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RIGID EXTENSIONS OF ℓ-GROUPS OF CONTINUOUS FUNCTIONS

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Abstract. Let $C(X,\mathbb{Z})$, $C(X,\mathbb{Q})$ and $C(X)$ denote the ℓ-groups of integer-valued, rational-valued and real-valued continuous functions on a topological space X, respectively. Characterizations are given for the extensions $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q}) \leqslant C(X)$ to be rigid, major, and dense.

Keywords: rigid extension, major extension, archimedean extension, dense extension

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1. Introduction

We begin by reminding the reader of some concepts from the theory of lattice-ordered groups (ℓ-groups). Our standard reference for the theory of ℓ-groups is [4]. For those familiar with archimedean ℓ-groups and the Yosida representation, skipping this section would be a prudent measure.

An abelian ℓ-group is an abelian group $(G,+,0)$ which is also a lattice $(G,\leqslant)$ such that for every $g,h,k \in G$, if $g \leqslant h$, then $g+k \leqslant h+k$. The positive cone of G is denoted by G^+ and it is the set of those $g \in G$ for which $g \geqslant 0$. An ℓ-subgroup of G is a subgroup that is also a sublattice. An ℓ-subgroup H is called *convex* if whenever $g \in G$ and $0 \leqslant g \leqslant h$ for some $h \in H^+$, then $g \in H$. We denote the collection of all such subgroups by $\mathfrak{C}(G)$. It is known that $\mathfrak{C}(G)$ is a lattice when it is partially ordered under inclusion. In particular, since an arbitrary intersection of convex ℓ-subgroups is again a convex ℓ-subgroup, it follows that for any $S \subseteq G$ there is a smallest convex ℓ-subgroup containing S denoted by $\langle S\rangle_G$. When $S=\{g\}$ we write $G(g)$ instead. A convex ℓ-subgroup of the form $G(g)$ is called a *principal* *convex ℓ-subgroup.* If H is an ℓ-subgroup of G and $\langle H\rangle_G = G$, then we say G is a *major extension of* H or H *majorizes* G.

If $S \subseteq G$, then the set

$$S^{\perp} = \{g \in G\colon \forall s \in S(|g| \wedge |s| = 0)\}$$

is called the *polar of* S. The polar of a set is a convex ℓ-subgroup. When $S = \{s\}$ we write $s^{\perp}$, and if $s^{\perp} = \{0\}$, then s is called a *weak order unit.* The *principal polar of* $s \in G$ is $s^{\perp\perp}$. We shall use $\mathscr{P}(G)$ to denote the collection of all polars of G. If $G = G(u)$ for some $u \in G^{+}$, then u is called a *strong order unit.* A strong order unit is necessarily a weak order unit.

The category where most of our discussion will take place is $\mathbf{W}$: the category whose objects are archimedean ℓ-groups with a designated weak order unit and whose morphisms are the ℓ-homomorphisms which preserve the designated unit. We will often refer to an object of $\mathbf{W}$ as (G, u).

Let $(G, u) \in \mathbf{W}$. By Zorn's Lemma, there exist convex ℓ-subgroups which are maximal with respect to not containing u. Such a convex ℓ-subgroup is called a *value of* u, and the set of all values of u is denoted by Y_uG. (When u is clear from the context, we will suppress the u and write YG.) We endow YG with the hull-kernel topology, which results in a compact Hausdorff space.

We let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be the two-point compactification of $\mathbb{R}$. For a topological space X, define

$$D(X) = \{f\colon X \to \overline{\mathbb{R}}\colon f \text{ is continuous and } f^{-1}(\mathbb{R}) \text{ is dense}\}.$$

$D(X)$ is a lattice under pointwise operations but is rarely a group. We abuse notation when we say a subset $H \subseteq D(X)$ is an ℓ-subgroup of $D(X)$. By this we mean that under pointwise operations H is an ℓ-group. For example, the ℓ-group of continuous real-valued functions $C(X)$ is an ℓ-subgroup of $D(X)$. For $f \in D(X)$, the *zero set* of f is $Z(f) = \{x \in X\colon f(x) = 0\}$, the *cozero set* of f is the set $\operatorname{coz}(f) = X \setminus Z(f)$, and the *reality set* of f is $f^{-1}(\mathbb{R})$.

Theorem 1.1 (The Yosida Embedding). *Let $(G, u) \in \mathbf{W}$. There is an ℓ-isomorphism of G onto an ℓ-subgroup of $D(YG)$, say $\widehat{G}$, which maps u to the constant function* 1 *and such that $\widehat{G}$ separates the points of YG.*

Henceforth we identify a $\mathbf{W}$-object with its image in $D(YG)$. If $(G, u) \in \mathbf{W}$, then we use G^{*} to denote $G(u)$. It is known that $\mathbf{W}$-objects are *saturated*, that is, if $K \subseteq YG$ is a clopen subset, then the characteristic function χ_K belongs to G. Furthermore, the collection $\{\operatorname{coz}(f)\colon f \in G\}$ forms a base for the topology of YG. We say G is a *convex ℓ-group* if whenever $f \in D(YG)$ and there are $g, h \in G$ such that $g \leqslant f \leqslant h$, then $f \in G$. (See [1] for more information on convex ℓ-groups).

Definition 1.2. Let $s \in G^+$. We call s *singular* if for every $0 \leqslant t \leqslant s$, $t \wedge (s - t) = 0$. An ℓ-group G is called *singular* if every positive element lies above a non-zero singular element. For more information on singular elements and singular ℓ-groups, we suggest the reader to refer to [3] and [7].

Let $(G, u) \in \mathbf{W}$ and let T be a subgroup of $\mathbb{R}$ containing 1. Define

$$W_T(G) = \{g \in G\colon\ g(p) \in T, \forall p \in g^{-1}(\mathbb{R})\}.$$

We are particularly interested in the cases when $T = \mathbb{Q}$ or $T = \mathbb{Z}$ (since our results could be stated in terms of whether T is a proper dense subgroup or a cyclic subgroup of $\mathbb{R}$). $W_{\mathbb{Z}}(G)$ is called the *singular part* of G and $W_{\mathbb{Q}}(G)$ is called the *rational part* of G. Observe that $(G, u) \in \mathbf{W}$ is singular if and only if $W_{\mathbb{Z}}(G) = G$ if and only if u is a singular element of G.

In this paper our aim is to consider the extensions $C(X, \mathbb{Z}) \leqslant C(X, \mathbb{Q}) \leqslant C(X)$ and determine when they are dense, essential, rigid, and major. Here $C(X)$ is the set of real-valued continuous functions whose domain is the topological space X, and $C(X, \mathbb{Q})$ and $C(X, \mathbb{Z})$ are the subsets consisting of functions in $C(X)$ which are rational-valued or integer-valued, respectively. When endowed with the pointwise operations, all three sets become archimedean ℓ-groups with the constant function $\mathbf{1}$ a weak-order unit, i.e., they are $\mathbf{W}$-objects. The convex ℓ-subgroups generated by $\mathbf{1}$ in $C(X)$, $C(X, \mathbb{Q})$ and $C(X, \mathbb{Z})$ are denoted by $C^*(X)$, $C^*(X, \mathbb{Q})$ and $C^*(X, \mathbb{Z})$, respectively. They are better known as the bounded continuous functions.

We conclude this section with some definitions from topology.

For a given $f \in C(X)$, the set $Z(f) = \{x \in X\colon\ f(x) = 0\}$ denotes the zero set of f. The complement of this set is called the cozero set of f and we use the notation $\operatorname{coz}(f)$. A set is called a zero set if it is of the form $Z(f)$ for some $f \in C(X)$. A cozero set is similarly defined, and we let $\operatorname{Coz}(X)$ denote the collection of cozero sets of X. For a given $(G, u) \in \mathbf{W}$, the collection $\{\operatorname{coz}(f)\colon\ f \in G\}$ is a subcollection of the base $\{\operatorname{coz}(f)\colon\ f \in C(YG)\}$. There are examples of when these bases coincide, e.g. if (G, u) is a convex ℓ-group.

We will assume that all spaces are Tychonoff, that is, completely regular and Hausdorff. For a Tychonoff space X, the zero sets form a base for the closed sets of X (and hence the cozero sets form a base for the open sets).

As usual the Stone-Čech compactification of X is denoted by βX, the realcompactification of X is denoted by υX, and the zero-dimensional (or Banaschewski) compactification is denoted by $\beta_0 X$. If X has a base of clopen sets, then X is said to be *zero-dimensional*, and if βX is zero-dimensional, then X is called *strongly zero-dimensional*. It is known for the $\mathbf{W}$-objects defined above that $YC(X) \cong \beta X$ and $YC(X, \mathbb{Z}) \cong YC(X, \mathbb{Q}) \cong \beta_0 X$. More information on βX and $\beta_0 X$ can be found in [9]. We will require several uses of the following characterization of strongly zero-dimensional spaces.

