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A MEASURE-THEORETIC CHARACTERIZATION OF THE HENSTOCK-KURZWEIL INTEGRAL REVISITED

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Abstract. In this paper we show that the measure generated by the indefinite Henstock-Kurzweil integral is $F_{\sigma\delta}$ regular. As a result, we give a shorter proof of the measure-theoretic characterization of the Henstock-Kurzweil integral.

Keywords: Henstock variational measure, Henstock-Kurzweil integral

MSC 2010: 26A39

1. Introduction

It is well known that a function F is absolutely continuous on $[a,b] \subset \mathbb{R}$ if and only if the Lebesgue-Stieltjes measure μ_F generated by F is absolutely continuous with respect to the Lebesgue measure. This remarkable result has been generalized by several authors; see, for example, [1], [6], [7], [10], [11] and the references therein. In this paper we give a shorter proof of the corresponding result for the multiple Henstock-Kurzweil (equivalently, the Perron) integral; see Theorem 4.5. As a by-product of our new techniques, we obtain a somewhat unexpected result that the measure $V_{\mathcal{HK}}F$ generated by the indefinite Henstock-Kurzweil integral is $F_{\sigma\delta}$ regular; see Theorem 3.8.

2. Preliminaries

Let $m \geqslant 1$ be an integer and let $\mathbb{R}^m$ be the m-dimensional Euclidean space equipped with the maximum norm $\|\cdot\|$. An *interval* in $\mathbb{R}^m$ is a set of the form $[\mathbf{u},\mathbf{v}] := \prod_{i=1}^{m} [u_i, v_i]$, where $\mathbf{u} = (u_1, \ldots, u_m)$, $\mathbf{v} = (v_1, \ldots, v_m)$ with $u_i, v_i \in \mathbb{R}$ and

$u_i < v_i$ for $i = 1, \dots, m$. Moreover, we set $(\mathbf{u}, \mathbf{v}) := \prod_{i=1}^{m} (u_i, v_i)$. Following [3], given any $[\mathbf{u}, \mathbf{v}] \in \mathcal{I}_m([\mathbf{a}, \mathbf{b}])$, we let $l([\mathbf{u}, \mathbf{v}]) = \max_{i=1,\dots,m} (v_i - u_i)$, $t([\mathbf{u}, \mathbf{v}]) = \min_{i=1,\dots,m} (v_i - u_i)$ and $\operatorname{reg}([\mathbf{u}, \mathbf{v}]) = (t([\mathbf{u}, \mathbf{v}]))(l([\mathbf{u}, \mathbf{v}]))^{-1}$.

Let $[\mathbf{a}, \mathbf{b}] := \prod_{i=1}^{m} [a_i, b_i]$ be a fixed interval in $\mathbb{R}^m$. For $\mathbf{x} \in \mathbb{R}^m$ and $r > 0$, set $B(\mathbf{x}, \mathbf{r}) := \{\mathbf{y} \in \mathbb{R}^m \colon \|\mathbf{y} - \mathbf{x}\| < r\}$, where $\mathbf{y} - \mathbf{x} = (y_1 - x_1, \dots, y_m - x_m)$. A *partition P in* $[\mathbf{a}, \mathbf{b}]$ (*of* $[\mathbf{a}, \mathbf{b}]$) is a finite collection $\{(I_1, \boldsymbol{\xi}_1), \dots, (I_p, \boldsymbol{\xi}_p)\}$ of interval-point pairs, where $I_1, \dots, I_p$ are non-overlapping intervals such that $\bigcup_{i=1}^{p} I_i \subseteq [\mathbf{a}, \mathbf{b}]$ (respectively $\bigcup_{i=1}^{p} I_i = [\mathbf{a}, \mathbf{b}]$) and $\boldsymbol{\xi}_i \in I_i$ for $i = 1, \dots, p$. Given that $Z \subseteq [\mathbf{a}, \mathbf{b}]$, a positive function δ on Z is called a *gauge* on Z. A partition P is said to be

(i) *anchored* in Z if $\{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_p\} \subset Z$,
(ii) *δ-fine* if $I_i \subset B(\boldsymbol{\xi}_i, \delta(\boldsymbol{\xi}_i))$ for $i = 1, \dots, p$.

