Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 887–897

Persistent URL: `http://dml.cz/dmlcz/140428`

Czechoslovak Mathematical Journal, 58 (133) (2008), 887–897

THE DIAMETER OF PAIRED-DOMINATION VERTEX CRITICAL GRAPHS

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(Received August 1, 2006)

Abstract. In this paper we continue the study of paired-domination in graphs introduced by Haynes and Slater (Networks *32* (1998), 199–206). A paired-dominating set of a graph G with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of G, denoted by $\gamma_{\rm pr}(G)$, is the minimum cardinality of a paired-dominating set of G. The graph G is paired-domination vertex critical if for every vertex v of G that is not adjacent to a vertex of degree one, $\gamma_{\rm pr}(G-v) < \gamma_{\rm pr}(G)$. We characterize the connected graphs with minimum degree one that are paired-domination vertex critical and we obtain sharp bounds on their maximum diameter. We provide an example which shows that the maximum diameter of a paired-domination vertex critical graph is at least $\frac{3}{2}(\gamma_{\rm pr}(G)-2)$. For $\gamma_{\rm pr}(G) \leqslant 8$, we show that this lower bound is precisely the maximum diameter of a paired-domination vertex critical graph.

Keywords: paired-domination, vertex critical, bounds, diameter

MSC 2010: 05C69

1. Introduction

Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [11], [12]. Brigham, Chinn, and Dutton [1] began the study of vertex domination critical graphs where the domination number decreases by the removal of any vertex. Further properties of these graphs were explored in [7], [8], [21], [22], [23], [24], but they have not been characterized. In [10] the same concept was introduced for total domination. In this paper we investigate paired-domination vertex critical graphs first studied by Edwards [5].

The first author was supported in part by the South African National Research Foundation and the University of KwaZulu-Natal, the second author was supported by the Natural Sciences and Engineering Research Council of Canada.

A *matching* M in a graph G is a set of independent edges in G. The number of edges in a maximum matching of G is called the *matching number* of G which we denote by $\alpha'(G)$. A vertex of G incident with an edge of the matching M is said to be matched by M, or simply M-matched. The matching M is called a *perfect matching* in G if every vertex of G is M-matched. A *paired-dominating set*, abbreviated PDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). Two vertices joined by an edge of M are said to be *paired* and are also called *partners* in S. Every graph without isolated vertices has a PDS since the end-vertices of any maximal matching form such a set. The *paired-domination number* of G, denoted by $\gamma_{\rm pr}(G)$, is the minimum cardinality of a PDS. A PDS of cardinality $\gamma_{\rm pr}(G)$ we call a $\gamma_{\rm pr}(G)$-set. Paired-domination was introduced by Haynes and Slater [14], [15] as a model for assigning backups to guards for security purposes, and is studied, for example, in [2], [3], [4], [6], [9], [13], [16], [17], [18], [19], [20] and elsewhere.

For notation and graph theory terminology we in general follow [11]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. The *open neighborhood* of $v \in V$ is $N(v) = \{u \in V\colon\ uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = N(S) \cup S$. For sets $S, T \subseteq V$, we say that S *dominates* T if $T \subseteq N[S]$ and that S *paired-dominates* T if S dominates T in G and $G[S]$ contains a perfect matching.

We denote the degree of a vertex v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum and maximum degrees of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. An *end-vertex* is a vertex of degree one and a *support vertex* is one that is adjacent to an end-vertex. The set of support vertices in G is denoted by $S(G)$, while the complement of G is denoted by $\overline{G}$. Two vertices at maximum distance apart in G are called *diametrical vertices* of G.

We call a vertex $v \in V$ *paired-critical* if $\gamma_{\rm pr}(G - v) < \gamma_{\rm pr}(G)$. Since paired-domination is undefined for a graph with isolated vertices, we say that a graph G is *paired-domination-vertex-critical*, or $\gamma_{\rm pr}$-vertex-critical, if every vertex of $V \setminus S(G)$ is paired-critical. If G is $\gamma_{\rm pr}$-vertex-critical and $\gamma_{\rm pr}(G) = k$, then we say that G is *k-$\gamma_{\rm pr}$-vertex-critical.* For example, the 5-cycle is 4-$\gamma_{\rm pr}$-vertex-critical. A graph is $\gamma_{\rm pr}$-vertex-critical if and only if each of its components is $\gamma_{\rm pr}$-vertex-critical. Also, K_2 is trivially 2-$\gamma_{\rm pr}$-vertex-critical. So henceforth we consider only connected graphs of order at least 3. The removal of a vertex can decrease the paired-domination number by at most two. Hence:

Observation 1. If G is a $\gamma_{\rm pr}$-vertex-critical graph, then $\gamma_{\rm pr}(G - v) = \gamma_{\rm pr}(G) - 2$ for every $v \in V(G) \setminus S(G)$. Furthermore, a $\gamma_{\rm pr}(G - v)$-set contains no neighbour of v.