Lemma 1.3. *Suppose X is a Tychonoff space. X is a strongly zero-dimensional space if and only if every cozero set is a countable union of clopen subsets of X.*

Recall that a *π-base for X* is a collection of open sets $\mathscr{B}$ such that for any open subset O of X there is a $B \in \mathscr{B}$ such that $B \subseteq O$. If there is a π-base consisting of clopen sets, then X is said to have a *clopen π-base*. These and other similar definitions can be found in [5].

2. Archimedean, essential and dense extensions

An ℓ-group G is called *archimedean* if $g, h \in G^+$ and $g \leqslant nh$ for every natural number n implies $g = 0$. Let $G \leqslant H$. We say H is an *archimedean extension* (or *a-extension*) of G if for each $0 < h \in H$ there exists a $0 < g \in G$ and a natural number n for which $g \leqslant nh$ and $h \leqslant ng$. In this case we can also say that G is an a-subgroup of H. An a-extension of an ℓ-group G shares many interesting properties with G. For example, an a-extension of an archimedean ℓ-group is archimedean. It is also known that $G \leqslant H$ being an archimedean extension is equivalent to saying that the contraction map of $\mathfrak{C}(H)$ onto $\mathfrak{C}(G)$ is an order isomorphism. Furthermore, H is an a-extension of G if and only if for each positive element $h \in H$ there is a positive element $g \in G$ which generates the same principal convex ℓ-subgroup in H as h does. For $g \in G^+$, $G(g) \leqslant G$ is an a-extension if and only if g is a strong order unit. For these and other similar results we direct the interested reader to [4].

We begin this section by giving an overview of results describing when the extensions $C(X, \mathbb{Z}) \leqslant C(X, \mathbb{Q}) \leqslant C(X)$ are a-extensions.

Theorem 2.1. *For a space X, the following statements are equivalent:*
(1) *$C(X, \mathbb{Z}) \leqslant C(X, \mathbb{Q})$ is an a-extension.*
(2) *For every $f \in C(X, \mathbb{Q})$, the image of f is finite.*
(3) *The boolean algebra of clopen sets of X is finite.*

Proof. That (3) implies (2) and (2) implies (1) are obvious. So we need only to show (1) implies (3). Suppose the boolean algebra of clopen sets of X is infinite, say $\{K_n\}$ is an infinite sequence of pairwise disjoint nonempty clopen subsets of X. Define $f\colon X \to \mathbb{Q}$ by

$$f(x) = \begin{cases} \dfrac{1}{n} & \text{if } x \in K_n, \\ 0 & \text{otherwise,} \end{cases}$$

then $f \in C(X,\mathbb{Q})^+$. If $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q})$ is an a-extension, then there exist an $N \in \mathbb{N}$ and a $g \in C(X,\mathbb{Z})^+$ such that $g \leqslant Nf$ and $f \leqslant Ng$. But whenever $x \in K_{N+1}$ we have $1 \leqslant g(x) \leqslant Nf(x) = N/(N+1) < 1$, a contradiction. Hence $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q})$ is not an a-extension. □

Proposition 2.2. *For a space X, $C(X,\mathbb{Z}) \leqslant C(X)$ is an a-extension if and only if X is finite.*

Proof. The reverse direction is clear. Suppose $C(X,\mathbb{Z}) \leqslant C(X)$ is an a-extension and assume, to get a contradiction, that X is infinite. Since $C(X,\mathbb{Z}) \leqslant C(X)$ is an a-extension, we also obtain that $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q})$ is an a-extension. It follows from the previous theorem that the boolean algebra of clopen sets of X is finite, say $\{K_1,\ldots,K_m\}$. Let $p \in X$ be a nonisolated point, then there is an infinite clopen set $K_i, 1 \leqslant i \leqslant m$, containing p. Moreover, because there are only finitely many clopen sets in X, we can choose i so that $K_j \subseteq K_i$ implies $K_j = K_i$. Without loss of generality, $i = 1$. Next let $f \in C(X)^+$ be such that f is nonconstant on K_1, then $f(K_1)$ is a connected subset of $\mathbb{R}$. Select $r \in \operatorname{int}_X f(K_1)$, and let $g = |f - r| \in C(X)$. We can find a countable sequence $\{r_n\} \subset K_1$ such that $0 < g(r_n) < 1/n$ for each $n \in \mathbb{N}$. By assumption, there exists $0 < h \in C(X,\mathbb{Z})$ and $N \in \mathbb{N}$ with $g \leqslant Nh$ and $h \leqslant Ng$. However, $1 \leqslant h(r_{N+1}) \leqslant Ng(r_{N+1}) < N/(N+1) < 1$, a contradiction. Therefore X is finite. □

Theorem 2.3 [6]**.** *Let (G,u) be a* **W***-object. If (G,u) is a convex ℓ-group, then $W_{\mathbb{Q}}(G) \leqslant G$ is an a-extension if and only if YG is zero-dimensional.*

Corollary 2.4. *For a space X, $C(X,\mathbb{Q}) \leqslant C(X)$ is an a-extension if and only if X is strongly zero-dimensional.*

We move on to essential and dense extensions. When $G \leqslant H$, H is said to be an *essential extension of* G or G is an *essential subgroup of* H if for any $K \in \mathfrak{C}(H)$, $K \cap G = 0$ implies that $K = 0$. (Essential extensions are also known as large extensions.) This notion is equivalent to the statement that for each $h \in H^+$ there exist a $g \in G^+$ and a natural number n for which $g \leqslant nh$. It follows that an a-extension is necessarily essential. If we can always choose $n = 1$ in the above definition, we obtain the notion of a dense extension. Specifically, H is said to be a *dense extension of* G, or G is a *dense ℓ-subgroup of* H, if for every $0 < h \in H$ there is a $0 < g \in G$ for which $g < h$. As just noted, it follows that a dense subgroup is an essential subgroup. Also, both notions are transitive, that is, if $G \leqslant H \leqslant K$ and $G \leqslant H$ and $H \leqslant K$ are essential (resp. dense) extensions, then $G \leqslant K$ is an essential (resp. dense) extension.

Example 2.5. Let H be a countable direct product of copies of $\mathbb{R}$ and let G be the direct sum. It is straightforward to check that G is a dense subgroup of H which is not an a-extension. Also, $\mathbb{Z} \leqslant \mathbb{R}$ is an a-extension (hence essential) and yet it is not dense, while $\mathbb{Q} \leqslant \mathbb{R}$ is both a dense subgroup and an a-subgroup. These examples show that the notions of dense, essential, and a-extensions are all different in the class of archimedean ℓ-groups.

Example 2.6. If $u \in G$ is a weak-order unit, then $G(u) \leqslant G$ is a dense embedding. The converse is also true. As noted before, it is an a-extension if and only if u is a strong order unit.

Example 2.7. Suppose $G \leqslant H$ and that G is a $\mathbb{Q}$-vector space. G is a dense subgroup of H if and only if it is an essential subgroup of H.

One should note that $W_{\mathbb{Q}}(C(X)) \leqslant C(X, \mathbb{Q})$, and in Theorem 5.1 of [6] it is shown that equality holds precisely when X is pseudocompact. At this point, given a **W**-object G, we will characterize when the extensions $W_{\mathbb{Z}}(G) \leqslant W_{\mathbb{Q}}(G) \leqslant G$ are dense or essential extensions. It is easy to see that $W_{\mathbb{Z}}(G) \leqslant W_{\mathbb{Q}}(G)$ is always an essential extension. The next proposition is also straightforward.

Proposition 2.8. *Let (G, u) be a* **W**-*object. Then the following statements are equivalent:*
(1) $W_{\mathbb{Z}}(G) \leqslant W_{\mathbb{Q}}(G)$ *is a dense extension.*
(2) $W_{\mathbb{Z}}(G) \leqslant G$ *is a dense extension.*
(3) G *is singular.*

Observe that $C(X)$ is never singular. It follows that $C(X, \mathbb{Z}) \leqslant C(X)$ and $C(X, \mathbb{Z}) \leqslant C(X, \mathbb{Q})$ are never dense extensions.

Theorem 2.9. *Let $(G, u) \in$* **W** *be a $\mathbb{Q}$-vector space, e.g. a convex ℓ-group. The following statements are equivalent:*
(1) $W_{\mathbb{Q}}(G) \leqslant G$ *is a dense extension.*
(2) $W_{\mathbb{Q}}(G) \leqslant G$ *is an essential extension.*
(3) YG *has a clopen π-base.*
(4) $W_{\mathbb{Q}}(G^*) \leqslant G^*$ *is a dense extension.*
(5) $W_{\mathbb{Q}}(G^*) \leqslant G^*$ *is an essential extension.*
(6) $W_{\mathbb{Z}}(G) \leqslant G$ *is an essential extension.*

Proof. By Example 2.7 it follows that (1) and (2) are equivalent and (4) and (5) are equivalent. It is clear that (3) implies (6) and (6) implies (2).

