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THE POSTAGE STAMP PROBLEM AND ARITHMETIC IN BASE r

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Abstract. Let h, k be fixed positive integers, and let A be any set of positive integers. Let $hA := \{a_1 + a_2 + \ldots + a_r : a_i \in A, r \leqslant h\}$ denote the set of all integers representable as a sum of no more than h elements of A, and let $n(h, A)$ denote the largest integer n such that $\{1, 2, \ldots, n\} \subseteq hA$. Let $n(h, k) := \max\limits_{A} : n(h, A)$, where the maximum is taken over all sets A with k elements. We determine $n(h, A)$ when the elements of A are in geometric progression. In particular, this results in the evaluation of $n(h, 2)$ and yields surprisingly sharp lower bounds for $n(h, k)$, particularly for $k = 3$.

Keywords: h-basis, extremal h-basis, geometric progression

MSC 2010: 11B13

The *Postage Stamp Problem* derives its name from the situation when we require the largest integer $n = n(h, k)$ such that all stamp values from 1 to n may be made up from a collection of k integer-valued stamp denominations with the restriction that there are no more than h stamps, repetitions being allowed. The problem of determining $n(h, k)$ is apparently due to Rohrbach [3], and has been studied often ever since. A large and extensive bibliography can be found in a paper of Alter and Barnett [1].

Let h, k be fixed positive integers, and let A be any set of positive integers. Let $hA := \{a_1 + a_2 + \ldots + a_r : a_i \in A, r \leqslant h\}$ denote the set of all integers representable as a sum of no more than h elements of A, and let $n(h, A)$ denote the largest integer n such that $\{1, 2, \ldots, n\} \subseteq hA$. Observe that in order for this to happen, it is necessary that $a_1 = 1$. Thus, $n(h, k) := \max\limits_{A} n(h, A)$, where the maximum is taken over all sets A with k elements. Any set A with k elements for which $n(h, A) = n(h, k)$ is called an *extremal h-basis* for $\{1, 2, \ldots, n(h, k)\}$, and it is natural to ask for all such extremal h-bases for a given k.

It is easy to see that $n(1,k)=k$ with unique extremal basis $\{1,2,\ldots,k\}$ and that $n(h,1)=h$ with unique extremal basis $\{1\}$. The result $n(h,2)=\lfloor\frac{1}{4}(h^2+6h+1)\rfloor$ with unique extremal basis $\{1,\frac{1}{2}(h+3)\}$ for odd h and $\{1,\frac{1}{2}(h+2)\}$ and $\{1,\frac{1}{2}(h+4)\}$ for even h has been rediscovered several times, for instance by Stöhr in [5, 6] and by Stanton, Bate and Mullin in [4]. No other closed-form formula is known for any other pair (h,k) where one of h, k is fixed.

The purpose of this note is to determine $n(h,A)$ when the elements of A are in geometric progression. In particular, this easily gives the value of $n(h,2)$. The study of this case naturally leads to the representation of positive integers in a fixed basis $r>1$. Suppose h,k,r are fixed positive integers, and let $A=\{1,r,r^2,\ldots,r^{k-1}\}$ be a k-term geometric progression. Since each positive integer n can are *uniquely* expressed in the form

$$n=d_0+d_1r+d_2r^2\ldots+d_{k-1}r^{k-1},$$

where $0\leqslant d_i\leqslant r-1$ for each i, $0\leqslant i\leqslant k-1$, it follows that

$$n\in hA \text{ if and only if } d_0+d_1+\ldots+d_{k-1}\leqslant h. \tag{1}$$

The determination of $n(h,A)$ in this case, and subsequently of $n(h,2)$, is an easy consequence of (1).

Theorem. *Let h,k,r be positive integers. Then*

$$n(h,\{1,r,r^2,\ldots,r^{k-1}\}) = \begin{cases} h & \text{if } h\leqslant r-2;\\ r^i(t+1)+(r^i-2) & \text{if } h=i(r-1)+t,\ 1\leqslant i\leqslant k-2,\\ & 0\leqslant t\leqslant r-2;\\ r^{k-1}(t+1)+(r^{k-1}-2) & \text{if } h=(k-1)(r-1)+t,\ t\geqslant 0.\end{cases}$$

Proof. We write $A=\{1,r,r^2,\ldots,r^{k-1}\}$. The case $h\leqslant r-2$ is easily dealt with. Henceforth, we assume $h\geqslant r-1$ and write $h=i(r-1)+t$ with $i\geqslant 1$ and $0\leqslant t\leqslant r-2$.

We first show that $N=r^i(t+1)+(r^i-1)=r^i(t+2)-1\notin hA$. Observe that $N<r^{i+1}$, and in base r it equals $d_i\,d_{i-1}\,\ldots\,d_0$, where $d_i=t+1$ and $d_j=r-1$ for $0\leqslant j\leqslant i-1$, since $N-r^i(t+1)=r^i-1=(r-1)(r^{i-1}+r^{i-2}+\ldots+r+1)$. By (1), $N\notin hA$ since $d_0+d_1+\ldots+d_{k-1}=i(r-1)+(t+1)=h+1$.