Let $\mathcal{I}_m([\mathbf{a}, \mathbf{b}])$ be the family of all subintervals of $[\mathbf{a}, \mathbf{b}]$. A function $F \colon \mathcal{I}_m([\mathbf{a}, \mathbf{b}]) \to \mathbb{R}$ is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each non-overlapping intervals $I, J \in \mathcal{I}_m([\mathbf{a}, \mathbf{b}])$ such that $I \cup J \in \mathcal{I}_m([\mathbf{a}, \mathbf{b}])$. Let F be an interval function on $\mathcal{I}_m([\mathbf{a}, \mathbf{b}])$ and X an arbitrary subset of $[\mathbf{a}, \mathbf{b}]$. For each gauge δ on X, we set

$$V(F, X, \delta) := \sup \left\{ \sum_{(I, \mathbf{x}) \in P} |F(I)| : \ P \text{ is a } \delta\text{-fine partition anchored in } X \right\}$$

and

$$V_{\mathcal{H}K} F(X) := \inf\{V(F, X, \delta) \colon \ \delta \text{ is a gauge on } X\}.$$

This extended real-valued set function $V_{\mathcal{H}K} F(\cdot)$ is a metric outer measure (cf. [3, Proposition 3.3]), known as the *Henstock variational measure generated by F*. We say that $V_{\mathcal{H}K} F$ is *absolutely continuous* with respect to the m-dimensional Lebesgue measure μ_m, in symbol $V_{\mathcal{H}K} F \ll \mu_m$, if $V_{\mathcal{H}K} F(Z) = 0$ for each set $Z \subset [\mathbf{a}, \mathbf{b}]$ such that $\mu_m(Z) = 0$.

3. Some results concerning the Henstock variational measure $V_{\mathcal{H}K} F$

In this section we will collect some useful results that will be required in the proof of our main result. First of all, we need the following refinement of [6, Lemma 6.2].

Lemma 3.1. *Let $F\colon \mathcal{I}_m([\mathbf{a},\mathbf{b}]) \to \mathbb{R}$ be an additive interval function. If $X \subseteq [\mathbf{a},\mathbf{b}]$ and $V_{\mathcal{HK}}F(X)$ is finite, then for each $\varepsilon > 0$ there exists a gauge δ on X such that*

$$\sum_{i=1}^{p} |F(I_i)| < V_{\mathcal{HK}}F\Bigl(X \cap \bigcup_{i=1}^{p} I_i\Bigr) + \varepsilon$$

for each δ-fine partition $\{(I_1,\mathbf{x}_1),\dots,(I_p,\mathbf{x}_p)\}$ anchored in X.

Proof. For each $\varepsilon > 0$ we choose a gauge δ on X such that

$$V(F,X,\delta) < V_{\mathcal{HK}}(X) + \frac{\varepsilon}{2}.$$

Consider any δ-fine partition $\{(I_1,\mathbf{x}_1),\dots,(I_p,\mathbf{x}_p)\}$ anchored in X. Clearly the set $X \setminus \bigcup\limits_{i=1}^{p} I_i$ may be assumed to be non-empty. Then the map $\mathbf{z} \mapsto \Delta(\mathbf{z}) := \min\Bigl\{\delta(\mathbf{z}), \operatorname{dist}\Bigl(\{\mathbf{z}\}, \bigcup\limits_{i=1}^{p} I_i\Bigr)\Bigr\}$ is a gauge on $X \setminus \bigcup\limits_{i=1}^{p} I_i$, where $\operatorname{dist}(U,V)$ denotes the distance between two subsets U, V of $[\mathbf{a},\mathbf{b}]$. Thus there exists a Δ-fine partition $\{(J_1,\mathbf{y}_1),\dots,(J_q,\mathbf{y}_q)\}$ anchored in $X \setminus \bigcup\limits_{i=1}^{p} I_i$ such that

$$V_{\mathcal{HK}}F\Bigl(X \setminus \bigcup_{i=1}^{p} I_i\Bigr) < \sum_{i=1}^{q} |F(J_i)| + \frac{\varepsilon}{2}.$$

Since it is clear that $\{(I_1,\mathbf{x}_1),\dots,(I_p,\mathbf{x}_p)\} \cup \{(J_1,\mathbf{y}_1),\dots,(J_q,\mathbf{y}_q)\}$ is a δ-fine partition anchored in X, the previous inequalities and the countable subadditivity of $V_{\mathcal{HK}}F$ yield

$$\sum_{i=1}^{p} |F(I_i)| < V_{\mathcal{HK}}F(X) - \sum_{i=1}^{q} |F(J_i)| + \frac{\varepsilon}{2} \leqslant V_{\mathcal{HK}}F\Bigl(X \cap \bigcup_{i=1}^{p} I_i\Bigr) + \varepsilon.$$

The proof is complete. □

Let $X = \prod\limits_{i=1}^{m} X_i$ and let $k \in \{1,\dots,m\}$. Following [6], we set $\Phi_{X,k}(T) := \prod\limits_{i=1}^{m} Y_i$, where $Y_k = T$ and $Y_i = X_i$ for each $i \in \{1,\dots,m\} \setminus \{k\}$.

Lemma 3.2. *Let $F\colon \mathcal{I}_m([\mathbf{a},\mathbf{b}]) \to \mathbb{R}$ be an additive interval function such that $V_{\mathcal{HK}}F \ll \mu_m$. If $k \in \{1,\dots,m\}$, $I \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$ and $\mathbf{x} \in I$, then for each $\varepsilon > 0$ there exists $\eta_{k,I}(\mathbf{x}) > 0$ such that*

$$|F(\Phi_{I,k}([u_k,v_k]))| < \varepsilon$$

whenever $0 < v_k - u_k < \eta_{k,I}(\mathbf{x})$.