Suppose that $W_{\mathbb{Q}}(G) \leqslant G$ is a dense extension and let O be an arbitrary open subset of YG. Without loss of generality, $O = \operatorname{coz}(f)$ for some $f \in G^+$. By hypothesis there is a $g \in W_{\mathbb{Q}}(G)$ such that $g \leqslant f$. Let r be any strictly positive real number in the image of f. It is straightforward to check that there are irrational numbers r_1, r_2 such that $0 < r_1 < r_2 < r$ and that the set $K = g^{-1}((r_1, r_2))$ is nonempty. From here it is clear that $K \subseteq O$ and so YG has a clopen π-base. Thus (1) implies (3).

Suppose $W_{\mathbb{Q}}(G) \leqslant G$ is a dense extension and pick $f \geqslant 0$ in G^*. By hypothesis there exists $0 \leqslant g \in W_{\mathbb{Q}}(G)$ such that $g < f$. We now have $g \wedge 1 \leqslant g < f$ where $g \wedge 1 \in W_{\mathbb{Q}}(G^*)$. Hence $W_{\mathbb{Q}}(G^*) \leqslant G^*$ is a dense extension.

Finally, suppose $W_{\mathbb{Q}}(G^*) \leqslant G^*$ is an essential extension and let $0 \leqslant f \in G$. Since $f \wedge 1 \in G^*$, we can find $0 \leqslant g \in W_{\mathbb{Q}}(G^*)$ and a natural number n such that $g \leqslant n(f \wedge 1) \leqslant nf$. Therefore $W_{\mathbb{Q}}(G) \leqslant G$ is an essential extension. $\square$

3. Rigid extensions

Definition 3.1. An extension $G \leqslant H$ is said to be *rigid* (or G is a *rigid subgroup* of H) if whenever $h \in H^+$ there is a $g \in G^+$ such that $h^{\perp\perp_H} = g^{\perp\perp_H}$. Observe that the property of being a rigid subgroup is transitive. It is also known that every rigid extension is an essential extension (see Theorem 11.1.15 [2]).

Example 3.2. Since $h^{\perp\perp_H} = H(h)^{\perp\perp_H}$, it follows that an a-extension is a rigid extension. It is straightforward to check that for any weak order unit $u \in G^+$, the extension $G(u) \leqslant G$ is a rigid extension that is not necessarily an a-extension.

The extension $\mathbb{Z} \leqslant \mathbb{R}$ is a rigid extension which is not dense. The extension of a direct sum of an infinite collection of non-trivial groups into the direct product of the said groups is a dense but not rigid extension.

In general, the embedding of $C(X)$ into its classical ring of quotients, $q(X)$, is a rigid extension. In [7] the authors determined when a $C(X)$ is rigidly embedded inside its maximum ring of quotients, $Q(X)$. The authors called a space for which this happens a *fraction dense space*. They obtained the following theorem.

Theorem 3.3 [7]**.** *The following statements are equivalent for Tychonoff space X.*

(1) *X is fraction dense.*
(2) *βX is fraction dense.*
(3) *Every polar of $C(X)$ is principal.*
(4) *The closure of any open set of X is the closure of a cozero set of X.*

In this section we will determine which topological properties of YG make $W_{\mathbb{Z}}(G) \leqslant G$ or $W_{\mathbb{Q}}(G) \leqslant G$ into a rigid extension. Since an a-extension is rigid, it follows that YG being zero-dimensional (for G convex) is a sufficient condition. We will show that it is not necessary in general. The following lemma is straightforward and has been used by several other authors.

Lemma 3.4. *Let $f, g \in G^+$. Then $f^{\perp\perp} = g^{\perp\perp}$ if and only if $\mathrm{cl}_{YG}\operatorname{coz}(f) = \mathrm{cl}_{YG}\operatorname{coz}(g)$.*

Lemma 3.4 enables us to quickly deduce the next proposition as well as the first two items listed in Theorem 3.6. We have found that the seventh item in the theorem is a much more useful topological property when it comes to determining whether a space satisfies the conditions of the theorem.

Proposition 3.5. *Let $(G, u) \in \mathbf{W}$ be a convex ℓ-group. Then $W_{\mathbb{Z}}(G) \leqslant G$ is a rigid extension if and only if YG is basically disconnected. Moreover, $C(X, \mathbb{Z}) \leqslant C(X)$ is rigid if and only if X is basically disconnected.*

Theorem 3.6. *Let $(G, u) \in \mathbf{W}$ be a convex ℓ-group. Then the following statements are equivalent:*
(1) *$W_{\mathbb{Q}}(G) \leqslant G$ is a rigid extension.*
(2) *$W_{\mathbb{Q}}(G^*) \leqslant G^*$ is a rigid extension.*
(3) *For each $f \in G$ there is a $g \in W_{\mathbb{Q}}(G)$ such that $\mathrm{cl}_{YG}\operatorname{coz}(f) = \mathrm{cl}_{YG}\operatorname{coz}(g)$.*
(4) *For each $f \in G^+$ there is a $g \in W_{\mathbb{Q}}(G)^+$ such that $\mathrm{cl}_{YG}\operatorname{coz}(f) = \mathrm{cl}_{YG}\operatorname{coz}(g)$.*
(5) *For every cozero set $C \subseteq YG$ there exists a sequence of pairwise disjoint clopen subsets of YG, say $\{K_n\}_{n\in\mathbb{N}}$, for which $\mathrm{cl}_{YG} C = \mathrm{cl}_{YG} \bigcup\limits_{n\in\mathbb{N}} K_n$.*
(6) *For every cozero set $C \subseteq YG$ there exists a sequence of clopen subsets of YG, say $\{K_n\}_{n\in\mathbb{N}}$, for which $\mathrm{cl}_{YG} C = \mathrm{cl}_{YG} \bigcup\limits_{n\in\mathbb{N}} K_n$.*
(7) *For every cozero set $C \subseteq YG$ there exists a sequence of clopen subsets of YG, say $\{K_n\}_{n\in\mathbb{N}}$, for which $K_n \subseteq C$ and $\mathrm{cl}_{YG} C = \mathrm{cl}_{YG} \bigcup\limits_{n\in\mathbb{N}} K_n$.*

Proof. That (1) and (3) are equivalent follows from Lemma 3.4. Proofs of the equivalences of (1) and (2), (3) and (4), and (5) and (6) are straightforward.

If (5) holds, then for any $f \in G^+$ choose a sequence of pairwise disjoint clopen sets $\{K_n\}$ such that $\mathrm{cl}_{YG}\operatorname{coz}(f) = \mathrm{cl}_{YG} \bigcup\limits_{n\in\mathbb{N}} K_n$. Define a function which maps K_n to the fraction $1/n$ and which maps $YG \setminus \bigcup\limits_{n} K_n$ to 0. This function is easily verified to belong to $C(YG, \mathbb{Q})$. It is also bounded, so convexity of G implies it is in $W_{\mathbb{Q}}(G)$. Hence (4) holds.

Next suppose that (4) is true and let C be a cozero set of YG. Since cozero sets of compact Hausdorff spaces are Lindelöf it is enough to assume that $C = \operatorname{coz}(f)$ for some $f \in G^+$. By hypothesis we select $g \in W_{\mathbb{Q}}(G)^+$ for which $\mathrm{cl}_{YG}\operatorname{coz}(f) = \mathrm{cl}_{YG}\operatorname{coz}(g)$. Define $K_n = g^{-1}((r_n, \infty])$ where $\{r_n\}$ is a decreasing sequence of positive irrational numbers converging to 0. Since $g \in W_{\mathbb{Q}}(G)$, it follows that each K_n is clopen and by Lemma 3.4 $\mathrm{cl}_{YG}\operatorname{coz}(f) = \mathrm{cl}_{YG} \bigcup\limits_{n\in\mathbb{N}} K_n$.

Clearly (7) implies (6). If (6) holds, then so does (1). Therefore, for each $f \in G^+$ and each natural number n there is a sequence $\{K_m^n\}_{m \in \mathbb{N}}$ such that $\mathrm{cl}\, f^{-1}((1/n, \infty]) = \mathrm{cl} \bigcup\limits_{m \in \mathbb{N}} K_m^n$. Observe that $K_m^n \subseteq \mathrm{coz}(f)$ for each natural m. It is apparent that $\mathrm{cl}\,\mathrm{coz}(f) = \mathrm{cl} \bigcup\limits_{n,m \in \mathbb{N}} K_m^n$. □

Definition 3.7. When X possesses the property that given $f \in C(X)^+$ there exists a countable sequence K_n of clopen subsets of X for which $\mathrm{cl}_X \mathrm{coz}(f) = \mathrm{cl}_X \bigcup\limits_{n \in \mathbb{N}} K_n$, we call X a *quasi strongly zero-dimensional space* or a *qsz-space* for short. Observe that by Theorem 2.3 a strongly zero-dimensional space (and hence a basically disconnected space) is a qsz-space. An application of Theorem 3.6 to $C(X)$ allows us to state the following corollary. Observe that there is something to be checked since in general $W_{\mathbb{Q}}(C(X))$ is different than $C(X, \mathbb{Q})$. We leave the details to the interested reader.

