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THE SHARPNESS OF CONVERGENCE RESULTS FOR q-BERNSTEIN POLYNOMIALS IN THE CASE $q > 1$

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Abstract. Due to the fact that in the case $q > 1$ the q-Bernstein polynomials are no longer positive linear operators on $C[0,1]$, the study of their convergence properties turns out to be essentially more difficult than that for $q < 1$. In this paper, new saturation theorems related to the convergence of q-Bernstein polynomials in the case $q > 1$ are proved.

Keywords: q-integers, q-binomial coefficients, q-Bernstein polynomials, uniform convergence, analytic function; Cauchy estimates

MSC 2010: 41A10, 30E10

1. Introduction

Given $q > 0, n \in \mathbb{Z}_+$, the *$q$-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \ldots + q^{k-1} (n \in \mathbb{N}), \quad [0]_q := 0;$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q [2]_q \ldots [n]_q (n \in \mathbb{N}), \quad [0]_q! := 1.$$

For integers k, n with $0 \leqslant k \leqslant n$, the *$q$-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Definition 1.1. Let $f\colon [0,1] \to \mathbb{C}$. The *$q$-Bernstein polynomials* of f are given by

$$B_{n,q}(f;z) := \sum_{k=0}^{n} f\Big(\frac{[k]_q}{[n]_q}\Big) p_{nk}(q;z), \quad n \in \mathbb{N},$$

where

$$p_{nk}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1} \left(1 - q^j z\right), \quad k = 0, 1, \dots n. \tag{1.1}$$

When $q = 1$, we recover the classical Bernstein polynomials:

$$B_{n,1}(f;z) = B_n(f;z) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} z^k (1-z)^{n-k}.$$

A comprehensive review of the results on q-Bernstein polynomials along with extensive bibliography on the subject is given in [15].

We would like to mention that a great number of various extensions and generalizations of the Bernstein polynomials have been introduced due to their high degree of importance, see e.g. [1], [3], [6], [9], [12], [13], [17]. A two-parametric generalization of q-Bernstein polynomials, as well as two versions of the Bernstein-Durrmeyer operator related to those polynomials have been considered in [23], and [5], [7] and [8].

The q-Bernstein polynomials inherit some properties of the classical Bernstein polynomials. Among those properties we mention the end-point interpolation property, the shape-preserving properties in the case $0 < q < 1$, and the representation via divided differences. Like the classical Bernstein polynomials, the q-Bernstein polynomials reproduce linear functions, and they are degree-reducing on the set of polynomials.

On the other hand, the examination of the convergence properties of the q-Bernstein polynomials reveals that these properties are essentially *different* from those of the classical ones. What is more, the cases $0 < q < 1$ and $q > 1$ are not similar to each other. This difference is caused by the fact that, for $0 < q < 1$, $B_{n,q}$ are *positive* linear operators on $C[0,1]$ while for $q > 1$, the positivity fails. The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult than that for $0 < q < 1$. As a result, the convergence of q-Bernstein polynomials in the case $0 < q < 1$ has been investigated in detail, including a Korovkin type theorem, the properties of the limit operator, the rate of convergence, and the saturation phenomenon (cf. [10], [15], [16], [18] - [21]). In contrast, there are only two papers, namely [14] and [22], dealing systematically with the convergence in the case $q > 1$. The results of [14] show, however, that for $q > 1$ the approximation with q-Bernstein polynomials may be *faster* than with the classical ones. The following respective result can be cited.

Theorem A ([14], Theorem 6)**.** *Let $q > 1$ be fixed. If a function f is analytic in $\{z\colon |z| < R\}$, $R > q$, then*

$$|B_{n,q}(f; z) - f(z)| \leqslant \frac{C_{f,q}}{[n]_q} \quad \text{for } |z| \leqslant 1,\ n \in \mathbb{N}. \tag{1.2}$$

That is, for functions analytic in $\{z\colon |z| < R\}$, $R > q$, the rate of approximation by the q-Bernstein polynomials ($q > 1$) is of order q^{-n} versus $1/n$ for the classical Bernstein polynomials.