P r o o f. We observe that $\mu_m(\Phi_{I,k}(\{x_k\}))=0$, so the assumption $V_{\mathcal{HK}}F\ll\mu_m$ implies that $V_{\mathcal{HK}}F(\Phi_{I,k}(\{x_k\}))=0$. Hence for each $\varepsilon>0$ there exists a gauge δ_k on $\Phi_{I,k}(\{x_k\})$ such that

$$V(F,\Phi_{I,k}(\{x_k\}),\delta_k)<\varepsilon.$$

Define a gauge δ on I by setting

$$\delta(\mathbf{z})=\begin{cases}\delta_k(\mathbf{z}) & \text{if } \mathbf{z}\in\Phi_{I,k}(\{x_k\}),\\ \operatorname{dist}(\{\mathbf{z}\},\Phi_{I,k}(\{x_k\})) & \text{if } \mathbf{z}\in I\setminus\Phi_{I,k}(\{x_k\}).\end{cases}$$

According to Cousin's lemma (cf. [6, Lemma 3.1]), δ-fine partitions of I exist. Hence, by our choice of δ, we may fix a δ_k-fine partition P_k anchored in $\Phi_{I,k}(\{x_k\})$ such that

$$\Phi_{I,k}(\{x_k\})\subseteq\bigcup_{(J,\xi)\in P_k}J\subseteq I.$$

Letting $\eta_{k,I}(\mathbf{x}):=\min\Big\{(v_k-u_k)\colon\Big(\prod_{i=1}^m[u_i,v_i],\boldsymbol{\xi}\Big)\in P_k\Big\}$, it is now easy to check that the conclusion of the lemma holds. □

Following [6, p. 688], an additive interval function F is continuous on $\mathcal{I}_m([\mathbf{a},\mathbf{b}])$ if for each $\varepsilon>0$ there exists $\eta>0$ such that $|F(I)-F(J)|<\varepsilon$ whenever $I,J\in\mathcal{I}_m([\mathbf{a},\mathbf{b}])$ satisfy $\mu_m((I\setminus J)\cup(J\setminus I))<\eta$. In view of Lemma 3.2, the proof of the next theorem is considerably simpler than that of [6, Lemma 5.17 and Theorem 5.18].

Theorem 3.3 [6, Theorem 5.18]. *Let $F\colon\mathcal{I}_m([\mathbf{a},\mathbf{b}])\to\mathbb{R}$ be an additive interval function. If $V_{\mathcal{HK}}F\ll\mu_m$, then F is continuous on $\mathcal{I}_m([\mathbf{a},\mathbf{b}])$.*

P r o o f. Let $G(\mathbf{x})=F([\mathbf{a},\mathbf{x}])$ $(\mathbf{x}\in[\mathbf{a},\mathbf{b}])$. According to Lemma 3.2 and due to the compactness of $[\mathbf{a},\mathbf{b}]$, G is uniformly continuous on $[\mathbf{a},\mathbf{b}]$. Therefore, by the additivity of F, we conclude that F is continuous on $\mathcal{I}_m([\mathbf{a},\mathbf{b}])$. □

In order to prove the next theorem, we need the following definitions.

Definition 3.4 [6, Definition 5.1]. Let $m=1$ and let $0<v-u<b-a<\infty$. An interval-point pair $([u,v],x)$ is said to be

(a) -1-special if $u=x\in(a,b)$;
(b) 0-special if any one of the following conditions is satisfied:
  (i) $x\in(u,v)$,
  (ii) $u=x=a$,
  (iii) $v=x=b$;
(c) 1-special if $v=x\in(a,b)$.

Definition 3.5 [6, Definition 5.3]. Let $\prod\limits_{i=1}^{m} I_i \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$. An interval-point pair $\Big(\prod\limits_{i=1}^{m} I_i, \mathbf{x}\Big)$ is said to be $\boldsymbol{\lambda}$-special if (I_i, x_i) is λ_i-special for $i = 1,\dots,m$.

The following important approximation theorem is a consequence of the above results.

Theorem 3.6. *Let $F\colon \mathcal{I}_m([\mathbf{a},\mathbf{b}]) \to \mathbb{R}$ be an additive interval function such that $V_{\mathcal{HK}}F \ll \mu_m$. If $X \subseteq [\mathbf{a},\mathbf{b}]$ and $\varepsilon > 0$, then there exist a F_σ-set Y and an upper semicontinuous gauge Δ on Y such that $X \subseteq Y \subseteq [\mathbf{a},\mathbf{b}]$ and*

$$V(F, Y, \Delta) \leqslant V_{\mathcal{HK}}F(X) + \varepsilon.$$

If, in addition, X is closed, then Y can be taken to be X.