Corollary 3.8. *For a space X the following statements are equivalent:*

(1) *X is a qsz-space.*
(2) *βX is a qsz-space.*
(3) *For each $f \in C(X)$ there is a $g \in C(X, \mathbb{Q})$ such that $\mathrm{cl}_X \mathrm{coz}(f) = \mathrm{cl}_X \mathrm{coz}(g)$.*
(4) *For each $f \in C(X)^+$ there is a $g \in C(X, \mathbb{Q})^+$ such that $\mathrm{cl}_X \mathrm{coz}(f) = \mathrm{cl}_X \mathrm{coz}(g)$.*
(5) *$C(X, \mathbb{Q}) \leqslant C(X)$ is a rigid extension.*
(6) *$C^*(X, \mathbb{Q}) \leqslant C^*(X)$ is a rigid extension.*

Proposition 3.9. *If X is a qsz-space, then X has a clopen π-base.*

P r o o f. The proof follows from the fact that a rigid extension is an essential extension together with an application of Theorem 3.6 and Theorem 2.9. □

Proposition 3.10. *If a space X has a countable clopen π-base, then X is a qsz-space.*

P r o o f. Suppose X has a countable clopen π-base. For each cozero set C there is a countable collection of clopen subsets of X whose union is dense in C. □

Proposition 3.11. *Let X be a hereditarily Lindelöf space with a clopen π-base. Then X is a qsz-space.*

P r o o f. Let C be an arbitrary cozero set of X. Since X has a clopen π-base, there exists a collection of clopen subsets $\{K_i\}_{i \in I}$ such that $K_i \subseteq C$ for each $i \in I$ and that $K = \bigcup K_i$ is a dense subset of C. Since X is hereditarily Lindelöf, $K = \bigcup K_{i_n}$ for some countable subset $\{i_n\}$ of I. It follows that the closure of C is the closure of a countable sequence of clopen sets. □

Proposition 3.12. *If X is a totally ordered space, then X is a qsz-space if and only if X is strongly zero-dimensional.*

Proof. As stated before, a strongly zero-dimensional space is a qsz-space. Conversely, it is known that for a totally ordered space, X is strongly zero-dimensional if and only if X has a clopen π-base. Hence if X is a qsz-space, then X is strongly zero-dimensional by Proposition 3.9.

Remark 3.13. We are now in a position to observe that every strongly zero-dimensional space is a zero-dimensional qsz-space, and in turn, each of them is a qsz-space. Furthermore, the class of qsz-spaces falls between the class of spaces with a countable clopen π-base and the class of spaces with a clopen π-base. We now demonstrate that all of these containments of classes are proper.

Example 3.14. A discrete uncountable space is an example of a qsz-space that does not have a countable clopen π-base.

Example 3.15. Let $X = \mathbb{Q}^2 \cup \{(a,b) \in \mathbb{R}^2 \colon b = 0\}$, and define a topology on X as follows. For a point $(a,b) \in X$ with $b = 0$, a base of neighborhoods is the usual base. Otherwise, (a,b) is isolated. X is not zero-dimensional as it has a subspace homeomorphic to $\mathbb{R}$. The set of isolated points forms a countable clopen π-base, and therefore X is a qsz-space which is not zero-dimensional.

Example 3.16. In this example we will see that a zero-dimensional space need not be a qsz-space. Details of this example can be found in Problem 4V of [9]. Let $J = \mathbb{R} \setminus \mathbb{Q}$. For $x \in J$, let $J_x = \{x + r \colon r \in Q\}$ and $\mathscr{J} = \{J_x \colon x \in J\}$. Re-index $\mathscr{J}$ by $\mathscr{J} = \{J_\alpha < 2^{\aleph_0}\}$ so that $J_\alpha \cap J_\beta = \emptyset$ whenever $\alpha \neq \beta$. For $\alpha < \omega_1$, let $U_\alpha = \mathbb{R} \setminus \cup\{J_\beta \colon \alpha < \beta < \omega_1\}$, and let $X = \cup\{\{\alpha\} \times U_\alpha \colon \alpha < \omega_1\}$. Equip X with the subspace topology from $W(\omega_1) \times \mathbb{R}$ (where $W(\omega_1)$ is the set of countable ordinals endowed with the order topology). It is known that X is zero-dimensional (and hence has a clopen π-base), but not strongly zero-dimensional. To show that X is not a qsz-space, we need the following two results.

Lemma 3.17. *Let $\{K_n\}$ be a countable sequence of clopen subsets of the space X in Example 3.16, and let $\pi \colon X \to W(\omega_1)$ be the continuous projection map. If $\pi(\bigcup K_n)$ is cofinal in $W(\omega_1)$, then $\pi(K_N)$ is cofinal in $W(\omega_1)$ for some $N \in \mathbb{N}$.*

Proposition 3.18. *Let K be a clopen subset of the space X in Example 3.16 such that $\pi(K)$ is cofinal in $W(\omega_1)$. Define the following sets for $r \in \mathbb{R}$ and $\varepsilon > 0$:*

$$S_r^\varepsilon = \{\sigma \in W(\omega_1) \colon (\{\sigma\} \times (r - \varepsilon, r + \varepsilon)) \cap X \subset K\}$$

and

$$T = \{r \in \mathbb{R} \colon\ S_r^\varepsilon \text{ is cofinal in } W(\omega_1) \text{ for some } \varepsilon > 0\}.$$

Then the set T is unbounded in $\mathbb{R}$.

Proof. Our first claim is that T is nonempty. In fact, we will show that T contains a rational number. Assume, by means of contradiction, that T contains no rational numbers and let $q \in \mathbb{Q} \setminus T$. By definition of T, for each $n \in \mathbb{N}$ there is an element $\sigma_n \in W(\omega_1)$ such that $\{\sigma\} \times (q - 1/n, q + 1/n)$ is not a subset of K for all $\sigma \geqslant \sigma_n$. Then the countable set $\{\sigma_n \colon\ n \in \mathbb{N}\}$ has a supremum in $W(\omega_1)$, let us call it τ. If there exists an element $(\sigma, q) \in K$ for some $\sigma \geqslant \tau$, then K being open implies $(\{\sigma\} \times (q - 1/k, q + 1/k)) \cap X \subseteq K$ for some $k \geqslant 1$, which cannot happen because τ is the supremum of the set $\{\sigma_n \colon\ n \in \mathbb{N}\}$. Hence for each $q \in \mathbb{Q}$ there exists $\sigma_q \in W(\omega_1)$ such that $(\sigma, q) \notin K$ for all $\sigma > \sigma_q$. The rational numbers are countable, so the set $\{\tau_q \colon\ q \in \mathbb{Q}\}$ has a supremum in $W(\omega_1)$, let us call it τ'. Recall that K was chosen such that $\pi(K)$ is cofinal in $W(\omega_1)$, from which it follows that there is an irrational number p and a $\sigma_1 > \tau'$ such that the point (σ_1, p) is in K. Since K is an open subset of X, there is an $\varepsilon > 0$ so that $(\{\sigma_1\} \times (p - \varepsilon, p + \varepsilon)) \cap X \subseteq K$. However, $\mathbb{Q}$ dense in $\mathbb{R}$ and $\{\sigma\} \times \mathbb{Q} \subset X$ for all $\sigma \in W(\omega_1)$ imply that $(\sigma_1, q) \in K$ for some $q \in \mathbb{Q} \cap (p - \varepsilon, p + \varepsilon)$. This is a contradiction. Consequently, there exists a rational number $q \in T$.

Next assume, by means of contradiction, that T is bounded, say T has a supremum in $\mathbb{R}$, call it t. We claim that $t \in T$. For each $n \in \mathbb{N}$, choose $t_n \in T$ such that $t_n \in (t - 1/n, t)$, which is possible because $t = \sup T$. Select, for each $n \in \mathbb{N}$, an element $\beta_n \in W(\omega_1)$ satisfying $\beta_n \leqslant \beta_{n+1}$ and $(\beta_n, t_n) \in K$. Now $((\beta_n, t_n))$ converges to the point (β, t) for some $\beta \in W(\omega_1)$, which implies that $(\beta, t) \in K$ because K is closed. However, K is also open, so there exists $\varepsilon_1 > 0$ such that $(\{\beta\} \times (t - \varepsilon_1, t + \varepsilon_1)) \cap X \subseteq K$. We need to show that there is an $\varepsilon > 0$ such that S_t^ε is cofinal in $W(\omega_1)$.