In this paper we discuss the sharpness of estimate (1.2). It is shown that

$$\max_{|z|\leqslant 1} |B_{n,q}(f; z) - f(z)| = o(1/[n]_q), \quad n \to \infty,$$

if and only if f is a linear function. Furthermore, estimate (1.2) is not possible for a function analytic in $\{z\colon |z| < R\}$, $1 < R < q$, which does not admit an analytic continuation to $\{z\colon |z| < R_1\}$ with $R_1 > R$. These results show that, in general, the estimate of Theorem A cannot be improved. We would like to mention that the problems of the impossibility of certain estimates for Bernstein-type operators have been considered in [2].

For functions analytic in a disc $\{z\colon |z| < R\}$, $R > 1$, the following statement has been known.

Theorem B ([14], Theorem 7)**.** *If a function f is analytic in $\{z\colon |z| > R\}$, $R > 1$, then the following estimate holds:*

$$|B_{n,q}(f; z) - f(z)| \leqslant \frac{C_f}{n} \quad \text{for } |z| \leqslant 1,\ n \in \mathbb{N}. \tag{1.3}$$

In this paper we generalize estimate (1.3) showing that for functions analytic in $\{z\colon |z| < R\}$, $R > 1$, the rate of approximation by the q-Bernstein polynomials in the closed unit disc is also exponential, but slower than that in (1.2).

The saturation phenomenon plays an important role in the approximation by positive linear operators, in particular by Bernstein polynomials. For the q-Bernstein polynomials it has been studied by H. Wang ([21]) in the case $0 < q < 1$; by H. Wang and X. Z. Wu ([22]) in the case $q > 1$. It is worth pointing out that [22] contains a new saturation result for the classical Bernstein polynomials in a complex domain. The present paper exhibits new results related to the saturation of convergence for the q-Bernstein polynomials in the case $q > 1$.

2. Statement of results

We start with the following statement showing the sharpness of estimate (1.2).

Theorem 2.1. *If*

$$\max_{|z|\leqslant 1} |B_{n,q}(f;z) - f(z)| = o\Big(\frac{1}{[n]_q}\Big), \quad n\to\infty, \tag{2.1}$$

then f is a linear function.

The next theorem shows that estimate (1.3) is not possible for functions analytic in a narrower disc than the one stated by Theorem A.

Theorem 2.2. *Let $f(z)$ be analytic in a disc $\{z\colon\ |z| < R\}$, $1 < R < q$. Then, for any $\varepsilon > 0$ and $1 \leqslant |z| < R-\varepsilon$, we have*

$$|B_{n,q}(f;z) - f(z)| \leqslant C\frac{|z|^n}{(R-\varepsilon)^n}, \tag{2.2}$$

where $C = C_{f,q,R,\varepsilon}$.

If f does not admit an analytic continuation into a disc $\{z\colon\ |z| < R_1\}$, $R_1 > R$, then (2.2) *ceases to be true if one replaces $R-\varepsilon$ by $R+\varepsilon$.*

Corollary 2.3. *If $f(z)$ is analytic in $\{z\colon\ |z| < R\}$, $1 < R < q$, and does not admit an analytic continuation into a disc $\{z\colon\ |z| < R_1\}$, $R_1 > R$, then estimate* (1.2) *is not possible.*