P r o o f. We may assume that X is uncountable and $V_{\mathcal{HK}}F(X)$ is finite. For each $\varepsilon > 0$ we select a gauge δ_0 on X corresponding to $\varepsilon/3^{m+1}$ in Lemma 3.1. For $k \in \mathbb{N}$, we let $X_k = \delta_0^{-1}([1/k,\infty))$, $Y_0 = \emptyset$ and $Y_k = \overline{X_k}$, where $\overline{W}$ denotes the closure of $W \subseteq [\mathbf{a},\mathbf{b}]$. Set $Y = \bigcup\limits_{n=1}^{\infty} Y_k$ so that Y is a F_σ-set containing X. Next, we define an upper semicontinuous gauge Δ on Y by letting

$$\Delta(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in Y_1, \\ \min\{1/k, \operatorname{dist}(\{\mathbf{x}\}, Y_{k-1})\} & \text{if } \mathbf{x} \in Y_k \setminus Y_{k-1} \text{ for some integer } k \geqslant 2 \end{cases}$$

and consider any Δ-fine partition P anchored in Y. As X is uncountable and $V_{\mathcal{HK}}F(X)$ is finite, we may suppose that $P_{\boldsymbol{\lambda}} := \{([\mathbf{u},\mathbf{v}],\mathbf{x}) \in P\colon ([\mathbf{u},\mathbf{v}],\mathbf{x}) \text{ is } \boldsymbol{\lambda}\text{-special}\}$ is non-empty for each $\boldsymbol{\lambda} \in \{-1,0,1\}^m$. Using Theorem 3.3, the condition $V_{\mathcal{HK}}F \ll \mu_m$ and the finiteness of $V_{\mathcal{HK}}F(X)$, there exists $\eta > 0$ such that the following conditions hold for each $\boldsymbol{\lambda} \in \{-1,0,1\}^m$:

(A) $[\mathbf{u}+\eta\boldsymbol{\lambda}, \mathbf{v}+\eta\boldsymbol{\lambda}] := \prod\limits_{i=1}^{m} [u_i+\eta\lambda_i, v_i+\eta\lambda_i] \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$ whenever $([\mathbf{u},\mathbf{v}],\mathbf{x}) \in P_{\boldsymbol{\lambda}}$,

(B) $|F([\mathbf{u}+\eta\boldsymbol{\lambda}, \mathbf{v}+\eta\boldsymbol{\lambda}]) - F([\mathbf{u},\mathbf{v}])| < \varepsilon/(4(3^m)\#P)$ whenever $([\mathbf{u},\mathbf{v}],\mathbf{x}) \in P_{\boldsymbol{\lambda}}$,

(C) $X_k \cap (\mathbf{u}+\eta\boldsymbol{\lambda}, \mathbf{v}+\eta\boldsymbol{\lambda})$ is non-empty whenever $([\mathbf{u},\mathbf{v}],\mathbf{x}) \in P_{\boldsymbol{\lambda}}$ and $\mathbf{x} \in Y_k \setminus Y_{k-1}$ for some $k \in \mathbb{N}$,

(D) $V_{\mathcal{HK}}F(X \cap W_{\boldsymbol{\lambda}}) < \varepsilon/4(3^m)$, where $W_{\boldsymbol{\lambda}} := \bigcup\limits_{([\mathbf{u},\mathbf{v}],\mathbf{x}) \in Q_{\boldsymbol{\lambda}}} [\mathbf{u}+\eta\boldsymbol{\lambda}, \mathbf{v}+\eta\boldsymbol{\lambda}]$ and $Q_{\boldsymbol{\lambda}} := \{([\mathbf{u},\mathbf{v}],\mathbf{x}) \in P_{\boldsymbol{\lambda}}\colon \mu_m([\mathbf{u},\mathbf{v}] \cap Y) = 0\}$.

Since $P = \bigcup\limits_{\boldsymbol{\lambda} \in \{-1,0,1\}^m} P_{\boldsymbol{\lambda}}$ and $\{P_{\boldsymbol{\lambda}}\colon \boldsymbol{\lambda} \in \{-1,0,1\}^m\}$ is pairwise disjoint, we infer from (A), the triangle inequality and (B) that

$$(1) \qquad \sum_{([\mathbf{u},\mathbf{v}],\mathbf{x}) \in P} |F([\mathbf{u},\mathbf{v}])| < \frac{\varepsilon}{4} + \sum_{\boldsymbol{\lambda} \in \{-1,0,1\}^m} \sum_{([\mathbf{u},\mathbf{v}],\mathbf{x}) \in P_{\boldsymbol{\lambda}}} |F([\mathbf{u}+\eta\lambda, \mathbf{v}+\eta\boldsymbol{\lambda}])|.$$