By definition of T, for each t_n there is an $\varepsilon_n > 0$ with $S_{t_n}^{\varepsilon_n}$ cofinal in $W(\omega_1)$. Cofinality and the argument in the previous paragraph imply that the set $B = \{\sigma \in W(\omega_1) \colon\ (\sigma, t) \in K\}$ is also cofinal in $W(\omega_1)$. Moreover, for each $\sigma \in B$ we can find an $\varepsilon_\sigma > 0$ satisfying $(\{\sigma\} \times (t - \varepsilon_\sigma, t + \varepsilon_\sigma)) \cap X \subset K$.

Suppose $A_n = \{\sigma \in B \colon\ (\{\sigma\} \times (t - 1/n, t + 1/n)) \cap X \subset K\}$ is bounded in $W(\omega_1)$ for each $n \geqslant 1$, say $\gamma_n = \sup A_n$. Let $\gamma = \sup\{\gamma_n\}$. For each n, there is an $\alpha_n > \gamma$ with $(\alpha_n, t_n) \in K$. Again, $((\alpha_n, t_n))$ converges to (α', t) for some $\alpha' \in W(\omega_1)$, which implies (α', t) is an element of the closed set K. Also, K open implies $(\{\alpha'\} \times (t - 1/m, t + 1/m)) \cap X$ for some $m \geqslant 1$, a contradiction. Hence A_N is cofinal in $W(\omega_1)$ for some $N \geqslant 1$. This means $S_t^{1/N}$ is cofinal in $W(\omega_1)$, i.e. $t \in T$.

Let $s = \frac{1}{2}(t + 1/N)$, so that $S_s^{1/(4N)}$ is cofinal in $W(\omega_1)$. As a result, $s \in T$ with $s > t$, which is a contradiction since t is the supremum of T. Hence T has no supremum. A similar argument yields that T has no infimum. $\square$

Now we can show that X is not a qsz-space. Let $g \in C(\mathbb{R})$ with $\mathrm{coz}(g) = (0,1)$, and define $G \in C(X)$ by $G(\sigma, r) = g(r)$. Assume, to get a contradiction, that there exists $\{K_n\}$ with each K_n clopen in X such that $\mathrm{cl}_X \mathrm{coz}(G) = \mathrm{cl}_X \bigcup K_n$. By Lemma 3.17, $\pi(K)$ is cofinal in $W(\omega_1)$ for some K in $\{K_n\}$. By Proposition 3.18, the set T defined in the proposition is unbounded in $\mathbb{R}$. However, $T \subset (0,1)$ because $K \subset \mathrm{coz}(G)$. This is a contradiction. Therefore X cannot be a qsz-space.

Example 3.19. This example is taken from [10]. Let C denote the Cantor set and $\mathscr{E}$ its usual topology. Observe that $(C, \mathscr{E})$ is homeomorphic to $\prod_{q \in \mathbb{Q}} \{0,1\}$, so it can be represented by the power set of the rationals by identifying each function with its support. Under this representation, the supremum function $\sup\colon C \to [-\infty, \infty]$ is a continuous function. Let ϱ be the topology generated by sup and $\mathscr{E}$, then ϱ is a separable metric topology.

Let A be a Bernstein set of (C, ϱ), that is, a set which intersects each uncountable closed set but does not contain any uncountable closed set. Such a set exists since every complete metric space contains a Bernstein set. Observe that A is dense in (C, ϱ).

Next fix a countable ϱ-dense subset D of A. To each point a of $A \setminus D$ assign a subset D_a of D such that $a \in \mathrm{cl}_\varrho D_a \cap \mathrm{cl}_\varrho(D \setminus D_a)$ for each $a \in A \setminus D$ and if $E \subset D$ and $|(A \setminus D) \cap \mathrm{cl}_\varrho(D \setminus E)| = 2^\omega$ then $E = D_a$ for some $a \in A \setminus D$. For each $a \in A \setminus D$, let $\{a_n \colon n \in \mathbb{N}\}$ be a sequence of points of D which ϱ-converges to a and has infinite intersection with D_a and $D \setminus D_a$. Now let ϱ^* be the topology on A generated by the base of open sets $\{\{a\} \cup \{a_n \colon n > m\} \colon a \in A \setminus D, m \in \mathbb{N}\} \cup D$.

Let $X = (A, \mathscr{E}|_A)$ and $Y = (C, \mathscr{E} \cup \varrho^*)$. In [10] it is shown that both X and Y are strongly zero-dimensional spaces while $X \times Y$ is zero-dimensional though not strongly zero-dimensional. The space X is a second countable, zero-dimensional space, so it has a countable clopen π-base. In Y, the set D is discrete and countable. Since D is ϱ-dense in A and A is ϱ-dense in C, D is dense in (C, ϱ). Moreover, because a basic open set in $\varrho^* \setminus (\mathscr{E} \cup \varrho)$ is of the form $\{a\} \cup \{a_n \colon n > m\}$ where $a \in A \setminus D$, $m \in \mathbb{N}$, and $\{a_n \colon n > m\} \subseteq D$, D is dense in Y. It follows that Y has a countable clopen π-base as well. Therefore the product $X \times Y$ has a countable clopen π-base and hence is a zero-dimensional qsz-space which is not strongly zero-dimensional.

Example 3.20. We can now point out that Proposition 3.11 cannot be weakened by removing hereditarily from the hypothesis. Let X be the space from Example 3.16.

Then βX is a compact (and hence Lindelöf) space with a clopen π-base which is not a qsz-space.

In the next section we will consider the question of when a subspace of a qsz-space is again a qsz-space. We will also investigate what happens when we take a product of qsz-spaces. For now we would like to point out that in Corollary 3.8 we could have also included the statement that for any countable subset $\{h_n\}_{n\in\mathbb{N}} \subseteq H^+$ there is a $g \in G^+$ such that $\{h_n\}_{n\in\mathbb{N}}^{\perp\perp} = g^{\perp\perp}$. This is true because the polar generated by a countable set is in fact a principal polar. This leads us to consider a strengthening of our definition of a qsz-space. In particular, we are interested in when the closure of any open subset of X is the closure of the union of a sequence of clopen sets. Obviously such a space is a qsz-space. Since the union of a sequence of clopen sets is a cozero set it follows that such a space is also fraction dense. These two properties characterize such a space. We recapitulate.

Proposition 3.21. *For a space X the following statements are equivalent.*

(1) *The closure of any open subset of X is the closure of the union of a sequence of clopen subsets of X.*
(2) *X is a fraction dense qsz-space.*
(3) *βX is a fraction dense qsz-space.*
(4) *X is a qsz-space for which every polar of $C(X,\mathbb{Q})$ is principal.*
(5) *X is a qsz-space for which $C(X,\mathbb{Q})$ is a fraction dense algebra.*
(6) *$C(X,\mathbb{Q}) \leqslant Q(X)$ is a rigid extension.*

Remark 3.22. Observe that the forcing of every open set to be a countable union of clopen sets is simply a characterization of a strongly zero-dimensional perfectly normal space.

Example 3.23. We should note that the classes of fraction dense spaces and qsz-spaces are distinct. Any connected metric space is a fraction dense space which is not a qsz-space. The space of ordinals $W(\omega_1)$ is a qsz-space which is not fraction dense.

Moreover, a fraction dense qsz-space is not necessarily strongly zero-dimensional. The space $X \times Y$ in Example 3.19 has a countable clopen π-base, so it is a fraction dense qsz-space. However, it is not strongly zero-dimensional.

We conclude this section with results regarding the rigidity of the extension $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q})$.

Definition 3.24. To characterize when $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q})$ is a rigid extension, we need the following definitions. An ℓ-group G is said to be *projectable* if $g^\perp$ is a cardinal summand of G for each $g \in G$, and G is *strongly projectable* if every polar is a cardinal summand of G. An ℓ-group G is said to be *conditionally (σ-)complete* if whenever $S \subseteq G$ is a (countable) bounded set; then the supremum of S exists in G. Since the conditional σ-completion of $\mathbb{Q}$ is $\mathbb{R}$, it follows that $C(X, \mathbb{Q})$ is never conditionally σ-complete. Recall the following Stone-Nakano Theorems.

Theorem 3.25. *For any Tychonoff space X, the following statements are equivalent:*
(1) *X is basically disconnected.*
(2) *$C(X)$ is a projectable ℓ-group.*
(3) *$C(X)$ is a conditionally σ-complete ℓ-group.*

Theorem 3.26. *For any Tychonoff space X, the following statements are equivalent:*
(1) *X is extremally disconnected.*
(2) *$C(X)$ is a strongly projectable ℓ-group.*
(3) *$C(X)$ is a conditionally complete ℓ-group.*

Now for any space X, $C(X, \mathbb{Z})$ is a projectable group. We can use the notion of rigidity to classify when $C(X, \mathbb{Q})$ is a (strongly) projectable ℓ-group.

