Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 1101–1106

Persistent URL: http://dml.cz/dmlcz/140443

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Czechoslovak Mathematical Journal, 58 (133) (2008), 1101–1106

ON THE DISTANCE FUNCTION OF A CONNECTED GRAPH

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(Received January 4, 2007)

Abstract. An axiomatic characterization of the distance function of a connected graph is given in this note. The triangle inequality is not contained in this characterization.

Keywords: connected graph, distance function

MSC 2010: 05C12

1. Introduction

By a graph is meant here a finite or infinite undirected graph with no loops or multiple edges. Obviously, if G is a connected graph, then the distance function of G is a mapping of $V(G) \times V(G)$ into the set of all non-negative integers.

In the present note, we assume that V is a nonempty set and f is a mapping of $V \times V$ into the set of all non-negative real numbers. By the well-known definition, f is a metric on V if it satisfies the following axioms:

(A) $f(u,v) = f(v,u)$ for all $u, v \in V$;

(B) $f(u,v) = 0$ if and only if $u = v$, for all $u, v \in V$;

(T) [the triangle inequality] $f(u,v) \leqslant f(u,w) + f(w,v)$ for all $u, v, w \in V$.

It is easy to see that if f is the distance function of a connected graph G with $V(G) = V$, then f is a metric on V. D. C. Kay and G. Chartrand [1] found a necessary and sufficient condition for a metric f on V to be the distance function of a connected graph G with $V(G) = V$. In their theorem, the following axioms were used:

(C) $f(u,v)$ is an integer for all $u, v \in V$;

(D) if $f(u,v) > 1$, then there exists $w \in V$ such that $u \neq w \neq v$ and $f(u,v) = f(u,w) + f(w,v)$ for all $u, v \in V(G)$.

Theorem A (D. C. Kay and G. Chartrand [1]). *Let f be a metric on V. Then f is the distance function of a connected graph G with $V(G) = V$ if and only if f satisfies the axioms* (C) *and* (D).

Remark 1. Kay and Chartrand have considered only finite graphs in [1] but their theorem also holds if V is infinite.

If f satisfies axioms (A) and (B), then we denote by G_f the graph such that $V(G_f) = V$ and

$$\text{the vertices } u \text{ and } v \text{ are adjacent in } G_f \text{ if and only if } f(u,v) = 1$$

for all $u, v \in V$.

Proposition 1. *Let f satisfy the axioms* (A) *and* (B). *If f is the distance function of a connected graph G with $V(G) = G$, then $G = G_f$.*

The proof is obvious.

Proposition 2. *If f satisfies the axioms* (A), (B), (C), *and* (D), *then there exists an $u - v$ walk of length $f(u,v)$ in G_f for all $u, v \in V(G)$.*

The proof is easy.

Proposition 3. *Let f satisfy the axioms* (A), (B), (C), *and* (D). *Then G_f is connected. If d denotes the distance function of G_f, then*

$$d(u,v) \leqslant f(u,v) \text{ for all } u, v \in V.$$

P r o o f. The proposition immediately follows from Proposition 2. □

The next theorem is a reformulation of a theorem of D. C. Kay and G. Chartrand [1]. We will present an explicit proof of it.

Theorem B. *Let f satisfy the axioms* (A), (B), (C), (D), *and let d denote the distance function of G_f. Then $f = d$ if and only if f satisfies the axiom* (T).

P r o o f. As we know, if $f = d$, then f satisfies axiom (T).