To this end, let $\boldsymbol{\lambda} \in \{-1,0,1\}^m$ and $R_{\boldsymbol{\lambda}} \in \{Q_{\boldsymbol{\lambda}}, P_{\boldsymbol{\lambda}} \setminus Q_{\boldsymbol{\lambda}}\}$. If $([\mathbf{u},\mathbf{v}],\mathbf{x}) \in R_{\boldsymbol{\lambda}}$, then it follows from (C) that there exists $\mathbf{z} \in X \cap (\mathbf{u}+\eta\boldsymbol{\lambda}, \mathbf{v}+\eta\boldsymbol{\lambda})$ such that $\{([\mathbf{u}+\eta\boldsymbol{\lambda}, \mathbf{v}+\eta\boldsymbol{\lambda}],\mathbf{z})\colon ([\mathbf{u},\mathbf{v}],\mathbf{x}) \in R_{\boldsymbol{\lambda}}\}$ is a Δ-fine, and hence δ_0-fine, partition anchored in X. Therefore our choice of δ_0 implies that

(2)

$$\sum_{([\mathbf{u},\mathbf{v}],\mathbf{x})\in R_{\boldsymbol{\lambda}}} |F([\mathbf{u}+\eta\boldsymbol{\lambda}, \mathbf{v}+\eta\boldsymbol{\lambda}])| < V_{\mathcal{HK}}F\Bigg(X \cap \bigcup_{([\mathbf{u},\mathbf{v}],\mathbf{x})\in R_{\boldsymbol{\lambda}}} [\mathbf{u}+\eta\boldsymbol{\lambda}, \mathbf{v}+\eta\boldsymbol{\lambda}]\Bigg) + \frac{\varepsilon}{4(3^m)}.$$

Combining (1), (2) and (D), we get the desired result. $\square$

Following [12, p. 20], we say that an outer measure μ is said to be $F_{\sigma\delta}$ regular if for every set X there is a $F_{\sigma\delta}$ set Y containing X with $\mu(X) = \mu(Y)$. The next theorem is an easy consequence of Theorem 3.6.

Theorem 3.7. *Let F be given as in Theorem 3.6. Then $V_{\mathcal{HK}}F$ is $F_{\sigma\delta}$ regular.*

The next theorem, which is the main result of this section, generalizes [6, Lemma 6.2] and [9, Theorem 3.6].

Theorem 3.8. *Let $F\colon \mathcal{I}_m([\mathbf{a},\mathbf{b}]) \to \mathbb{R}$ be an additive interval function such that $V_{\mathcal{HK}}F \ll \mu_m$. If $X \subseteq [\mathbf{a},\mathbf{b}]$ is closed and $V_{\mathcal{HK}}F(X)$ is finite, then for each $\varepsilon > 0$ there exists an upper semicontinuous gauge δ on X such that*

$$\sum_{i=1}^{p} |F(I_i)| < V_{\mathcal{HK}}F\Bigg(X \cap \bigcup_{i=1}^{p} I_i\Bigg) + \varepsilon$$

for each δ-fine partition $\{(I_1,\mathbf{x}_1),\ldots,(I_p,\mathbf{x}_p)\}$ anchored in X.

P r o o f. In view of Theorem 3.6, the proof is similar to that of Lemma 3.1. $\square$

4. A measure-theoretic characterization of the Henstock-Kurzweil integral

In this section we give a shorter proof of the measure-theoretic characterization of the Henstock-Kurzweil integral established in [6]; see Theorem 4.5.

Definition 4.1. A function $f\colon [\mathbf{a},\mathbf{b}] \to \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* on $[\mathbf{a},\mathbf{b}]$ if there exists $A \in \mathbb{R}$ with the following property: given $\varepsilon > 0$ there exists a gauge δ on $[\mathbf{a},\mathbf{b}]$ such that

$$\Bigg|\sum_{(I,\mathbf{x})\in P} f(\mathbf{x})\mu_m(I) - A\Bigg| < \varepsilon$$

for each δ-fine partition P of $[\mathbf{a},\mathbf{b}]$. Here A is called the Henstock-Kurzweil integral of f over $[\mathbf{a},\mathbf{b}]$, and we write A as $(HK)\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$.

For further properties of the Henstock-Kurzweil integral, consult, for instance, [8] and references therein.

The next theorem, whose original proof is long and involved (cf. [6, Section 5]), is now an easy consequence of [2, Theorem 1] and Theorem 3.6.

Theorem 4.2 [6, Theorem 4.1]**.** *Let $F\colon \mathcal{I}_m([\mathbf{a},\mathbf{b}]) \to \mathbb{R}$ be an additive interval function such that $V_{\mathcal{HK}}F \ll \mu_m$. Then there exists a sequence $\{X_k\}_{k=1}^{\infty}$ of closed sets such that $[\mathbf{a},\mathbf{b}] = \bigcup\limits_{k=1}^{\infty} X_k$ and $V_{\mathcal{HK}}F(X_k)$ is finite for each $k \in \mathbb{N}$.*

Let F be given as in Theorem 4.2. According to [3, 3.10 Theorem], F is derivable μ_m-almost everywhere on $[\mathbf{a},\mathbf{b}]$, that is, $F'(\mathbf{x})$ exists for μ_m-almost all $\mathbf{x} \in [\mathbf{a},\mathbf{b}]$. Recall that $F'(\mathbf{x})$ exists if for each $\varepsilon > 0$ and $0 < \alpha \leqslant 1$ there exists $\delta(\mathbf{x}) > 0$ such that

$$|F'(\mathbf{x})\mu_m(I) - F(I)| < \varepsilon\mu_m(I)$$

whenever $\mathbf{x} \in I \subset B(\mathbf{x},\delta(\mathbf{x}))$, $I \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$ and $\operatorname{reg}(I) \geqslant \alpha$. The next lemma is a modest extension of [5, Corollary 2].

