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THE MAXIMUM CLIQUE AND THE SIGNLESS LAPLACIAN EIGENVALUES

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Abstract. Lower and upper bounds are obtained for the clique number $\omega(G)$ and the independence number $\alpha(G)$, in terms of the eigenvalues of the signless Laplacian matrix of a graph G.

Keywords: bound, clique number, independence number, signless Laplacian eigenvalues

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1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices. Denote $V(G) = \{v_1, v_2, \dots v_n\}$. For any two vertices $v_i, v_j \in V(G)$ with $i < j$ we will use the symbol $i \sim j$ to denote the edge $v_i v_j$. Let A be the adjacency matrix of G and $D = \operatorname{diag}(d_1, d_2, \dots d_n)$ the diagonal matrix of vertex degrees. The matrix $L = D - A$ is known as the Laplacian of G and has been studied extensively in literature (see, e.g., [1]). The matrix $Q = D + A$ is called the signless Laplacian in [4] and appears very rarely in published papers (see [1]), the paper [2] being one of the very few research papers concerning this matrix. Since the signless Laplacian is a positive semi-definite matrix, all its eigenvalues are non-negative. Throughout the paper, we order the eigenvalues of the signless Laplacian matrix Q (Q-eigenvalues for short) of a graph G as $q_1 \geqslant q_2 \geqslant \dots \geqslant q_n \geqslant 0$, and let $u_1, \dots, u_n$ be the corresponding normalized eigenvectors. We can suppose that $u_1 > 0$.

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Given a graph G, define $\omega(G)$ and $\alpha(G)$ (ω and α for short), the clique number and the independence number of G to be the numbers of vertices of the largest clique and the largest independent set in G, respectively. Obviously, $\omega(G) = \alpha(G^c)$ where G^c is the complement of G.

In this paper we obtain lower and upper bounds for the independence number α and the clique number ω of a graph G. The bounds involve the signless Laplacian eigenvalues.

2. Bounds on the clique number and independence number of arbitrary graphs

We give bounds for the clique number ω and for the independence number α. They involve the largest Q-eigenvalue q_1 and the least Q-eigenvalue q_n of the graph G. The derivation of these bounds rests on a theorem of Motzkin and Straus. We recall the following important result.

Theorem 2.1 ([6]). *Let S be the simplex in E^n given by $x_i \geqslant 0$, $\sum\limits_{i=1}^{n} x_i = 1$. Let ω be the order of the maximal clique in G and let A be the adjacency matrix of G. Then*

$$1 - \frac{1}{w} = \max_{x \in S} \langle x, Ax \rangle . \tag{1}$$

Theorem 2.1 offers great possibilities for the investigation of the clique number by means of spectra. It will be used many times in the sequel. The dual theorem of Theorem 2.1 also holds, as shown by the following theorem.

Theorem 2.2. *Let α be the size of the largest independent set of G. Then*

$$\frac{1}{\alpha} = \min_{x \in S} \langle x, (I + A)x \rangle \tag{1'}$$

where S and A are the same as in Theorem 2.1.

Lemma 2.1. *Let G be a graph with minimum degree δ. Then $q_n \leqslant \delta$.*

Proof. By the min-max characterization of the eigenvalues of a symmetric matrix, we have $q_n = \min\limits_{x \neq 0} (x^t Q x / x^t x)$. Let $e_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)$, where j is a vertex with the minimum degree δ. Then $q_n \leqslant {e_j}^t Q e_j / {e_j}^t e_j = \delta$. $\square$

Theorem 2.3. *Let G be a graph with n vertices, m edges and minimum degree δ. Then*

$$w \geqslant \frac{2m}{2m-(q_n-\delta)^2}. \tag{2}$$

Proof. Let $u_n = (y_1, y_2, \ldots, y_n)$ be the normalized eigenvector corresponding to q_n. Then

$$\begin{aligned} q_n = (u_n)^T Q u_n &= \sum_{i\sim j}(y_i+y_j)^2 = \sum_{i=1}^{n} d_i y_i^2 + \sum_{i\sim j} 2y_iy_j \\ &\geqslant \delta \sum_{i=1}^{n} y_i^2 + \sum_{i\sim j} 2y_iy_j = \delta + \sum_{i\sim j} 2y_iy_j. \end{aligned}$$

By Lemma 2.1, $q_n - \delta \leqslant 0$. We have $(q_n-\delta)^2 \leqslant \Bigl(\sum\limits_{i\sim j} 2y_iy_j\Bigr)^2$.