Proposition 3.27. *For a space X the following statements are equivalent:*
(1) *The closure of every countable union of clopen subsets of X is clopen.*
(2) *$C(X, \mathbb{Z}) \leqslant C(X, \mathbb{Q})$ is a rigid extension.*
(3) *Whenever $\{K_n\}$ is a countable collection of clopen sets and O is an open set disjoint from each K_n, then O and $\bigcup\limits_{n \in \mathbb{N}} K_n$ have disjoint closures.*
(4) *$C(X, \mathbb{Z})$ is conditionally σ-complete.*
(5) *$C(X, \mathbb{Q})$ is a projectable ℓ-group.*

Proof. Obviously (1) through (3) are equivalent. Suppose (1) and let $g \in C(X, \mathbb{Q})^+$. We know that $\operatorname{coz}(g)$ is a countable union of clopen sets and thus has clopen closure. Let us call the cloure K. It follows that we can write $\mathbf{1} = \chi_K + \chi_{X \setminus K}$. Observe that by Lemma 3.4, $\chi_K \in g^{\perp\perp}$. It is also straightforward to show that$\chi_{(X \setminus K)} \in g^{\perp}$. It follows that $C(X, \mathbb{Q})$ is a projectable ℓ-group.

Next suppose (5) and let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of clopen sets. Set $O = \bigcup\limits_{n \in \mathbb{N}} K_n$ and let $g \in C(X, \mathbb{Q})^+$ satisfy $O = \operatorname{coz}(g)$. Since $C(X, \mathbb{Q})$ is projectable it follows that $\mathbf{1} = g_1 + g_2$ for some $g_1 \in g^{\perp\perp}$ and $g_2 \in g^{\perp}$. Furthermore, it is straightforward to check that both g_1 and g_2 are characteristic functions of complementary clopen sets. Finally, by Lemma 3.4 we obtain that $\operatorname{coz}(g_1) = \operatorname{cl}_X \operatorname{coz}(g) = \operatorname{cl}_X \bigcup\limits_{n \in \mathbb{N}} K_n$. So (5) implies (1).

Assume (1) holds, and we will show (4). Let $\{f_n\colon n\in\mathbb{N}\}\subset C(X,\mathbb{Z})$ with $f_n\leqslant g$ for some $g\in C(X,\mathbb{Z})$. For each $m\in\mathbb{Z}$, define sets $A_m=\operatorname{cl}\bigcup\limits_{n=1}^{\infty}f_n^{-1}(m)$ and $B_m=\operatorname{cl}\bigcup\limits_{n=1}^{\infty}f_n^{-1}([m+1,\infty))$. By (1), both of these sets are clopen. First we claim that $X=\bigcup\limits_{m\in\mathbb{Z}}A_m\setminus B_m$. To see this, fix $x\in X$ and let $M=\sup\{f_n(x)\colon n\in\mathbb{N}\}$, so $x\in A_M$. If $x\notin B_M$, then we are done, so suppose $x\in B_M$. Let $g(x)=k$ and let $L=\sup\{i\colon x\in A_i, i=M\dots k\}$. Then $x\in A_L$ and $x\notin A_i$ for all $i\geqslant L+1$, so $x\in A_L\setminus B_L$.

Observe that $(A_i\setminus B_i)\cap(A_j\setminus B_j)=\emptyset$ whenever $i\neq j$. It follows that the function $f\colon X\to\mathbb{Z}$ defined by $f(x)=m$ if $x\in A_m\setminus B_m$ is a continuous function. We claim that $f_n\leqslant f$ for all n. Let $M=\sup\{f_n(x)\colon n\in\mathbb{N}\}$, so $x\in A_M$. It follows that $f_n(x)\leqslant M\leqslant f(x)$ for all n.

At this point we will show that $f=\bigvee f_n$. Assume, by means of contradiction, that there exists an $h\in C(X,\mathbb{Z})$ such that $f_n\leqslant h<f$ for all n. There exists an open set O on which $h<f$. We must have that $O\cap(A_M\setminus B_M)\neq\emptyset$ for some $M\in\mathbb{Z}$, so $h(x)<M$ on that set. Since $O\cap A_M\neq\emptyset$, it follows that $O\cap\bigcup\limits_{n=1}^{\infty}f_n^{-1}(M)\neq\emptyset$. Select $p\in O\cap\bigcup\limits_{n=1}^{\infty}f_n^{-1}(M)$, then we have $f_n(p)=M=f(p)\leqslant h$, a contradiction. Hence (4) holds.

Finally, assume (4) holds. Let $\{C_n\}$ be a countable collection of clopen subsets of X. Then $\chi_{C_n}\in C(X,\mathbb{Z})$ for each n. This set of functions has a supremum by hypothesis, say $f\in C(X,\mathbb{Z})$. It is easy to show that $\operatorname{coz}(f)=\operatorname{cl}\bigcup\limits_{n=1}^{\infty}C_n$, and this set is clopen. $\square$

Corollary 3.28. *Suppose X is a qsz-space. Then $C(X,\mathbb{Z})\leqslant C(X,\mathbb{Q})$ is a rigid extension if and only if X is basically disconnected.*

Proposition 3.29. *For a space X the following statements are equivalent:*

(1) *The closure of every union of clopen subsets of X is clopen.*
(2) *Whenever $\{K_i\}_{i\in I}$ is a collection of clopen sets and O is an open set disjoint from each K_i, then O and $\bigcup\limits_{i\in I}K_n$ have disjoint closures.*
(3) *$C(X,\mathbb{Z})$ is conditionally complete.*
(4) *$C(X,\mathbb{Q})$ is a strongly projectable ℓ-group.*

Proof. Clearly (1) and (2) are equivalent. The proof of the equivalence of (1) and (3) is similar to the proof of the equivalence of (1) and (4) in Proposition 3.27 and is omitted. Suppose (1) holds, and we will prove (4). Let A be a subset of $C(X,\mathbb{Q})$. We need to show that $A^{\perp}$ is a cardinal summand of $C(X,\mathbb{Q})$. Without loss of generality, we may assume $f \geqslant 0$ for all $f \in A$. Let $U = \bigcup\{f^{-1}((r,\infty))\colon f \in A, r \in (0,\infty) \setminus \mathbb{Q}\}$, then $\operatorname{cl} U$ is clopen. Since $\mathbf{1} = \chi_{\operatorname{cl} U} + \chi_{X\setminus \operatorname{cl} U}$ where $\chi_{\operatorname{cl} U} \in A^{\perp\perp}$ and $\chi_{X\setminus\operatorname{cl} U} \in A^{\perp}$, $A^{\perp}$ is a cardinal summand of $C(X,\mathbb{Q})$ as needed.

Finally, assume that $C(X,\mathbb{Q})$ is a strongly projectable ℓ-group, and we will show (1). Let $\{K_i\}_{i\in I}$ be a collection of clopen subsets of X, and let $U = \bigcup\limits_{i\in I} K_i$. Let $A = \{\chi_{K_i}\colon i \in I\} \subseteq C(X,\mathbb{Q})$, then $A^{\perp}$ is a cardinal summand of $C(X,\mathbb{Q})$. We can write $1 = f + g$ where $f \in A^{\perp\perp}$ and $g \in A^{\perp}$, hence $f = \chi_{\operatorname{cl} U}$ and $g = \chi_{X\setminus\operatorname{cl} U}$. As a result, $\operatorname{cl} U$ is clopen in X. □

Corollary 3.30. *If X has a clopen π-base, then $C(X,\mathbb{Z})$ is conditionally complete if and only if X is extremally disconnected.*

4. Subspaces of qsz-spaces

It is natural to ask if a subspace of a qsz-space is again a qsz-space. In general, the answer is no. The space in Example 3.15 is a qsz-space, but it contains a copy of $\mathbb{R}$ which is not a qsz-space. However, certain subspaces of qsz-spaces are again qsz-spaces, e.g., cozersets. If X has some additional structure or the subspace in question is embedded in a nice way, then we can show that the qsz-property is preserved.

Lemma 4.1. *Let X be a fraction dense qsz-space and let $Y \subseteq X$. Then Y is a qsz-space.*

Proof. Let $U \in \operatorname{Coz}(Y)$, then $U = V \cap Y$ for an open subspace V of X. Since X is a fraction dense space, $V \in \operatorname{Coz}(X)$. Then there exists a sequence of clopen sets $\{K_n\}_{n\in\mathbb{N}}$ such that $\operatorname{cl}_X V = \operatorname{cl}_X \bigcup\limits_{n\in\mathbb{N}} K_n$. For each n let $C_n = K_n \cap Y$, then each C_n is a clopen subset of Y. Observe that $\operatorname{cl}_Y U = \operatorname{cl}_Y \bigcup\limits_{n\in\mathbb{N}} C_n$. Hence Y is a qsz-space. □

Definition 4.2. Let X be a topological space. In [8] the authors define the set $\mathscr{Z}^{\sharp}(X) = \{\operatorname{cl}(\operatorname{int} Z(f))\colon f \in C(X)\}$. If Y is a subspace of X such that for each $f \in C(Y)$ there exists a $g \in C(X)$ with $\operatorname{cl}_Y \operatorname{int}_Y Z(f) = Y \cap \operatorname{cl}_X(\operatorname{int}_X Z(g))$, then Y is said to be *$\mathscr{Z}^{\sharp}$-embedded* in X. The notion of a $\mathcal{Z}^{\sharp}$-embedded subspace is a generalization of the notion of a z-embedded subspace, which in turn is a generalization of a C^*-embeddded subspace.