Conversely, assume that f satisfies (T). Consider arbitrary $u, v \in V$. Put $n = d(u,v)$. We will prove that $f(u,v) = n$. We proceed by induction on n. The case when $n \leqslant 1$ is obvious. Let $n \geqslant 2$. Obviously, there exists $w \in V$ such that $d(u,w) = 1$ and $d(w,v) = n - 1$. By the induction hypothesis, $f(w,v) = n - 1$ and, moreover, $f(u,w) = 1$. Axiom (T) implies that $f(u,v) \leqslant n$. As follows from Proposition 3, $f(u,v) \geqslant n$. Thus $f(u,v) = n$, which completes the proof. □

Obviously, the following axiom is stronger than the axiom (D):

(E) if $f(u,v) > 1$, then there exists $w \in V$ such that $f(u,v) = f(u,w) + f(w,v)$ and $(f(u,w) = 1$ or $f(w,v) = 1)$ for all $u, v \in V(G)$.

In the present note, we will prove that if the axiom (D) is replaced by the axiom (E) in Theorem B, then the triangle inequality can be replaced by an axiom containing equalities only. The new axiom is inspired by some results of the present author.

2. New axiom

We will be interested in the following axiom (S):

(S) if $f(u,v) = 1 = f(x,y)$ and $f(u,x) = 1 + f(v,x)$, then $f(u,x) = 1 + f(u,y)$ or $f(u,y) = 1 + f(v,y)$ or $f(v,y) = 1 + f(v,x)$, for all $u, v, x, y \in V$.

Lemma 1. *Let f satisfy the axioms* (A), (C), *and* (T). *Then f satisfies the axiom* (S).

Proof. Suppose, to the contrary, that there exist $u, v, x, y \in V$ such that $f(u,v) = 1 = f(x,y)$, $f(u,x) = 1 + f(v,x)$, $f(u,x) \neq 1 + f(u,y)$, $f(u,y) \neq 1 + f(v,y)$, and $f(v,y) \neq 1 + f(v,x)$. Combining the axioms (A) and (T), we get $f(u,x) < 1 + f(u,y)$, $f(u,y) < 1 + f(v,y)$, and $f(v,y) < 1 + f(v,x)$. By virtue of axiom (C), $f(u,x) \leqslant f(v,x)$. This is a contradiction with the fact that $f(u,x) = 1 + f(v,x)$. □

Remark 2. Axiom (S) can be compared to Axiom VIII of the characterization of the set of all shortest paths in a connected graph in [3], to Axiom VIII of the characterization of the interval function of a finite connected graph in [4], to Axiom Y of the characterization of the interval function of an infinite connected graph in [6], and to Axiom G of the characterization of the set all steps in a finite connected graph in [5]. (By the interval function of a graph is meant the interval function in the sense of H. M. Mulder [2]. By a step in a connected graph G is meant an ordered triple (u,v,w) of vertices u, v, w in G such that $d(u,v) = 1$ and $d(v,w) = d(u,v) - 1$, where d denotes the distance function of G).

Lemma 2. *Let f satisfy the axioms* (A), *and* (S), *and let $u, v, x, y \in V$. Assume that $f(u,v) = 1 = f(x,y)$, $f(u,y) \geqslant 3$, $f(u,x) = f(u,y) - 1$, and $f(v,x) = f(u,x) - 1$. Then $f(v,y) = f(u,y) - 1$.*

Proof. Obviously, $f(u,x) \neq 1 + f(u,y)$. Thus, by the axiom (S),

$$f(u,y) = 1 + f(v,y) \text{ or } f(v,y) = 1 + f(v,x).$$

Since $f(v,x) = f(u,x) - 1 = f(u,y) - 2$, we get $f(v,y) = f(u,y) - 1$. □

By an f-path we mean a sequence $(u_0, \ldots, u_n)$, where $u_0, \ldots, u_n \in V$ and $n \geqslant 0$, such that

$$f(u_i, u_j) = j - i \quad \text{for all integers } i \text{ and } j \text{ such that } 0 \leqslant i \leqslant j \leqslant n.$$

Assume that f satisfies the axioms (A) and (B). Obviously, if $(u_0, \ldots, u_n)$ is an f-path, then $(u_n, \ldots, u_0)$ is an f-path as well. Moreover, it is clear that every f-path is a path in G_f.