By the Cauchy inequality we obtain

$$\Bigl(\sum_{i\sim j} 2y_iy_j\Bigr)^2 \leqslant 4\Bigl(\sum_{i=1}^{m} 1^2\Bigr)\Bigl(\sum_{i\sim j}(y_iy_j)^2\Bigr) = 2m\Bigl(2\sum_{i\sim j} y_i^2y_j^2\Bigr).$$

Since $(y_1^2, y_2^2, \ldots, y_n^2) \geqslant 0$ and $y_1^2 + y_2^2 + \ldots + y_n^2 = 1$, by Theorem 2.1 we have

$$2\sum_{i\sim j} y_i^2y_j^2 \leqslant 1 - \frac{1}{w}.$$

Therefore

$$\frac{(q_n-\delta)^2}{2m} \leqslant 1 - \frac{1}{w},$$

that is

$$w \geqslant \frac{2m}{2m-(q_n-\delta)^2}.$$

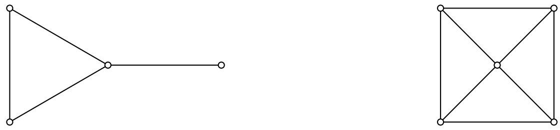
□

Remark 2.1. In [5] it was proved that

$$w \geqslant \frac{2m}{2m-(\lambda_1-\Delta)^2} \tag{3}$$

where Δ is the maximum degree and λ_1 is the largest Laplacian eigenvalue. We now show that the bounds (3) and (2) are incomparable.

Let G_1, G_2 be the graphs shown in Fig 1.



G_1 G_2

Fig. 1

By direct computation, we find that for G_1, the bound (3) is 1.143 and better than the bound (2) 1.041 numerically, but the bounds are in practice equivalent (both bounds imply that the clique number is at least 2); and for G_2, the bound (2) is 1.33 and better than the bound (3) 1.067 numerically, but the bounds are in practice equivalent (both bounds imply that the clique number is at least 2). And if $G = K_{1,n-1}$ (the star) or $G = K_n$ (the complete graph), the two bounds are equivalent numerically.

Note that the eigenvector u_1 is positive number. Let U be the sum of the entries of u_1. Then the vector $x = u_1/U$ belongs to the simplex S of Eq. (1).

Theorem 2.4. *Let G be a graph with maximum degree Δ. Then*

$$w \geqslant \frac{n}{n - q_1 + \Delta}. \tag{4}$$

Proof. Let $u_1 = (r_1, r_2, \ldots, r_n)$ be the normalized eigenvector corresponding to q_1. Let $U = \sum\limits_{i=1}^{n} r_i$ and $x = u_1/U$. Then

$$\langle x, Qx\rangle = \langle x, Dx\rangle + \langle x, Ax\rangle \leqslant \Delta\,\langle x, x\rangle + \Big(1 - \frac{1}{w}\Big) = \frac{\Delta}{U^2} + 1 - \frac{1}{w}.$$

On the other hand,

$$\langle x, Qx\rangle = \sum_{i\sim j}(x_i + x_j)^2 = \frac{1}{U^2}\sum_{i\sim j}(r_i + r_j)^2 = \frac{q_1}{U^2}.$$

Therefore

$$\frac{q_1}{U^2} \leqslant \frac{\Delta}{U^2} + \Big(1 - \frac{1}{w}\Big).$$

Since

$$U^2 = \Big(\sum_{i=1}^{n} r_i\Big)^2 \leqslant n\sum_{i=1}^{n} r_i^2 = n,$$

we obtain,

$$\frac{q_1-\Delta}{n} \leqslant \frac{q_1-\Delta}{U^2} \leqslant 1-\frac{1}{w}.$$

That is,

$$w \geqslant \frac{n}{n-q_1+\Delta}.$$

□

Remark 2.2. We now show that in many cases the bound (4) is better than both (2) and (3). For example, for $G = K_n$ the bound (4) is nicely equal to n and is better than (2) and (3), which gives $\omega(K_n) \geqslant (n^2-n)/(n^2-n-1)$ (this implies that the clique number is at least 2, which is very trivial!); and for $n \leqslant 5$ the bound (4) is always better than (3), except for the P_5 (path of order 5). The bound (3) gives $\omega(P_5) \geqslant 1.486$, while the bound (4) gives $\omega(P_5) \geqslant 1.478$ (both bounds imply that the clique number is at least 2). And if G is a bipartite regular graph, the three bounds are equivalent.