3. Proofs of the theorems

It has been proved in Lemma 2 of [14] that for $n \geqslant m$, one has

$$B_{n,q}(t^m;z) = \alpha_{m,m}^{(n)} z^m + \alpha_{m-1,m}^{(n)} z^{m-1} + \ldots + \alpha_{1,m}^{(n)} z, \tag{3.1}$$

where $\alpha_{i,m}^{(n)} \geqslant 0$, $i = 1,\ldots,m$ with $\sum\limits_{i=1}^{m} \alpha_{i,m}^{(n)} = 1$ and

$$\alpha_{0,0}^{(n)} = \alpha_{1,1}^{(n)} = 1,\ \alpha_{m,m}^{(n)} = \Big(1-\frac{1}{[n]_q}\Big)\Big(1-\frac{[2]_q}{[n]_q}\Big)\ldots\Big(1-\frac{[m-1]_q}{[n]_q}\Big) =: \lambda_{mn}. \tag{3.2}$$

Remark 3.1. We notice that λ_{mn} are the eigenvalues of the q-Bernstein operator (cf. [14], Lemma 5). For $q = 1$, the numbers λ_{mn}, $0 \leqslant m \leqslant n$, are the eigenvalues of the Bernstein operator, see [4]. The latter result has been extended to the case $q \neq 1$ in [14].

The q-Bernstein polynomials admit the following representation via the divided differences of f (see [14], formulae (6) and (7)):

$$B_n(f,q;x) = \sum_{k=0}^{n} \lambda_{kn} f\left[0; \frac{1}{[n]_q}; \ldots; \frac{[k]_q}{[n]_q}\right] x^k, \tag{3.3}$$

where λ_{kn} are given by (3.2).

Note that

$$\lambda_{0n} = \lambda_{1n} = 1, \tag{3.4}$$

and it is clear from (3.2) that

$$0 \leqslant \lambda_{kn} \leqslant 1, \quad k = 0, 1, \ldots, n. \tag{3.5}$$

When $q = 1$, we get the known representation for the classical Bernstein polynomials, see [11], Chapter 4, §4.1.

If is known (cf. e.g. [11], §2.7, p. 44) that the divided differences of an analytic function f can be expressed by

$$f[x_0; x_1; \ldots; x_k] = \frac{1}{2\pi i} \oint_{\mathscr{L}} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - x_0)\ldots(\zeta - x_k)}, \tag{3.6}$$

where $\mathscr{L}$ is a contour encircling $x_0, \ldots, x_k$ and f is assumed to be analytic on and within $\mathscr{L}$.

We start with a saturation-type result for analytic functions with non-negative coefficients.

Lemma 3.1. *Let $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ with $c_k \geqslant 0$ and $\sum\limits_{k=0}^{\infty} c_k < \infty$. Then for any $m \geqslant 2$, $n \geqslant m$, one has*

$$\max_{x\in[0,1]} |B_{n,q}(f;x) - f(x)| \geqslant \frac{c_m}{me[n]_q}. \tag{3.7}$$

Proof of Lemma 3.1. Due to the fact that $B_{n,q}$ is a bounded linear operator on $C[0,1]$, we have

$$B_{n,q}(f;x) = \sum_{k=0}^{\infty} c_k B_{n,q}(t^k;x).$$

Since $B_{n,q}$ leaves invariant linear functions, it follows that

$$B_{n,q}(f;x) - f(x) = \sum_{k=2}^{\infty} c_k \{B_{n,q}(t^k;x) - x^k\}.$$

We fix $m \geqslant 2$. By virtue of (3.1), for $n \geqslant m$,

$$\begin{aligned} B_{n,q}(t^m;x) - x^m &= (\alpha_{m,m}^{(n)} - 1)x^m + \alpha_{m-1,m}^{(n)}x^{m-1} + \ldots + \alpha_{1,m}^{(n)}x \\ &\geqslant (\alpha_{m,m}^{(n)} - 1)x^m + (\alpha_{m-1,m}^{(n)} + \ldots + \alpha_{1,m}^{(n)})x^{m-1} \\ &= (1 - \alpha_{m,m}^{(n)})x^{m-1}(1-x). \end{aligned}$$

