigma_n = n$ and (32). Therefore (31) follows from (7), (30) and the assumption conditions above.

Remark. The corollary is just the result of Theorem 2 in the paper of Liu and Yang (1996). We consider the problem of the generalized random selection for a arbitrary stochastic sequence. We choose a subsequence of $X_0, X_1, \ldots, X_n, \ldots$ and $(X_0,X_1),(X_1,X_2),\ldots,(X_{n-1},X_n),\ldots$ (where $(X_0,X_1),(X_1,X_2),\ldots,(X_{n-1},X_n),\ldots$ are the ordered couples of random variables $\{X_n, n \geqslant 0\}$) according to the value Y_n takes. We select X_n and (X_{n-1},X_n) if and only if $Y_n \in [-m,0)\cup(0,m]$. Therefore we obtain a subsequence of the above sequence. We let for $i,j \in S_0$, $S_n(i;\omega)$ be the number of the states i in the sequence $X_1, X_2, \ldots, X_n$, which are selected by $\{Y_k, 1 \leqslant k \leqslant n\}$; $S_n(i,j;\omega)$ be the number of the ordered couples (i,j) in the ordered couples $(X_0,X_1),(X_1,X_2),\ldots,(X_{n-1},X_n)$, which are selected by $\{Y_k, 1 \leqslant k \leqslant n\}$. That is $S_n(i;\omega) = \sum\limits_{k=1}^{n} Y_k\delta_i(X_k)$, $S_n(i,j;\omega) = \sum\limits_{k=1}^{n} Y_k\delta_i(X_{k-1})\delta_j(X_k)$, where $\delta_i(j)$ is defined as before. Then we can conclude the following results.

Corollary 4. *Let $\{X_n, n \geqslant 0\}$ be a nonhomogeneous Markov chain with the initial distribution* (2) *and the transition matrix* (3) *and let $\{\sigma_n, n \geqslant 1\}$, $S_n(i;\omega)$ and $S_n(i,j;\omega)$ be defined as before. $|Y_n| \leqslant m$, $n \geqslant 1$. assume that*

$$\limsup_{n\to\infty} \frac{n}{\sigma_n} \leqslant M \quad \text{a.s.} \tag{33}$$

Then

$$\lim_{n\to\infty} \frac{1}{\sigma_n}\left\{S_n(i;\omega) - \sum_{k=1}^{n} Y_k p_k(X_{k-1},i)\right\} = 0 \quad \text{a.s.}, \tag{34}$$

$$\lim_{n\to\infty} \frac{1}{\sigma_n}\left\{S_n(i,j;\omega) - \sum_{k=1}^{n} Y_k \delta_i(X_{k-1}) p_k(i,j)\right\} = 0 \quad \text{a.s.} \tag{35}$$

P r o o f. Letting $g_k(X_{k-1},X_k) = \delta_i(X_k)$, $k \geqslant 1$, then

$$\begin{aligned}\sum_{k=1}^{n} Y_k\{g_k(X_{k-1},X_k) - E(g_k(X_{k-1},X_k)|X_{k-1})\} \\ = \sum_{k=1}^{n} Y_k\{\delta_i(X_k) - E(\delta_i(X_k)|X_{k-1})\} = S_n(i;\omega) - \sum_{k=1}^{n} Y_k p_k(X_{k-1},i).\end{aligned} \tag{36}$$

Noticing that $|Y_n| \leqslant m$, $n \geqslant 1$. We have by (33) that

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E[\exp\{\alpha|Y_k g_k(X_{k-1},X_k)|\}|X_{k-1}] \\ \leqslant \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} E[\mathrm{e}^{m\alpha}|X_{k-1}] \leqslant \limsup_{n\to\infty} \mathrm{e}^{m\alpha}\frac{n}{\sigma_n} \leqslant M\mathrm{e}^{\alpha m} \quad \text{a.s.}\end{aligned} \tag{37}$$

It is also easy to see from (33) that $\lim\limits_{n\to\infty} \sigma_n = \infty$. Therefore $D = \Omega$. It is easy to see from (7) and (36) that (34) holds. Similarly, let $g_k(X_{k-1},X_k) = \delta_i(X_{k-1})\delta_j(X_k)$,