By adapting a similar technique, we find the analogous result for α.

Theorem 2.5. *Let G be a graph with minimum degree δ. Then*

$$\alpha \geqslant \frac{U^2}{q_1^2-\delta+1},$$

where U is the same as in Theorem 2.4.

3. Some results for regular graphs

We proceed now to the investigation of the clique number of regular graphs. For d-regular graphs we have $u_1 = e/\sqrt{n}$, where $e = (1, 1, \ldots, 1)$ and $q_1 = 2d$. Naturally, all theorems of Section 2 hold also for regular graphs.

Theorem 3.1. *Let G be a d-regular graph. Then*

$$w \geqslant \frac{n^2}{n^2-nd+(d-q_n)M^2},$$

where $M = \min\limits_{y_i \neq 0}\{1/|y_i|\}$ and $u_n = (y_1, y_2, \ldots, y_n)$ is the normalized eigenvector corresponding to q_n.

P r o o f. Let $\theta = M/n$ and $x = n^{-1}e + \theta u_n$. Then $\theta y_i \geqslant -1/n$ $(i = 1, 2, \ldots, n)$, and since $\langle e, u_n\rangle = 0$, it follows that x belongs to the simplex S of Eq. (1).

By Eq. (1) we have

$$\begin{aligned}\langle x,Qx\rangle &= \langle x,Dx\rangle+\langle x,Ax\rangle\\ &\leqslant d\,\langle x,x\rangle+\Big(1-\frac{1}{w}\Big)=d\Big(\frac{1}{n}+\theta^2\Big)+\Big(1-\frac{1}{w}\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle x,Qx\rangle &= \Big\langle \frac{e}{n}+\theta u_n, Q\Big(\frac{e}{n}+\theta u_n\Big)\Big\rangle\\ &= \Big\langle \frac{e}{n}+Q\frac{e}{n}\Big\rangle+\Big\langle \frac{e}{n},Q\theta u_n\Big\rangle+\Big\langle \theta u_n,Q\frac{e}{n}\Big\rangle+\Big\langle \theta u_n,Q\theta u_n\Big\rangle\\ &= \frac{1}{n^2}\,\langle e,Qe\rangle+0+0+\theta^2\,\langle u_n,Qu_n\rangle=\frac{2nd}{n^2}+\theta^2 q_n.\end{aligned}$$

Therefore $2d/n+\theta^2 q_n\leqslant d(1/n+\theta^2)+1-1/w$, that is

$$w\geqslant\frac{1}{1-\frac{d}{n}+\theta^2(d-q_n)}.$$

Since $\theta=M/n$, $M=\min\limits_{y_i\neq 0}\{1/|y_i|\}$, we find

$$w\geqslant\frac{n^2}{n^2-nd+(d-q_n)M^2},$$

which gives the required inequality. □

By adapting a similar technique, we find the following similar result for α.

Theorem 3.2. *Let G be a d-regular graph. Let $u_n=(y_1,y_2,\ldots,y_n)$ be the normalized eigenvector corresponding to q_n, and let $M=\min\limits_{y_i\neq 0}\{1/|y_i|\}$. Then*

$$\alpha\geqslant\frac{n^2}{nd+n+M^2(q_n-d+1)}.$$

Given an $n\times n$ matrix A and an ordered partition $(X_1,\ldots,X_m)$ of the ordered set $\{1,2,\ldots,n\}$, $A_{n\times n}$ can be presented as a partitioned matrix

$$\begin{pmatrix}A_{1,1} & \ldots & A_{1,m}\\ \vdots & \ddots & \vdots\\ A_{m,1} & \ldots & A_{m,m}\end{pmatrix},$$

where $A_{i,j}$ has X_i as the set of its row numbers and X_j as the set of its column numbers. We always use Q_A hereafter to denote the quotient matrix of the partitioned matrix A, which is defined to be the $m \times m$ matrix whose entries are the average row sums of the blocks of $A_{n\times n}$; namely, each entry $x_{i,j}$ is obtained by dividing the sum of all row sums of $A_{i,j}$ by $|X_i|$, where $1 \leqslant i, j \leqslant m$.