Lemma 4.3. *Let $F\colon \mathcal{I}_m([\mathbf{a},\mathbf{b}]) \to \mathbb{R}$ be an additive interval function and suppose that F is derivable at each point of a non-empty closed set $X \subset (\mathbf{a},\mathbf{b})$. Then given $\varepsilon > 0$ there exists an upper semicontinuous gauge Δ on X such that*

$$|F'(\mathbf{x})\mu_m(I) - F(I)| < \varepsilon(l(I))^m \tag{3}$$

whenever $\mathbf{x} \in X$, $I \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$ and $I \subset B(\mathbf{x},\Delta(\mathbf{x})) \cap [\mathbf{a},\mathbf{b}]$.

P r o o f. According to our assumptions and [5, Corollary 2], for each $\varepsilon > 0$ there exists a gauge δ_0 on X such that

$$|F'(\mathbf{z})\mu_m(J) - F(J)| < \frac{\varepsilon(l(J))^m}{3} \tag{4}$$

provided that $\mathbf{z} \in X$, $J \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$ and $J \subset B(\mathbf{z},\delta_0(\mathbf{z}))$.

We may further assume that $\delta_0(\mathbf{z}) < \operatorname{dist}(\{\mathbf{z}\},[\mathbf{a},\mathbf{b}] \setminus (\mathbf{a},\mathbf{b}))$ for each $\mathbf{z} \in X$. Let $X_k := \overline{\delta_0^{-1}([1/k,\infty))}$ $(k \in \mathbb{N})$ and define an upper semicontinuous gauge Δ on X by setting

$$\Delta(\mathbf{x}) = \begin{cases} \frac12 & \text{if } \mathbf{x} \in X_1,\\ \min\{\frac12 k^{-1}, \frac12 \operatorname{dist}(\{\mathbf{x}\},X_{k-1})\} & \text{if } \mathbf{x} \in X_k \setminus X_{k-1} \text{ for some integer } k \geqslant 2.\end{cases}$$

Now we prove (3). Select an $\mathbf{x} \in X$ and choose $I \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$ and $I \subset B(\mathbf{x}, \Delta(\mathbf{x}))$. Next, we pick a $\mathbf{y} \in \delta_0^{-1}([1/q(\mathbf{x}), \infty))$ such that $\|\mathbf{x}-\mathbf{y}\| < \min\{\delta_0(\mathbf{x}), \Delta(\mathbf{x})\}$, where $q(\mathbf{x}) := \min\{k \in \mathbb{N} \colon \mathbf{x} \in X_k\}$. Then it follows from the triangle inequality that $I \subset B(\mathbf{y}, 2\Delta(\mathbf{x})) \subset B(\mathbf{y}, \delta_0(\mathbf{y}))$; in particular, (4) implies that

$$|F'(\mathbf{y})\mu_m(I) - F(I)| < \frac{\varepsilon(l(I))^m}{3}.$$

It remains to prove that $|F'(\mathbf{x}) - F'(\mathbf{y})| < \frac{2}{3}\varepsilon$. According to our choice of $\mathbf{y}$ and Δ, we get $\|\mathbf{x}-\mathbf{y}\| < \min\{\delta_0(\mathbf{x}), \delta_0(\mathbf{y})\}$. Hence we may select a sufficiently small cube $K \subset B(\mathbf{x}, \delta_0(\mathbf{x})) \cap B(\mathbf{y}, \delta_0(\mathbf{y}))$ so that $\mu_m(K) = (l(K))^m$. By (4) again, we conclude that $|F'(\mathbf{x}) - F'(\mathbf{y})| < \frac{2}{3}\varepsilon$. The proof is complete. □

The following simple modification of [9, Theorem 3.11] is sufficient for the purpose of this paper. Moreover, the proof is considerably simpler and shorter than that of [9, Theorem 3.11].

Theorem 4.4. *Let $F\colon \mathcal{I}_m([\mathbf{a},\mathbf{b}]) \to \mathbb{R}$ be an additive interval function such that $V_{\mathcal{HK}}F \ll \mu_m$. If F is derivable at each point of a non-empty closed set $X \subseteq (\mathbf{a},\mathbf{b})$, $F'|_X$ is bounded and $V_{\mathcal{HK}}F(X)$ is finite, then for each $\varepsilon > 0$ there exists an upper semicontinuous gauge δ on X such that*

$$\sum_{(I,\mathbf{x})\in P} |F'(\mathbf{x})\mu_m(I) - F(I)| < \varepsilon$$

for each δ-fine partition P anchored in X.