Therefore,

$$|B_{n,q}(f;x) - f(x)| \geqslant c_m(1 - \alpha_{m,m}^{(n)})x^{m-1}(1-x).$$

For $m \geqslant 2$, (3.2) implies that $\alpha_{m,m}^{(n)} \leqslant (1 - 1/[n]_q)$ and we have

$$\begin{aligned} \max_{x\in[0,1]} |B_{n,q}(f;x) - f(x)| &\geqslant \frac{c_m}{[n]_q} \max_{x\in[0,1]} x^{m-1}(1-x) \\ &= \frac{c_m}{m[n]_q}\Big(1 - \frac{1}{m}\Big)^{m-1} \geqslant \frac{c_m}{me} \cdot \frac{1}{[n]_q}. \end{aligned}$$

$\square$

Proof of Theorem 2.1. By condition (2.1), a function f is analytic in $\{z\colon\ |z| < 1\}$, that is $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$. By virtue of (3.3), the Cauchy estimates together with (2.1) imply that for each fixed $k \geqslant 2$, we have

$$\lambda_{kn} f\left[0; \frac{1}{[n]_q}; \ldots; \frac{[k]_q}{[n]_q}\right] - \frac{f^{(k)}(0)}{k!} = o\Big(\frac{1}{[n]_q}\Big), \quad n \to \infty$$

or

$$\lambda_{kn}\left\{f\left[0; \frac{1}{[n]_q}; \ldots; \frac{[k]_q}{[n]_q}\right] - \frac{f^{(k)}(0)}{k!}\right\} + (\lambda_{kn} - 1)\frac{f^{(k)}(0)}{k!} = o\Big(\frac{1}{[n]_q}\Big), \quad n \to \infty.$$

Using the Cauchy Theorem and (3.6), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \frac{\lambda_{kn}[n]_q}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \left\{\frac{1}{\zeta(\zeta - 1/[n]_q)\ldots(\zeta - [k]_q/[n]_q)} - 1/\zeta^{k+1}\right\} f(\zeta)\,\mathrm{d}\zeta \\ &\quad = \lim_{n\to\infty} \frac{\lambda_{kn}[n]_q}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \left\{\frac{\zeta^k - (\zeta - 1/[n]_q)\ldots(\zeta - [k]_q/[n]_q)}{\zeta^{k+1}(\zeta - 1/[n]_q)\ldots(\zeta - [k]_q/[n]_q)}\right\} f(\zeta)\,\mathrm{d}\zeta \\ &\quad = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\zeta^{k-1}(1 + [2]_q + \ldots + [k]_q)}{\zeta^{2k+1}} f(\zeta)\,\mathrm{d}\zeta \\ &\quad = \frac{1 + [2]_q + \ldots + [k]_q}{(k+1)!} f^{(k+1)}(0) = (1 + [2]_q + \ldots + [k]_q)c_{k+1}. \end{aligned}$$

On the other hand,

$$\lim_{n\to\infty}[n]_q\,(1-\lambda_{kn}) = 1+[2]_q+\ldots+[k-1]_q.$$

As a result, we obtain

$$c_{k+1}\,(1+[2]_q+\ldots+[k]_q) = c_k(1+[2]_q+\ldots+[k-1]_q), \quad k\geqslant 2,$$

whence

$$c_{k+1} = \frac{c_2}{1+[2]_q+\ldots+[k]_q}, \quad k\geqslant 2.$$

Therefore, if (2.1) is true, then $\arg c_k = \arg c_2$ for all $k\geqslant 2$. Without loss of generality, we may assume that all $c_k\geqslant 0$. Applying Lemma 3.1, we derive the required statement. □

Now we prove a technical lemma needed for the proof of Theorem 2.2.