Lemma 4.3. *Assume X is a qsz-space, and let Y be a dense or open subspace of X. If Y is $\mathscr{Z}^\sharp$-embedded in X, then Y is a qsz-space.*

Proof. Let $U \in \mathrm{Coz}(Y)$. By Lemma 2.3 of [8], there exists $V \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_Y U = Y \cap \mathrm{cl}_X V$. Select a sequence $\{K_n\}_{n\in\mathbb{N}}$ of clopen sets of X with $\mathrm{cl}_X V = \mathrm{cl}_X \bigcup K_n$. For each $n \in \mathbb{N}$, let $C_n = Y \cap K_n$. Then we have $\mathrm{cl}_Y U = \mathrm{cl}_Y \bigcup\limits_{n\in\mathbb{N}} C_n$ where each C_n is clopen on Y. Therefore Y is a qsz-space. □

Example 4.4. The assumption that Y be $\mathscr{Z}^\sharp$-embedded in X cannot be dropped from the previous lemma. The space X in Example 3.16 is a dense subspace of its zero-dimensional compactification $\beta_0 X$, so it is a dense subspace of a qsz-space. However, X is not a qsz-space.

We would also like to investigate products of qsz-spaces. We do not know if the product of two qsz-spaces is again a qsz-space, but we do have some partial results.

Lemma 4.5. *If X is compact and $X \times Y$ is a qsz-space, then Y is a qsz-space.*

Proof. Suppose U is a cozero set of Y, then $X \times U \in \mathrm{Coz}(X \times Y)$. There exists a sequence $\{K_n\}$ of clopen subsets of $X \times Y$ such that $\mathrm{cl}_{X\times Y} \bigcup\limits_{n\in\mathbb{N}} K_n = \mathrm{cl}_{X\times Y}(X \times U) = X \times \mathrm{cl}_Y U$. Let $\pi\colon X \times Y \to Y$ be the projection map. For each $n \in \mathbb{N}$, let $C_n = \pi(K_n)$. Note that π is always an open map, and that π is a closed map in this case because X is compact. Hence each C_n is a clopen subset of Y. Observe that $\mathrm{cl}_Y U = \mathrm{cl}_Y \Bigl(\bigcup\limits_{n\in\mathbb{N}} C_n\Bigr)$, therefore Y is a qsz-space. □

Lemma 4.6. *If X and Y are qsz-spaces such that $X \times Y$ is a Lindelöf space, then $X \times Y$ is a qsz-space.*

Proof. Let $U \in \mathrm{Coz}(X \times Y)$. By Lemma 3.4 of [8], U is a σ-rectangle, say $U = \bigcup\limits_{n\in\mathbb{N}} A_n \times B_n$ where $\{A_n\} \subset \mathrm{Coz}(X)$ and $\{B_n\} \subset \mathrm{Coz}(Y)$. Since X and Y are qsz-spaces, for each n there exist sequences of clopen sets $\{K_m^n\}_{m\in\mathbb{N}}$ in X and $\{C_m^n\}_{m\in\mathbb{N}}$ in Y such that $\mathrm{cl}_X A_n = \mathrm{cl}_X \bigcup\limits_{m\in\mathbb{N}} K_m^n$ and $\mathrm{cl}_Y B_n = \mathrm{cl}_Y \bigcup\limits_{m\in\mathbb{N}} C_m^n$. We claim that $\mathrm{cl}_{X\times Y} U = \mathrm{cl}_{X\times Y} \bigcup\limits_{n,m,l\in\mathbb{N}} K_m^n \times C_l^n$. To see this, note that $\mathrm{cl}_{X\times Y} U = \mathrm{cl}_{X\times Y} \bigcup\limits_{n,m,l\in\mathbb{N}} K_m^n \times C_l^n$. Hence $X \times Y$ is a qsz-space. □

Lemma 4.7. *If Y has a countable dense set D of isolated points, then $X \times Y$ is a qsz-space if and only if X is a qsz-space.*

Proof. The proof of the forward direction is straightforward. Suppose X is a qsz-space, and let $U \in \mathrm{Coz}(X \times Y)$. For each $d \in D$ we have that $X \times \{d\} \in$

$\operatorname{Coz}(X \times Y)$, so $U_d = U \cap (X \times \{d\}) \in \operatorname{Coz}(X \times Y)$. It follows that for each d there exists a sequence of clopen sets $\{K_n^d\}_{n\in\mathbb{N}}$ with $\operatorname{cl} U_d = \operatorname{cl} \bigcup\limits_{n\in\mathbb{N}} K_n^d$. Now

$$\operatorname{cl} U = \operatorname{cl} \bigcup_{d\in D} U_d = \operatorname{cl} \bigcup_{d\in D} \bigcup_{n\in\mathbb{N}} K_n^d$$

where $\bigcup\limits_{d\in D} \bigcup\limits_{n\in\mathbb{N}} K_n^d$ is a countable union of clopen sets. Hence $X \times Y$ is a qsz-space.

$\square$

5. Major extensions

We conclude this article with a section on when the extensions $W_{\mathbb{Z}}(G) \leqslant W_{\mathbb{Q}}(G) \leqslant G$ are major extensions (for a **W**-object G). For the record, $G \leqslant H$ is a major extension if for each $h \in H^+$ there exists $g \in G^+$ such that $h \leqslant g$. It is straightforward to check that an a-extension is a major extension. Also, if $(G,u) \in \mathbf{W}$ and u is a strong order unit, then it is obvious that $W_{\mathbb{Z}}(G) \leqslant W_{\mathbb{Q}}(G) \leqslant G$ are major extensions. After our initial result we look at the situation of $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q}) \leqslant C(X)$. Note that $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q})$ is always a major extension, so we are more interested in when $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension. We begin with the following theorem.

Theorem 5.1. *Let (G,u) be a convex ℓ-group. The following statements are equivalent:*

(1) *$W_{\mathbb{Q}}(G) \leqslant G$ is a major extension.*
(2) *$W_{\mathbb{Z}}(G) \leqslant G$ is a major extension.*
(3) *For every $g \in G^+$ there exists a sequence of pairwise disjoint clopen subsets of YG, say $\{K_n\}_{n\in\mathbb{N}}$, whose union is dense in YG and such that $g\chi_{K_n} \leqslant n\chi_{K_n}$.*

Example 5.2. Let $H = C(\alpha\mathbb{N})$. Let $g\colon \mathbb{N} \to \mathbb{R}$ be the function defined by $g(n) = \frac{1}{4}\pi n$. Finally, let $G = \langle H, g\rangle$ be the ℓ-group generated by H and g and consider $(G,\mathbf{1}) \in \mathbf{W}$. Since g has an obvious extension lying in $D(\alpha\mathbb{N})$, it follows by the Yosida Representation that $YG = \alpha\mathbb{N}$. Furthermore, $W_{\mathbb{Z}}(G) = C(\alpha\mathbb{N},\mathbb{Z})$. Note that G satisfies condition (iii) of Theorem 5.1 yet $W_{\mathbb{Q}}(G) \leqslant G$ is not a major extension. It should be apparent that our example is not a convex ℓ-group.

Turning to the case of $C(X)$, we obtain the following theorem. In this special case we are able to add more equivalences.

Theorem 5.3. *For any space X the following statements are equivalent:*

(1) $C(X,\mathbb{Q}) \leqslant C(X)$ *is a major extension.*

(2) $C(X,\mathbb{Z}) \leqslant C(X)$ *is a major extension.*

(3) *For every $f \in C(X)$ there exists a sequence of pairwise disjoint clopen subsets of X, say $\{K_n\}_{n\in\mathbb{N}}$, for which $X = \bigcup\limits_{n\in\mathbb{N}} K_n$ and $f|_{K_n} \in C^*(K_n)$.*

(4) *For every $f \in C(X)$ there exists a sequence of clopen subsets of X, say $\{K_n\}_{n\in\mathbb{N}}$, for which $X = \bigcup\limits_{n\in\mathbb{N}} K_n$ and $f|_{K_n} \in C^*(K_n)$.*

(5) $C(\upsilon X,\mathbb{Q}) \leqslant C(\upsilon X)$ *is a major extension.*

Proof. It is easy to see that (1) and (2) are equivalent. Suppose (2) holds and let $f \in C(X)^+$. Then there is a function $g \in C(X,\mathbb{Z})$ such that $f \leqslant g$. Now let $K_1 = g^{-1}([0,\frac{1}{2}])$, and for each $n \geqslant 2$ let $K_n = g^{-1}([n-\frac{1}{2}, n+\frac{1}{2}])$. These sets are clopen and pairwise disjoint because g is integer-valued. Moreover, $X = \bigcup\limits_{n\in\mathbb{N}} K_n$, and whenever $x \in K_n$ we have $0 \leqslant f(x) \leqslant g(x) \leqslant n+\frac{1}{2}$, i.e. $f|_{K_n} \in C^*(K_n)$. Hence (3) holds.