It remains to show that every positive integer less than or equal to $r^i(t+1)+(r^i-2)=r^i(t+2)-2$ is an element of hA. We employ the notation $(a_k,a_{k-1},\ldots,$

$a_1, a_0)_r$ to denote the number $a_k r^k + a_{k-1} r^{k-1} + \ldots + a_1 r + a_0$. Since the base r representation of N is $(t+1, r-1, r-1, \ldots, r-1)_r$ (i occurrences of $r-1$), each positive integer less than N must be in hA by (1) since *at least* one digit in base r representation of such an integer must be less than the corresponding one for N and none can be greater. This completes the proof. □

Corollary 1 is a special case of the theorem, which we single out in order to prove the result stated in Corollary 2, due to Stöhr in [5]. Our proof of the result in Corollary 2 is therefore a consequence of a more general result, whereas Stöhr proved his result directly.

Corollary 1. *For* $h \geqslant 1$,

$$n(h, \{1, r\}) = \begin{cases} h & \text{if } h \leqslant r-2; \\ r(h-r+3)-2 & \text{if } h \geqslant r-1. \end{cases}$$

Corollary 2 (Stöhr, [5])**.** *For* $h \geqslant 1$,

$$n(h, 2) = \left\lfloor \frac{h^2 + 6h + 1}{4} \right\rfloor.$$

Moreover, the only extremal basis is $\{1, \frac{1}{2}(h+3)\}$ *if* h *is odd, and* $\{1, \frac{1}{2}(h+2)\}$ *and* $\{1, \frac{1}{2}(h+4)\}$ *if* h *is even.*

Proof. From Corollary 1,

$$n(h, 2) = \max_{2 \leqslant r \leqslant h+2} r(h-r+3) - 2 = \left\lfloor \frac{(h+3)^2}{4} \right\rfloor - 2 = \left\lfloor \frac{h^2 + 6h + 1}{4} \right\rfloor.$$

Since the maximum product of two positive real numbers x and y with a fixed sum $x + y = c$ is attained at $x = y$, the maximum in the displayed equation above is achieved at $r = \frac{1}{2}(h+3)$. Thus, there is only one extremal basis if h is odd and two such bases if h is even. □

We close this paper with a remark on the lower bound on $n(h, k)$ provided by the theorem when $k \geqslant 3$. By the theorem, substituting $t = (k-1)(r-1) - h$, we get

$$n(h, k) \geqslant \max_r r^{k-1}(h - (k-1)(r-1) + 2) - 2. \tag{2}$$

If we now maximize $f(r) := r^{k-1}(h - (k-1)(r-1) + 2)$ in the interval $[2, \infty)$, a simple computation shows that it attains its maximum at $r = (h+k+1)/k$. Further computation shows that $f(h,k)$ at $r=(h+k+1)/k$ equals $(h+k+1)^k/k^k$. Note that this is the best possible when $k=2$, as seen in Corollary 2, but gives a lower bound in the general case

$$n(h,k)\geqslant\left(\frac{h+k+1}{k}\right)^k, \tag{3}$$

which is surprisingly close to the best known lower bounds for $n(h,k)$ for $k\geqslant 3$, obtained by Hofmeister [2]. For instance, for $k=3$, (3) gives the lower bound

$$n(h,3)\geqslant\tfrac{1}{27}(h+4)^3=\tfrac{1}{27}h^3+\tfrac{4}{9}h^2+\tfrac{16}{9}h+\tfrac{64}{27}$$

against the lower bound

$$n(h,3)\geqslant\tfrac{4}{81}h^3+\tfrac{2}{3}h^2+\tfrac{66}{27}h$$

obtained in [2].

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References

[1] *R. Alter and J. A. Barnett*: A postage stamp problem. Amer. Math. Monthly *87* (1980), 206–210.
[2] *G. Hofmeister*: Asymptotische Abschätzungen für dreielementige Extremalbasen in natürlichen Zahlen. J. reine angew. Math. *232* (1968), 77–101.
[3] *H. Rohrbach*: Ein Beitrag zur additiven Zahlentheorie. Math. Z. *42* (1937), 1–30.
[4] *R. G. Stanton, J. A. Bate and R. C. Mullin*: Some tables for the postage stamp problem. Congr. Numer., Proceedings of the Fourth Manitoba Conference on Numerical Mathematics, Winnipeg *12* (1974), 351–356.
[5] *A. Stöhr*: Gelöste and ungelöste Fragen über Basen der natürlichen Zahlenreihe, I. J. reine Angew. Math. *194* (1955), 40–65.
[6] *A. Stöhr*: Gelöste and ungelöste Fragen über Basen der natürlichen Zahlenreihe, II. J. reine Angew. Math. *194* (1955), 111–140.

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