Lemma 3.2. *Let $q>1$. Then*

$$\lim_{n\to\infty}\prod_{k=1}^{n-1}\Bigl(1-\frac{[k]_q}{a[n-k]_q}\Bigr) = \prod_{k=1}^{\infty}\Bigl(1-\frac{1}{aq^k}\Bigr).$$

P r o o f of Lemma 3.2. Consider

$$\left|\ln\prod_{k=1}^{n-1}\Bigl(1-\frac{[k]_q}{a[n-k]_q}\Bigr)\right| = \sum_{k=1}^{n-1}\left|\ln\Bigl(1-\frac{[n-k]_q}{a[n]_q}\Bigr)\right| =: \sum_{k=0}^{\infty} d_{kn},$$

where

$$d_{kn} = \begin{cases} \left|\ln\Bigl(1-\dfrac{[n-k]_q}{a[n]_q}\Bigr)\right| & \text{if } 1\leqslant k\leqslant n-1,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly

$$0\leqslant d_{kn}\leqslant \left|\ln\Bigl(1-\frac{1}{aq^k}\Bigr)\right| \quad \text{for all } k,n\in\mathbb{N}$$

and

$$\lim_{n\to\infty} d_{kn} = \left|\ln\Bigl(1-\frac{1}{aq^k}\Bigr)\right| \quad \text{for each } k\in\mathbb{N}.$$

Since $\sum\limits_{k=1}^{\infty}|\ln(1-1/aq^k)|<\infty$, we may apply the Lebesgue Dominated Convergence Theorem to obtain

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} d_{kn} = \sum_{k=1}^{\infty}(\lim_{n\to\infty} d_{kn}) = \sum_{k=1}^{\infty}\left|\ln\Bigl(1-\frac{1}{aq^k}\Bigr)\right|.$$

Equivalently,

$$\lim_{n\to\infty}\left|\ln\prod_{k=1}^{n-1}\Big(1-\frac{[n-k]_q}{a[n]_q}\Big)\right| = \left|\ln\prod_{k=1}^{\infty}\Big(1-\frac{1}{aq^k}\Big)\right|.$$

The statement now follows. $\square$

In the sequel, we use either the letter C or the abbreviation Const to denote positive constants whose value may not be specified explicitly. We notice that constants denoted by the same symbol need not be equal to each other.

P r o o f of Theorem 2.2. Let

$$f(z)=\sum_{k=0}^{\infty} c_k z^k, \quad B_{n,q}(f;z)=\sum_{k=0}^{\infty} c_{kn} z^k.$$

Then, for $1\leqslant |z| < R-\varepsilon$,

$$|B_{n,q}(f;z)-f(z)| \leqslant |z|^n \sum_{k=0}^{n} |c_{kn}-c_k| + \sum_{k=n+1}^{\infty} |c_k|\cdot|z|^k =: \sigma_1+\sigma_2. \tag{3.8}$$

To estimate σ_1, we write using (3.6) and (3.3):

$$c_k - c_{kn} = \frac{1}{2\pi\mathrm{i}}\oint_{|\zeta|=R-\varepsilon_1}\Big\{\frac{1}{\zeta^{k+1}} - \frac{\lambda_{kn}}{\zeta(\zeta-1/[n]_q)\dots(\zeta-[k]_q/[n]_q)}\Big\} f(\zeta)\,\mathrm{d}\zeta,$$

where $1 < R-\varepsilon < R-\varepsilon_1 < q$. Hence

$$\begin{aligned}|c_k-c_{kn}| \leqslant{}& \frac{1}{2\pi}\left|\oint_{|\zeta|=R-\varepsilon_1}\frac{1-\lambda_{kn}}{\zeta(\zeta-1/[n]_q)\dots(\zeta-[k]_q/[n]_q)} f(\zeta)\,\mathrm{d}\zeta\right|\\ &+\frac{1}{2\pi}\left|\oint_{|\zeta|=R-\varepsilon_1}\Big\{\frac{1}{\zeta^{k+1}}-\frac{1}{\zeta(\zeta-1/[n]_q)\dots(\zeta-[k]_q/[n]_q)}\Big\} f(\zeta)\,\mathrm{d}\zeta\right|\\ &=: I_{kn}+J_{kn}.\end{aligned}$$