Consider two sequences of real numbers: $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n$ and $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_m$ with $m < n$. The latter sequence is said to interlace the former whenever

$$\alpha_i \geqslant \beta_i \geqslant \alpha_{n-m+i} \quad \text{for} \quad i = 1, 2 \ldots, m.$$

The interlacing is called tight if there exists an integer $k \in [0, m]$ such that

$$\lambda_i = \mu_i \ \text{ for } \ 1 \leqslant i \leqslant k \quad \text{and} \quad \lambda_{n-m+i} = \mu_i \ \text{ for } \ k+1 \leqslant i \leqslant m.$$

Lemma 3.1 ([3]). *For a symmetric partitioned matrix A, the eigenvalues of the quotient matrix Q_A interlace the eigenvalues of A.*

Theorem 3.3. *Suppose $\{v_1, v_2, \ldots, v_\omega\}$ is the largest clique vertices set of the d-regular graph G. If $n > q_1 - q_n$, then*

$$\frac{n(q_n + 1 - d)}{n - 2d + q_n} \leqslant \omega \leqslant \frac{n(q_2 + 1 - d)}{n - 2d + q_2}.$$

The equalities $n(q_n + 1 - d)/(n - 2d + q_n) = \omega = n(q_2 + 1 - d)/(n - 2d + q_2)$ hold if and only if $G = O_n$ (the empty graph of order n).

Proof. The clique vertices set $\{v_1, v_2, \ldots, v_\omega\}$ gives rise to a partition of $Q(G)$ with the quotient matrix

$$Q_{Q(G)} = \begin{pmatrix} d + (\omega - 1) & d - (\omega - 1) \\ \dfrac{[d - (\omega - 1)]\omega}{n - \omega} & 2d - \dfrac{[d - (\omega - 1)]\omega}{n - \omega} \end{pmatrix}.$$

$Q_{Q(G)}$ has eigenvalues $\lambda_1 = 2d$, $\lambda_2 = d + (\omega - 1) - ([d - (\omega - 1)]\omega)/(n - \omega)$. By Lemma 3.1 we have $q_n \leqslant \lambda_2 \leqslant q_2$, which gives the required inequality (note that $n > q_1 - q_n$). If the right equality holds, then $\lambda_2 = q_n$, and since $\lambda_1 = q_1 = 2d$ the interlacing is tight. Since $n(q_n+1-d)/(n-2d+q_n) = \omega = n(q_2+1-d)/(n-2d+q_2)$ holds, if and only if $q_n = \lambda_2 = q_2$, and since $\sum\limits_{i=1}^{n} q_i = nd$, we obtain $\omega = 1$, that is $G = O_n$. $\square$

Note that the condition $n > q_1 - q_n$ is necessary. For example, if $G = K_n$ or G is a bipartite $n/2$-regular graph then $n - 2d + q_n = 0$, the inequalities in Theorem 3.3 are invalid. However, in these two cases the clique number ω is known.

Now we will give some bounds for the independence number α by adapting a similar technique.

Theorem 3.4. *Suppose $\{v_1, v_2, \ldots, v_\alpha\}$ is a largest independent vertices set of a d-regular graph G, then*

$$\frac{n(d-q_2)}{2d-q_2} \leqslant \alpha \leqslant \frac{n(d-q_n)}{2d-q_n}.$$

The equalities $n(d-q_2)/(2d-q_2) = \alpha = n(d-q_n)/(2d-q_n)$ hold, if and only if $G = K_n$ (the complete graph).

P r o o f. The independent set $\{v_1, v_2, \ldots, v_\alpha\}$ gives rise to a partition of $Q(G)$ with the quotient matrix

$$Q_{Q(G)} = \begin{pmatrix} d & d \\ \dfrac{d\alpha}{n-\alpha} & 2d - \dfrac{d\alpha}{n-\alpha} \end{pmatrix}.$$

$Q_{Q(G)}$ has eigenvalues $\lambda_1 = 2d$, $\lambda_2 = d - d\alpha/(n-\alpha)$ and so $q_n \leqslant \lambda_2 \leqslant q_2$ gives the required inequality. If the right equality holds, then $\lambda_2 = q_n$, and since $\lambda_1 = q_1 = 2d$, the interlacing is tight. Since $(n(d-q_2))/(2d-q_2) = \alpha = (n(d-q_n))/(2d-q_n)$ holds if and only if $q_n = \lambda_2 = d - d\alpha/(n-\alpha) = q_2$, and since $\sum\limits_{i=1}^{n} q_i = 2m = nd$, we obtain $\alpha = 1$, that is $G = K_n$. $\square$

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