Lemma 3. *Let f satisfy the axioms* (A) *and* (S). *Consider $u_0, \ldots, u_{n-1}, u_n \in V$ such that $n \geqslant 3$, $(u_0, \ldots, u_{n-1})$ is an f-path, $f(u_0, u_n) = n$, and $f(u_{n-1}, u_n) = 1$. Then $(u_0, \ldots, u_{n-1}, u_n)$ is an f-path as well.*

Proof. As follows from the definition of an f-path, it is sufficient to prove that

$$(1) \qquad f(u_i, u_n) = n - i \quad \text{for all integers } i \text{ such that } 0 \leqslant i \leqslant n - 2.$$

We proceed by induction on i. The case when $i = 0$ is obvious. Assume that $1 \leqslant i \leqslant n - 2$. By the induction hypothesis, $f(u_{i-1}, u_n) = n - i + 1$. Put $u = u_{i-1}$, $v = u_i$, $x = u_{n-1}$, and $y = u_n$. By Lemma 2, $f(u_i, u_n) = f(u_{i-1}, u_n) - 1 = n - i$. Thus (1) is proved. $\square$

Lemma 4. *Let f satisfy the axioms* (A), (B), (C), (E), (S), *and let $u, v \in V(G)$. Then there exists an $u - v$ path α in G_f such that α is an f-path.*

Proof. Consider arbitrary $u, v \in V$. Put $n = f(u, v)$. We wish to prove that there exists a $u - v$ path α in G_f such that α is an f-path. We proceed by induction on n. The case when $n = 0$ or 1 is obvious. If $n = 2$, then the result immediately follows from Axiom (E). Let $n \geqslant 3$. By Axiom (E), there exists $w \in V$ such that $f(u, v) = f(u, w) + f(w, v)$ and $(f(u, w) = 1$ or $f(w, v) = 1)$.

We first assume that $f(w, v) = 1$. Then $f(u, w) = n - 1$. The induction hypothesis implies that there exist $u_0, \ldots, u_{n-1} \in V$ such that $u_0 = u$, $u_{n-1} = w$, and $(u_0, \ldots, u_{n-1})$ is an f-path. Since $f(u_0, v) = n$ and $f(u_{n-1}, v) = 1$, Lemma 3 implies that $(u_0, \ldots, u_{n-1}, v)$ is an f-path as well.

We now assume that $f(u, w) = 1$. By Axiom (A), $f(v, w) = n - 1$ and $f(w, u) = 1$. Then we can prove analogously that there exist $v_0, \ldots, v_{n-1} \in V$ such that $v_0 = v$, $v_{n-1} = w$ and $(v_0, \ldots, v_{n-1}, u)$ is an f-path. Hence $(u, v_{n-1}, \ldots, v_0)$ is an f-path, which completes the proof. $\square$

3. Main result

The next theorem gives a new characterization of the distance function of a connected graph.

Theorem 1. *Let f satisfy the axioms* (A), (B), *and* (C), *and let d denote the distance function of G_f. Then $f = d$ if and only if f satisfies the axioms* (E) *and* (S).

P r o o f. If $f = d$, then f satisfies the axioms (E) and (T). By Lemma 1, f satisfies the axiom (S).

Conversely, let f satisfy the axioms (E) and (S). Suppose, to the contrary, that $f \neq d$. Then there exist $r, s \in V$ such that $f(r,s) \neq d(r,s)$ and

$$f(r_0, s_0) = d(r_0, s_0) \text{ for all } r_0, s_0 \in V \text{ such that } d(r_0, s_0) < d(r,s). \tag{2}$$

Put $j = d(r,s)$ and $k = f(r,s)$. Then $j \neq k$ and therefore, by Proposition 3, $j < k$. As follows from Lemma 4, there exist $t_0, t_1, \ldots, t_k \in V$ such that $t_0 = r$, $t_k = s$ and $(t_0, t_1, \ldots, t_k)$ is an f-path. Moreover, there exist $t_{k+1}, \ldots, t_{k+j} \in V$ such that $t_{k+j} = t_0$ and $(t_{k+j}, \ldots, t_{k+1}, t_k)$ is a shortest $r - s$ path in G_f. Put $t_{k+j+1} = t_1, \ldots, t_{k+2j} = t_j$.