Proof. For each $\varepsilon > 0$ we let δ_1 and δ_2 be upper semicontinuous gauges on X corresponding to $\varepsilon_0 := 2^{-(m+1)}\varepsilon(4 + \mu_m([\mathbf{a},\mathbf{b}]) + \sup\limits_{\mathbf{x}\in X} |F'(\mathbf{x})|)^{-1}$ in Theorem 3.8 and Lemma 4.3, respectively. Then our assumptions imply that there exists $N \in \mathbb{N}$ such that $X_N := \delta_1^{-1}([1/N, \infty)) \cap \delta_2^{-1}([1/N, \infty))$ is compact, non-empty, $\mu_m(X \setminus X_N) < \varepsilon_0$ and $V_{\mathcal{HK}}F(X \setminus X_N) < \varepsilon_0$. Next, we fix an open set $G \supset X$ such that $\mu_m(G \setminus X_N) < \varepsilon_0$ and define an upper semicontinuous gauge δ on X by letting

$$\delta(\mathbf{x}) = \begin{cases} 1/N & \text{if } \mathbf{x} \in X_N, \\ \min\{\delta_1(\mathbf{x}), \operatorname{dist}\big(\{\mathbf{x}\}, X_N \cup ([\mathbf{a},\mathbf{b}] \setminus G)\big)\} & \text{if } \mathbf{x} \in X \setminus X_N. \end{cases}$$

Let $f(\mathbf{x}) := F'(\mathbf{x})$ $(\mathbf{x} \in X)$ and consider any δ-fine partition P anchored in X. For any given $(I, \mathbf{x}) \in P$, we follow the proof of Ward [13, 3. Lemma] to fix a $\frac{1}{2}$-regular net $\mathcal{N}(I)$ of I. Further, for each $J \in \mathcal{N}(I)$ satisfying $J \cap X_N \neq \emptyset$, we fix a

$\mathbf{x}_J \in J \cap X_N$. Then, by the triangle inequality, we get

$$
\begin{aligned}
&\sum_{(I,\mathbf{x})\in P} |f(\mathbf{x})\mu_m(I) - F(I)| \\
&\quad \leqslant \Bigg\{ \sum_{\substack{(I,\mathbf{x})\in P\\ \mathbf{x}\in X\setminus X_N}} |f(\mathbf{x})|\mu_m(I) + \sum_{\substack{(I,\mathbf{x})\in P\\ \mathbf{x}\in X\setminus X_N}} |F(I)| + \sum_{\substack{(I,\mathbf{x})\in P\\ \mathbf{x}\in X_N}} \sum_{\substack{J\in\mathcal{N}(I)\\ J\cap X_N=\emptyset}} |f(\mathbf{x})|\mu_m(J) \Bigg\} \\
&\qquad + \Bigg\{ \sum_{\substack{(I,\mathbf{x})\in P\\ \mathbf{x}\in X_N}} \sum_{\substack{J\in\mathcal{N}(I)\\ J\cap X_N\neq\emptyset}} |f(\mathbf{x}_J)\mu_m(J) - F(J)| + \sum_{\substack{(I,\mathbf{x})\in P\\ \mathbf{x}\in X_N}} \sum_{\substack{J\in\mathcal{N}(I)\\ J\cap X_N\neq\emptyset}} |f(\mathbf{x}_J) - f(\mathbf{x})|\mu_m(J) \Bigg\} \\
&\qquad + \sum_{\substack{(I,\mathbf{x})\in P\\ \mathbf{x}\in X_N}} \Bigg| \sum_{\substack{J\in\mathcal{N}(I)\\ J\cap X_N=\emptyset}} F(J) \Bigg| \\
&\qquad := S_1 + S_2 + S_3.
\end{aligned}
$$

Our choice of δ implies that $S_1 < \varepsilon_0 \sup\limits_{\mathbf{x}\in X} |f(\mathbf{x})| + 2\varepsilon_0 + \varepsilon_0 \sup\limits_{\mathbf{x}\in X} |f(\mathbf{x})|$. Also, $S_2 \leqslant (2^{m-1}\varepsilon_0 + 2\varepsilon_0)\mu_m([\mathbf{a},\mathbf{b}])$, since $\delta(\mathbf{x}) = 1/N \leqslant \delta_2(\mathbf{x})$ for all $\mathbf{x} \in X_N$ and $(l(J))^m \leqslant 2^{m-1}\mu_m(J)$ for each $\frac{1}{2}$-regular interval $J \subseteq [\mathbf{a},\mathbf{b}]$.