$k \geqslant 1$. Noticing that

$$\sum_{k=1}^{n} Y_k\{g_k(X_{k-1}, X_k) - E(g_k(X_{k-1}, X_k)|X_{k-1})\} \tag{38}$$
$$= \sum_{k=1}^{n} Y_k\{\delta_i(X_{k-1})\delta_j(X_k) - E(\delta_i(X_{k-1})\delta_j(X_k)|X_{k-1})\}$$
$$= S_n(i,j;\omega) - \sum_{k=1}^{n} Y_k\delta_i(X_{k-1})p_k(X_{k-1}, j),$$

it can also be seen that (37) holds. Therefore, $D = \Omega$ holds too. (35) follows from (7) and (38). $\square$

Theorem 2. *Let $\{X_n, n \geqslant 0\}$ be a nonhomogeneous Markov chain with the initial distribution (41) and the transition matrix (42). Let $S_n(i;\omega)$ and $S_n(i,j;\omega)$ be defined as before. Let $\sigma_n = \sum\limits_{k=1}^{n} Y_k$, where Y_k is defined as in Corollary 4. Let $P = (p(i,j))$ be another transition matrix and be ergodic. $\{g(x,y)\}$ be a real-valued function defined on S_0^2. If*

$$\lim_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} |Y_k p_k(i,j) - Y_{k-1}p(i,j)| = 0 \quad \textit{for all } i,j \in S_0, \tag{39}$$

Then

$$\lim_{n\to\infty} \frac{1}{\sigma_n} S_n(i;\omega) = \pi_i \quad \text{a.s.}, \tag{40}$$

$$\lim_{n\to\infty} \frac{1}{\sigma_n} S_n(i,j;\omega) = \pi_i p(i,j) \quad \text{a.s.}, \tag{41}$$

$$\lim_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} Y_k g(X_{k-1}, X_k) = \sum_{i\in S_0}\sum_{j\in S_0} \pi_i g(i,j)p(i,j) \quad \text{a.s.}, \tag{42}$$

where $(\pi_{s_1}, \pi_{s_2}, \ldots, \pi_{s_N})$ is the unique stationary distribution determined by P.

P r o o f. (1) We have by (34) in Corollary 4,

$$\lim_{n\to\infty} \frac{1}{\sigma_n} \left\{ S_n(j;\omega) - \sum_{k=1}^{n} Y_k p_k(X_{k-1}, j) \right\} = 0 \quad \text{a.s.} \tag{43}$$

Since

$$\sum_{k=1}^{n} Y_k p_k(X_{k-1}, j) = \sum_{k=1}^{n}\sum_{i\in S_0} Y_k \delta_i(X_{k-1}) p_k(i,j), \tag{44}$$

by (39) and the definition of σ_n in Corollary 4, we have

$$\begin{aligned}(45)\qquad &\lim_{n\to\infty}\frac{1}{\sigma_n}\left|\sum_{k=1}^{n}\sum_{i\in S_0}\delta_i(X_{k-1})(Y_k p_k(i,j)-Y_{k-1}p(i,j))\right| \\ &\leqslant \sum_{i\in S_0}\lim_{n\to\infty}\frac{1}{\sigma_n}\left|\sum_{k=1}^{n}(Y_k p_k(i,j)-Y_{k-1}p(i,j))\right| \\ &\leqslant \sum_{i\in S_0}\lim_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{n}|Y_k p_k(i,j)-Y_{k-1}p(i,j)|.\end{aligned}$$

By (43), (44), (45) and the definition of $S_n(i;\omega)$ we have

$$\begin{aligned}(46)\qquad &\lim_{n\to\infty}\frac{1}{\sigma_n}\left\{S_n(j;\omega)-\sum_{i\in S_0}S_n(i;\omega)p(i,j)\right\} \\ &=\lim_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{n}\sum_{i\in S_0}\delta_i(X_{k-1})(Y_k p_k(i,j)-Y_{k-1}p(i,j))=0 \quad \text{a.s.}\end{aligned}$$