Assume that $(t_j, t_{j+1}, \ldots, t_{j+k})$ is an f-path. By the definition of an f-path, $f(t_j, t_{j+k}) = k$. Clearly, $t_{j+k} = t_0$ and $(t_0, t_1, \ldots, t_j)$ is also an f-path. We get $f(t_0, t_j) = j \neq k$; a contradiction. Thus $(t_j, t_{j+1}, \ldots, t_{j+k})$ is not an f-path.

Recall that $(t_0, t_1, \ldots, t_k)$ is an f-path. We see that there exists i, $0 \leqslant i < j$, such that

$$(t_i, t_{i+1}, \ldots, t_{i+k}) \text{ is an } f\text{-path}$$

and

$$\alpha =_{df} (t_{i+1}, t_{i+2}, \ldots, t_{i+k+1}) \text{ is not an } f\text{-path}.$$

As follows from the definition, $(t_{i+1}, \ldots, t_{i+k})$ is an f-path.

Put $u = t_i$, $v = t_{i+1}$, $x = t_{i+k}$, and $y = t_{i+k+1}$. Obviously, $f(u,x) = k$,

$$f(u,v) = 1 = f(x,y) \text{ and } f(u,x) = 1 + f(v,x). \tag{3}$$

Since α is not an f-path, Lemma 3 implies that

$$f(v,y) \neq 1 + f(v,x). \tag{4}$$

Note that $t_{i+k+j} = t_i$. It is clear that $\beta =_{df} (t_{i+k+j}, \ldots, t_{i+k+1}, t_{i+k})$ is an $u - x$ walk in G_f. Hence $d(u,x) \leqslant j$. If $d(u,x) < j$, then, by (2), $k = f(u,x) = d(u,x) < j$

and therefore $k < j$, which is a contradiction. Thus $d(u,x) = j$. We see that β is a shortest $u-x$ path in G_f. Since $f(u,x) = k$, β is not an f-path. Recall that $y = t_{i+k+1}$. Since $d(u,y) = j-1$, it follows from (2) that $f(u,y) = j-1$ and therefore $(t_{i+k+j}, \ldots, t_{i+k+2}, t_{i+k+1})$ is an f-path. Lemma 3 implies that

$$f(u,x) \neq 1 + f(u,y). \tag{5}$$

Combining (3), (4), and (5) with Axiom (E), we get

$$f(u,y) = 1 + f(v,y). \tag{6}$$

Since $d(u,y) = j-1$, (2) implies that $f(u,y) = j-1$. By (6), $f(v,y) = j-2$. As follows from Proposition 3, $d(v,y) \leqslant j-2$. Hence $d(v,x) \leqslant j-1$. By (2), $f(v,x) = d(v,x) \leqslant j-1$. As follows from (3), $f(v,x) = k-1$. We get $k \leqslant j$, which is a contradiction. Thus the theorem is proved. $\square$

Corollary 1. *Let f satisfy axioms the* (A), (B), *and* (C). *Then f satisfies the axioms* (D) *and* (T) *if and only if f satisfies the axioms* (E) *and* (S).

Remark 3. Let $|V| = 4$, let $V = \{r,s,t,u\}$, and let f satisfy the axioms (A) and (B). Assume that $f(r,u) = f(s,u) = f(t,u) = 1$, $f(r,s) = f(s,t) = 2$, and $f(r,t) = 4$. Clearly, G_f is a star. It is easy to see that f satisfies Axioms (D) and (S) but f is not the distance of G_f. Thus Axiom (E) cannot be replaced by Axiom (D) in Theorem 1.

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