We set

$$M_f(r) := \max_{|z|\leqslant r} |f(z)|.$$

With this notation, we write

$$I_{kn} \leqslant \frac{(1-\lambda_{kn})M_f(R-\varepsilon_1)}{(R-\varepsilon_1)^k\,(1-((R-\varepsilon_1)[n]_q)^{-1})\dots(1-[k]_q((R-\varepsilon_1)[n]_q)^{-1})}.$$

Consider the product in the denominator. Evidently, for n large enough, we have

$$\Big(1-\frac{1}{(R-\varepsilon_1)[n]_q}\Big)\dots\Big(1-\frac{[k]_q}{(R-\varepsilon_1)[n]_q}\Big) \geqslant \Big(1-\frac{1}{(R-\varepsilon_1)[n]_q}\Big)\dots\Big(1-\frac{[n-1]_q}{(R-\varepsilon_1)[n]_q}\Big).$$

By virtue of Lemma 3.2,

$$\lim_{n\to\infty}\prod_{j=1}^{n-1}\Big(1-\frac{[j]_q}{(R-\varepsilon_1)[n]_q}\Big)=\prod_{j=1}^{\infty}\Big(1-\frac{1}{(R-\varepsilon_1)q^j}\Big)\neq 0.$$

Therefore,

$$\Big(1-\frac{1}{(R-\varepsilon_1)[n]_q}\Big)\dots\Big(1-\frac{[k]_q}{(R-\varepsilon_1)[n]_q}\Big)\geqslant C>0. \tag{3.9}$$

As a result, we obtain

$$\begin{aligned} I_{kn} &\leqslant \frac{\mathrm{Const}(1-\lambda_k^{(n)})}{(R-\varepsilon_1)^k}\leqslant\frac{\mathrm{Const}(1+[2]_q+\ldots+[k-1]_q)}{[n]_q(R-\varepsilon_1)^k}\\ &=\frac{\mathrm{Const}(q^k-k)}{(R-\varepsilon_1)^k\cdot[n]_q\cdot(q-1)}\leqslant\frac{\mathrm{Const}\,q^k}{(R-\varepsilon_1)^kq^n}=\frac{C}{(R-\varepsilon_1)}\Big(\frac{R-\varepsilon_1}{q}\Big)^{n-k}. \end{aligned}$$

Hence

$$|z|^n\sum_{k=0}^{n}I_{kn}\leqslant \mathrm{Const}\,\frac{|z|^n}{(R-\varepsilon_1)^n}\sum_{k=0}^{\infty}\Big(\frac{R-\varepsilon_1}{q}\Big)^k=:\mathrm{Const}\,\frac{|z|^n}{(R-\varepsilon_1)^n}. \tag{3.10}$$

Now, we estimate the second term

$$\begin{aligned} J_{kn}&=\frac{1}{2\pi}\oint_{|\zeta|=R-\varepsilon_1}\left|\frac{(\zeta-1/[n]_q)\dots(\zeta-1/[n]_q)-\zeta^k}{\zeta^{k+1}(\zeta-1/[n]_q)\dots(\zeta-1/[n]_q)}f(\zeta)\,\mathrm{d}\zeta\right|\\ &\leqslant\frac{M_f(R-\varepsilon_1)}{2\pi(R-\varepsilon_1)^{k+1}}\oint_{|\zeta|=R-\varepsilon_1}\frac{|(1-1/\zeta[n]_q)\dots(1-[k]_q/\zeta[n]_q)-1|}{|(1-1/\zeta[n]_q)\dots(1-[k]_q/[n]_q)|}|\mathrm{d}\zeta|\\ &\leqslant\frac{\mathrm{Const}}{(R-\varepsilon)^k}\max_{|\zeta|=R-\varepsilon_1}\frac{|(1-1/\zeta[n]_q)\dots(1-[k]_q/\zeta[n]_q)-1|}{|(1-1/\zeta[n]_q)\dots(1-[k]_q/[n]_q)|}\\ &\leqslant\frac{\mathrm{Const}}{(R-\varepsilon)^k}\max_{|\zeta|=R-\varepsilon_1}\Big|\Big(1-\frac{1}{\zeta[n]_q}\Big)\dots\Big(1-\frac{[k]_q}{\zeta[n]_q}\Big)-1\Big|, \end{aligned}$$

by virtue of (3.9).