It remains to prove that $S_3 < 3m\varepsilon_0$. Choose a $([\mathbf{u},\mathbf{v}],\mathbf{x}) \in P$ such that $\mathbf{x} \in X_N$. Let $\mathcal{A}_0([\mathbf{u},\mathbf{v}]) = [\mathbf{u},\mathbf{v}]$ and for $r = 1,\dots,m$, we let

$$
\begin{aligned}
\mathcal{A}_r([\mathbf{u},\mathbf{v}]) = \Bigg\{ \bigcap_{k=1}^{r} \Phi_{[\mathbf{u},\mathbf{v}],k}([s_k,t_k])\colon\ [\mathbf{s},\mathbf{t}] \in \mathcal{N}([\mathbf{u},\mathbf{v}]) \\
\text{and } X_N \cap \bigcap_{k=1}^{r} \Phi_{[\mathbf{u},\mathbf{v}],k}([s_k,t_k]) \neq \emptyset \Bigg\},
\end{aligned}
$$

where $\Phi_{[\mathbf{u},\mathbf{v}],k}([s_k,t_k]) = \prod\limits_{i=1}^{m} W_i$, $W_k = [s_k,t_k]$ and $W_i = [u_i,v_i]$ whenever $i \in \{1,\dots,m\} \setminus \{k\}$. For any given $r \in \{1,\dots,m\}$ and $K \in \mathcal{A}_{r-1}([\mathbf{u},\mathbf{v}])$, we let $\mathcal{B}(K)$ be the smallest collection of subinterval of K such that $\overline{K \setminus \bigcup\limits_{J\in\mathcal{A}_r([\mathbf{u},\mathbf{v}])} J} = \bigcup\limits_{U\in\mathcal{B}(K)} U$. Consequently,

$$
\begin{aligned}
S_3 &= \sum_{\substack{(I,\mathbf{x})\in P\\ \mathbf{x}\in X_N}} \Bigg| \sum_{r=1}^{m} \sum_{K\in\mathcal{A}_{r-1}(I)} \sum_{J\in\mathcal{B}(K)} F(J) \Bigg| \\
&\leqslant \sum_{r=1}^{m} \sum_{\substack{(I,\mathbf{x})\in P\\ \mathbf{x}\in X_N}} \sum_{K\in\mathcal{A}_{r-1}(I)} \sum_{J\in\mathcal{B}(K)} |F(J)| < 3m\varepsilon_0.
\end{aligned}
$$

Combining the above inequalities completes the proof. $\square$

The collection of all functions that are Henstock-Kurzweil integrable on $[\mathbf{a},\mathbf{b}]$ will be denoted by $\mathrm{HK}[\mathbf{a},\mathbf{b}]$. We are now ready to state and prove the main result of this section.

Theorem 4.5 [6, Theorem 4.3] **.** *Let $F\colon \mathcal{I}_m([\mathbf{a},\mathbf{b}]) \to \mathbb{R}$ be an additive interval function. The following conditions are equivalent.*

(i) *There exists $f \in \mathrm{HK}[\mathbf{a},\mathbf{b}]$ such that $F(I) = (\mathrm{HK})\int_I f(\mathbf{x})\,\mathrm{d}\mathbf{x}$ for each $I \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$.*

(ii) *$V_{\mathcal{HK}}F \ll \mu_m$.*

P r o o f. The implication (i) $\Longrightarrow$ (ii) is just [6, Theorem 3.8]. For the converse, suppose that (ii) holds. In view of Theorem 4.2, [3, 3.10 Theorem] and Lemma 4.3, there exists a sequence $\{X_k\}_{k=1}^{\infty}$ of pairwise disjoint of closed subsets of $[\mathbf{a},\mathbf{b}]$ so that $\mu_m\Bigl([\mathbf{a},\mathbf{b}] \setminus \bigcup\limits_{k=1}^{\infty} X_k\Bigr) = 0$, F is derivable at each point of the set $\bigcup\limits_{k=1}^{\infty} X_k$ and the sum $\sup\limits_{\mathbf{x}\in X_k} |F'(\mathbf{x})| + V_{\mathcal{HK}}F(X_k)$ is finite for each $k \in \mathbb{N}$. Define a function f on $[\mathbf{a},\mathbf{b}]$ by letting $f(\mathbf{x}) = F'(\mathbf{x})$ if $\mathbf{x} \in \bigcup\limits_{k=1}^{\infty} X_k$ and 0 otherwise. As a consequence of condition (ii) and Theorem 4.4, for each $\varepsilon > 0$ there exists a gauge δ on $[\mathbf{a},\mathbf{b}]$ such that

$$\sum_{(I,\mathbf{x})\in P} |f(\mathbf{x})\mu_m(I) - F(I)| < \varepsilon$$

for each δ-fine partition P in $[\mathbf{a},\mathbf{b}]$. Hence the additivity of F implies that (i) holds. The proof is complete. $\square$

The proof of Theorem 4.5 relies on Theorem 4.2, for which no satisfactory analogue in infinite dimension is known. Thus it is an open question whether Theorem 4.5 holds for infinite-dimensional generalized Riemann integrals defined in [4].

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