If (3) holds, then it is obvious that (4) holds. Any sequence of clopen sets can be used to create a sequence of pairwise disjoint clopen sets, so (4) implies (3). Finally, assume (3) is true and let $f \in C(X)^+$. Select a sequence $\{K_n\}_{n\in\mathbb{N}}$ of clopen subsets of X for which $X = \bigcup\limits_{n\in\mathbb{N}} K_n$ and $f|_{K_n} \in C^*(K_n)$. For each n let $r_n \in \mathbb{Q}$ be such that $f|_{K_n} \leqslant r_n$. Define a function $g\colon X \to \mathbb{Q}$ by $g(x) = r_n$ if $x \in K_n$. This function is continuous because the sets K_n are pairwise disjoint clopen sets, and g was constructed to satisfy $f \leqslant g$. Therefore $C(X,\mathbb{Q}) \leqslant C(X)$ is a major extension.

Before we show that (5) and (1) are equivalent, observe that

$$C(X,\mathbb{Z}) \leqslant C(\upsilon X,\mathbb{Q}) \leqslant C(X,\mathbb{Q}) \leqslant C(X) = C(\upsilon X)$$

where $C(X,\mathbb{Z}) \leqslant C(X,\mathbb{Q})$ is always a major extension. Hence $C(\upsilon X,\mathbb{Q}) \leqslant C(X,\mathbb{Q})$ and $C(X,\mathbb{Q}) \leqslant C(\upsilon X)$ being major extensions imply $C(\upsilon X,\mathbb{Q}) \leqslant C(X) = C(\upsilon X)$ is a major extension. If $C(\upsilon X,\mathbb{Q}) \leqslant C(\upsilon X)$ is a major extension, then $C(X,\mathbb{Q}) \leqslant C(X)$ is a major extension because $C(\upsilon X,\mathbb{Q}) \leqslant C(X,\mathbb{Q}) \leqslant C(\upsilon X)$. □

Proposition 5.4. *If X is pseudocompact, then $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension. The converse holds if X is a connected space.*

Proof. The first statement is clear. Suppose that X is connected and $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension. Assume, by means of contradiction, that X is not pseudocompact. Then there exists an unbounded function $f \in C(X)^+$. We can select a function $g \in C(X,\mathbb{Z})$ such that $f \leqslant g$. In particular, g is unbounded because f is unbounded. So g takes on more than one integer value; let n be one such value.

The set $g^{-1}(\{n\})$ is a nonempty clopen subset, and it is not all of X since g is not constant. However, this contradicts the connectedness of X. Therefore X is pseudocompact. $\square$

If X has the property that $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension, then X need not be zero-dimensional. Simply take any pseudocompact space which is not zero-dimensional. The space in Example 3.16 is an example of a zero-dimensional space which is not strongly zero-dimensional. We claim that $C(X,\mathbb{Z}) \leqslant C(X)$ is not a major extension. Let $f\colon X \to \mathbb{R}$ be the continuous projection map. If $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension, then there exists a countable sequence of pairwise disjoint clopen subsets of X, say $\{K_n\}_{n\in\mathbb{N}}$, for which $X = \bigcup\limits_{n\in\mathbb{N}} K_n$ and $f|_{K_n} \in C^*(K_n)$. By Lemma 3.17, there exists $K \in \{K_n\}$ with $\pi(K)$ cofinal in $W(\omega_1)$. Then by Proposition 3.18, the set T defined in the proposition is unbounded in $\mathbb{R}$. However, T should be bounded because $f|K$ is a bounded function, which is a contradiction.

Our next result adds to the intrigue of major extensions. It demonstrates that we can characterize a-extensions by using major extensions.

Proposition 5.5. *$C(Y,\mathbb{Z}) \leqslant C(Y)$ is a major extension for every cozero set Y of X if and only if X is strongly zero-dimensional.*

Proof. If X is strongly zero-dimensional then so is every cozero set of X and thus $C(Y,\mathbb{Z}) \leqslant C(Y)$ is a major extension for each cozero set Y of X.

As for the converse suppose Y is an arbitrary cozero set of X, say $Y = \operatorname{coz}(f)$ for some $f \in C(X)^+$. For each $n \in \mathbb{N}$ set $Y_n = f^{-1}((\frac{1}{n},\infty))$, then each Y_n is also a cozero set of X such that $\operatorname{cl}_X Y_n \subseteq Y$. By hypothesis, for each n there exists a sequence of clopen sets $\{K_n^m\}$ of Y such that $Y_n = \bigcup\limits_{m=1}^{\infty} K_n^m$ and $f|_{K_n^m}$ is bounded on each K_n^m. It is straighforward to check that each K_n^m is clopen in X. Thus, $Y = \bigcup\limits_{n\in\mathbb{N}} Y_n = \bigcup\limits_{n,m\in\mathbb{N}} K_n^m$ is a countable union of clopen sets, whence X is strongly zero-dimensional. $\square$

Proposition 5.6. *Suppose $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension. If Y is a C-embedded subspace of X, then $C(Y,\mathbb{Z}) \leqslant C(Y)$ is also a major extension.*

Proof. Let $f \in C(Y)$, then there exists $F \in C(X)$ such that $F|_Y = f$. Since $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension, there is a sequence $\{K_n\}_{n\in\mathbb{N}}$ of clopen subsets of X for which $X = \bigcup\limits_{n\in\mathbb{N}} K_n$ and $F|_{K_n} \in C^*(K_n)$ for all n. Then $Y = \bigcup\limits_{n\in\mathbb{N}} (Y \cap K_n)$ where each $Y \cap K_n$ is a clopen subset of Y and $f|_{Y\cap K_n} \in C^*(Y \cap K_n)$ for each n. $\square$

Corollary 5.7. *Suppose X is a normal space. If $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension then so is $C(K,\mathbb{Z}) \leqslant C(X)$ for every closed subset $K \subseteq X$.*

Corollary 5.8. *Let $\{X_i\}_{i\in I}$ be a collection of Tychonoff spaces and set $X = \bigoplus_{i\in I} X_i$, the topological sum. $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension if and only if $C(X_i,\mathbb{Z}) \leqslant C(X_i)$ is a major extension for every $i \in I$.*

Corollary 5.9. *Let $X = \prod_{i\in I} X_i$. If $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension, then $C(X_i,\mathbb{Z}) \leqslant C(X_i)$ is a major extension for each $i \in I$.*

Example 5.10. Notice that since $\mathbb{R}$ is locally compact and σ-compact, $\mathbb{R}$ is a cozero set of $\beta\mathbb{R}$. It is also a C^*-embedded subspace. Since $\beta\mathbb{R}$ is compact, $C(\beta\mathbb{R},\mathbb{Z}) \leqslant C(\beta\mathbb{R})$ is a major extension. However, $C(\mathbb{R},\mathbb{Z}) \leqslant C(\mathbb{R})$ is not a major extension. Hence Proposition 5.6 cannot be generalized to C^*-embedded subspaces.

Example 5.11. It should be apparent now that an a-extension is both a rigid extension and a major extension. Thus, it is natural to wonder whether a rigid and major extension is an a-extension. Consider the space X from Example 3.15. There it is shown that X is a qsz-space and hence so is βX. This means that $C(\beta X,\mathbb{Z}) \leqslant C(\beta X)$ is a rigid extension and (clearly) a major extension. But since βX is not zero-dimensional it follows that the extension is not an a-extension.

Moreover, since X has a connected C-embedded subspace which is not pseudocompact we can conclude that $C(X,\mathbb{Z}) \leqslant C(X)$ is not a major extension.

Questions.

(1) Is the product of two qsz-spaces again a qsz-space?
(2) Is the converse to Corollary 5.9 true?
(3) If X is Lindelöf, is it true that $C(X,\mathbb{Z}) \leqslant C(X)$ is a major extension? What about for locally compact or σ-compact?
(4) Is there a nice characterization of those spaces X for which $C(K,\mathbb{Z}) \leqslant C(X)$ is a major extension for every closed set (zeroset, open set)?

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