In Section 2 we characterize the connected $\gamma_{\rm pr}$-vertex-critical graphs that have an end-vertex, and we obtain sharp bounds on their maximum diameter. In Section 3 we show that the maximum diameter of a k-$\gamma_{\rm pr}$-vertex-critical graph is at least $\frac{3}{2}(k-2)$. For $k \leqslant 8$ we show in Section 4 that this maximum diameter is achieved.

2. Graphs with end-vertices

We can readily characterize the $\gamma_{\rm pr}$-vertex-critical graphs with end-vertices. For this purpose, we recall that the *corona* $\operatorname{cor}(H)$ of a graph H (also denoted $H \circ K_1$ in [11]) is the graph obtained from H by adding a pendant edge to each vertex of H.

Theorem 2. *Let G be a connected graph of order at least 3 with at least one end-vertex. Then G is $\gamma_{\rm pr}$-vertex-critical if and only if $G = \operatorname{cor}(H)$ for some connected graph H satisfying $\alpha'(H) = \alpha'(H-v)$ for every $v \in V(H)$.*

Proof. First we consider sufficiency. Suppose $G = \operatorname{cor}(H)$ for some connected graph H satisfying $\alpha'(H) = \alpha'(H-v)$ for every $v \in V(H)$. Since every minimal PDS contains every support vertex in the graph, and since $S(G) = V(H)$,

$$\gamma_{\rm pr}(G) = 2\alpha'(H) + 2(|V(H)| - 2\alpha'(H)) = 2(|V(H)| - \alpha'(H)). \tag{1}$$

To show that G is $\gamma_{\rm pr}$-vertex-critical, let $u \in V(G) - S(G)$. Then $d_G(u) = 1$ and u is adjacent to a unique vertex v of H. Let M_v be a maximum matching in $H - v$. Then $|M_v| = \alpha'(H-v) = \alpha'(H)$. Let V_1 be the set of vertices in H incident with an edge of M_v and let $V_2 = V(H) \setminus (V_1 \cup \{v\})$. Then $|V_1| = 2\alpha'(H)$, $|V_2| = |V(H)| - 2\alpha'(H) - 1$ and V_2 is an independent set. Let V_2' be the set of end-vertices of G dominated by V_2; thus, $|V_2'| = |V_2|$. Notice that since H is a connected graph, v is adjacent to at least one other vertex of H. Therefore, $(V(H) \setminus \{v\}) \cup V_2'$ is a PDS of $G - u$, so that

$$\gamma_{\rm pr}(G-u) \leqslant |V(H)|-1+|V_2| = 2(|V(H)|-\alpha'(H))-2 = \gamma_{\rm pr}(G)-2 \leqslant \gamma_{\rm pr}(G-u). \tag{2}$$

Hence equality holds throughout the inequality chain (2) and by Observation 1, G is $\gamma_{\rm pr}$-vertex-critical. This establishes sufficiency.

Next we consider necessity. Suppose that G is a $\gamma_{\rm pr}$-vertex-critical graph that contains an end-vertex. Let v' be an end-vertex and let v be its neighbor. Suppose there exists $w \in N(v) \setminus \{v'\}$ with $w \notin S(G)$. Then by Observation 1, there is a $\gamma_{\rm pr}(G-w)$-set not containing v, but since v is a support vertex in $G-w$, the vertex v belongs to every $\gamma_{\rm pr}(G-w)$-set, a contradiction. Thus each vertex in $N(v) \setminus \{v'\}$ is a support vertex. It follows that $G = \mathrm{cor}(H)$ for some connected graph H. Thus, as in (1), $\gamma_{\mathrm{pr}}(G) = 2(|V(H)| - \alpha'(H))$.

It remains for us to show that $\alpha'(H) = \alpha'(H - v)$ for every $v \in V(H)$. Let $v \in V(H)$ and let u be the end-vertex adjacent to v. Let M_v be a maximum matching in $H - v$. Then $|M_v| = \alpha'(H - v)$. Let V_1 be the set of vertices in H incident with an edge of M_v and let $V_2 = V(H) \setminus (V_1 \cup \{v\})$. Then $|V_1| = 2\alpha'(H-v)$, $|V_2| = |V(H)| - 2\alpha'(H - v) - 1$ and V_2 is an independent set. Let V_2' be the set of end-vertices dominated by V_2; thus, $|V_2'| = |V_2|$. Let $S = (V(H) \setminus \{v\}) \cup V_2'$. Then S is a minimum PDS of $G - u$. Hence, $\gamma_{\mathrm{pr}}(G - u) = |S| = |V(H)| - 1 + |V_2| = 2(|V(H)| - \alpha'(H - v)) - 2$. However, since G is a γ_{pr}-vertex-critical graph, $\gamma_{\mathrm{pr}}(G-u) = \gamma_{\mathrm{pr}}(G) - 2 = 2(|V(H)| - \alpha'(H)) - 2$. Consequently, $\alpha'(H) = \alpha'(H-v)$, as desired. □