Setting

$$u(\zeta):=\ln\prod_{j=1}^{k}\Big(1-\frac{[j]_q}{\zeta[n]_q}\Big),$$

we obtain

$$
\begin{aligned}
|u(\zeta)| &\leqslant \sum_{j=1}^{k}\Big|\ln\Big(1-\frac{[j]_q}{\zeta[n]_q}\Big)\Big| \leqslant \sum_{j=1}^{k}\frac{[j]_q/(R-\varepsilon_1)[n]_q}{1-[j]_q/(R-\varepsilon_1)[n]_q}\\
&= \sum_{j=1}^{k}\frac{[j]_q}{(R-\varepsilon_1)[n]_q-[j]_q} = \frac{1}{[n]_q}\sum_{j=1}^{k}\frac{[j]_q}{(R-\varepsilon_1)-[j]_q/[n]_q}\\
&\leqslant \frac{1}{[n]_q}\sum_{j=1}^{k}\frac{[j]_q}{(R-\varepsilon_1)-1} =: \mathrm{Const}\,\frac{[k+1]_q-(k+1)}{[n]_q(q-1)}\\
&\leqslant \mathrm{Const}\, q^{k-n}.
\end{aligned}
$$

Using the inequality $|\mathrm{e}^u-1|\leqslant |u|\mathrm{e}^{|u|}$, we derive

$$
\Big|\Big(1-\frac{1}{\zeta[n]_q}\Big)\dots\Big(1-\frac{[k]_q}{\zeta[n]_q}\Big)-1\Big| \leqslant C_q\cdot q^{k-n}.
$$

Therefore,

$$
J_{kn} \leqslant \frac{\mathrm{Const}}{(R-\varepsilon_1)^n}\Big(\frac{q}{(R-\varepsilon_1)}\Big)^{k-n}
$$

and

$$
\sigma_1 \leqslant \frac{\mathrm{Const}}{(R-\varepsilon_1)^n}. \tag{3.11}
$$

On the other hand, the Cauchy estimates imply that

$$
|c_k| \leqslant \frac{M_f(R-\varepsilon_1)}{(R-\varepsilon_1)^k},
$$

whence

$$
\sigma_2 \leqslant M_f(R-\varepsilon_1)\Big(\frac{|z|}{R-\varepsilon_1}\Big)^{n+1}\sum_{k=0}^{\infty}\Big(\frac{R-\varepsilon}{R-\varepsilon_1}\Big)^k \leqslant \mathrm{Const}\Big(\frac{|z|}{R-\varepsilon_1}\Big)^n. \tag{3.12}
$$

Substituting estimates (3.11) and (3.12) with $\varepsilon_1=\varepsilon/2$ into (3.8), we derive (2.2).

Now, we assume that for $1\leqslant |z| < R-\varepsilon$, the following estimate holds for some $\varepsilon>0$:

$$
|B_{n,q}(f;z)-f(z)| \leqslant \frac{|z|^n}{(R+\varepsilon)^n}, \quad n\in\mathbb{N}.
$$

With the Cauchy Theorem this implies that the Taylor coefficients of f satisfy

$$
|c_{n+1}| \leqslant \frac{\mathrm{Const}}{(R+\varepsilon)^n}, \quad n\in\mathbb{N}.
$$

Consequently, $f(z)$ admits an analytic continuation into $\{z\colon\ |z|<R+\varepsilon\}$, contrary to the condition of the theorem. $\square$

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