Multiplying (46) by $p(j,l)$, adding them together for $j\in S_0$ and using (46) once again, we have

$$\begin{aligned}(47)\qquad 0 &=\sum_{j\in S_0}p(j,l)\lim_{n\to\infty}\frac{1}{\sigma_n}\left\{S_n(j;\omega)-\sum_{i\in S_0}S_n(i;\omega)p(i,j)\right\} \\ &=\lim_{n\to\infty}\frac{1}{\sigma_n}\left\{\sum_{j\in S_0}S_n(j;\omega)p(j,l)-S_n(l;\omega)\right\} \\ &\quad+\lim_{n\to\infty}\frac{1}{\sigma_n}\left\{S_n(l;\omega)-\sum_{i\in S_0}\sum_{j\in S_0}S_n(i;\omega)p(i,j)p(j,l)\right\} \\ &=\lim_{n\to\infty}\frac{1}{\sigma_n}\left\{S_n(l;\omega)-\sum_{i\in S_0}S_n(i;\omega)p^{(2)}(i,l)\right\} \quad \text{a.s.,}\end{aligned}$$

where $p^{(k)}(i,l)$ (k is an integer) is the k-step transition probability of P. By induction, we have

$$(48)\qquad \lim_{n\to\infty}\frac{1}{\sigma_n}\left\{S_n(l;\omega)-\sum_{i\in S_0}S_n(i;\omega)p^{(k)}(i,l)\right\}=0 \quad \text{a.s.}$$

By (48) we have

$$(49)\qquad \lim_{n\to\infty}\frac{1}{\sigma_n}\left\{S_n(l;\omega)-\sum_{i\in S_0}S_n(i;\omega)\frac{1}{N}\sum_{k=1}^{N}p^{(k)}(i,l)\right\}=0 \quad \text{a.s.}$$

Since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}p^{(k)}(i,l)=\pi_l \tag{50}$$

and

$$\frac{1}{\sigma_n}\sum_{i\in S_0}S_n(i;\omega)=\sum_{k=1}^{n}\sum_{i\in S_0}Y_k\delta_i(X_k)\Big/\sum_{k=1}^{n}Y_k=1, \tag{51}$$

it is easy to see from (49)–(51) that (40) holds .

(2) By (35) in Corollary 4, we have

$$\lim_{n\to\infty}\frac{1}{\sigma_n}\left\{S_n(i,j;\omega)-\sum_{k=1}^{n}Y_k\delta_i(X_{k-1})p_k(i,j)\right\}=0\ \text{ a.s.} \tag{52}$$

It is easy to show by (39) and (45)

$$\lim_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{n}\delta_i(X_{k-1})(Y_kp_k(i,j)-Y_{k-1}p(i,j))=0. \tag{53}$$

By (52), (53) and the definition of $S_n(i;\omega)$ we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\sigma_n}\{S_n(i,j;\omega)-S_n(i;\omega)p(i,j)\}\\&=\lim_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{n}\delta_i(X_{k-1})(Y_kp_k(i,j)-Y_{k-1}p(i,j))=0\ \text{ a.s.}\end{aligned} \tag{54}$$

It can be seen from (40) and (54) that (41) holds.

(3) Since

$$\begin{aligned}\frac{1}{\sigma_n}\sum_{k=1}^{n}Y_kg(X_{k-1},X_k)&=\frac{1}{\sigma_n}\sum_{k=1}^{n}\sum_{i\in S_0}\sum_{j\in S_0}Y_k\delta_i(X_{k-1})\delta_j(X_k)g(i,j)\\&=\sum_{i\in S_0}\sum_{j\in S_0}g(i,j)\frac{1}{\sigma_n}S_n(i,j;\omega),\end{aligned} \tag{55}$$

(42) follows from (41) and (55). □

Corollary 5. *Let* $\{X_n, n \geqslant 0\}$ *be a nonhomogeneous Markov chain with the initial distribution* (2) *and the transition matrix* (3) *and* $f_n(\omega)$ *be the relative entropy density defined as in* (30). *Let* $P = (p(i,j))$ *be defined as in Theorem* 2. *If* (39) *holds, then*

$$\lim_{n\to\infty} f_n(\omega) = -\sum_{i\in S_0} \pi_i \sum_{j\in S_0} p(i,j)\log p(i,j) \quad \text{a.s.} \tag{56}$$

Proof. Let $Y_n \equiv 1$, $g(x,y) = -\log p(x,y)$ in Theorem 2, then $\sigma_n = n$. Moreover,

$$\limsup_{n\to\infty}\left(\frac{n}{\sigma_n}\right) = \limsup_{n\to\infty}\frac{n}{n} \leqslant 1 \quad \text{a.s.} \tag{57}$$

From (42) in Theorem 2 it follows immediately that (56) holds.

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