We remark that there are infinite families of connected graphs H satisfying $\alpha'(H) = \alpha'(H - v)$ for every $v \in V(H)$. For example, let H be any hamiltonian graph of odd order. We observe further that the diameter of such graphs H cannot be too large.

Proposition 3. *If H is a connected graph satisfying $\alpha'(H) = \alpha'(H-v)$ for every $v \in V(H)$, then every maximum matching in $H - v$ matches every neighbor of v. In particular, H is a 2-edge-connected graph.*

P r o o f. Suppose that $H - v$ contains a maximum matching M that does not match a neighbor u of v. Then $M \cup \{uv\}$ is a matching in H, and so $\alpha'(H) \geqslant |M| + 1 = \alpha'(H - v) + 1$, a contradiction. Hence every maximum matching in $H - v$ matches every neighbor of v.

Suppose that H has a bridge $e = uv$. Let H_u and H_v be the two components of $H-e$, where $u \in V(H_u)$ and $v \in V(H_v)$. Then $\alpha'(H) \geqslant \alpha'(H_u)+\alpha'(H_v)$. Since every maximum matching of $H-u$ matches every neighbor of u, the vertex v is matched in every maximum matching of $H-u$. This implies that $\alpha'(H_v - v) = \alpha'(H_v) - 1$. But then $\alpha'(H) = \alpha'(H - v) = \alpha'(H_u) + \alpha'(H_v - v) = \alpha'(H_u) + \alpha'(H_v) - 1$, producing a contradiction. Hence, H is 2-edge-connected. □

Proposition 4. *If H is a connected graph of order n satisfying $\alpha'(H) = \alpha'(H-v)$ for every $v \in V(H)$, then $\mathrm{diam}(H) \leqslant \frac{1}{2}(n-1)$.*

P r o o f. We proceed by induction on the number of blocks $b(H)$ in H. Suppose $b(H) = 1$. Let u and v be two diametrical vertices in H, and so $\mathrm{diam}(H) = d(u, v)$. Since H is 2-connected, every two vertices of H lie on a common cycle of H. In particular, there is a cycle C containing u and v. Hence, $|V(C)| \geqslant 2d(u,v) = 2\,\mathrm{diam}(H)$. On the one hand, if $|V(C)| \geqslant 2\,\mathrm{diam}(H) + 1$, then $n \geqslant |V(C)| \geqslant$

$2\operatorname{diam}(H)+1$. On the other hand, suppose $|V(C)| = 2\operatorname{diam}(H)$. Since $\alpha'(H) = \alpha'(H-w)$ for every $w \in V(H)$, the graph H is not a hamiltonian graph of even order. Thus H contains at least one vertex not on C, implying that $n \geqslant |V(C)|+1 = 2\operatorname{diam}(H)+1$. In both cases, $n \geqslant 2\operatorname{diam}(H)+1$, or, equivalently, $\operatorname{diam}(H) \leqslant \frac{1}{2}(n-1)$. This establishes the base case.

Assume that $b \geqslant 1$ and that if H' is a connected graph of order n' satisfying $b(H') \leqslant b$ and $\alpha'(H') = \alpha'(H'-v)$ for every $v \in V(H')$, then $\operatorname{diam}(H') \leqslant \frac{1}{2}(n'-1)$. Let H be a connected graph of order n satisfying $b(H) = b+1$ and $\alpha'(H) = \alpha'(H-v)$ for every $v \in V(H)$. Let B be an end-block of H and v the unique cut-vertex of H contained in B. Let $F = H - (V(B) \setminus \{v\})$. Then F is a connected graph satisfying $b(F) = b$. We proceed further with three claims.

Claim 1. $\alpha'(H) = \alpha'(B) + \alpha'(F)$.

P r o o f. We show first that $\alpha'(B) = \alpha'(B-v)$. Suppose $\alpha'(B) > \alpha'(B-v)$. Then $\alpha'(B) = \alpha'(B-v)+1$ and every maximum matching of B matches the vertex v. Let $e = uv$ be an edge of such a maximum matching M_B of B. Then $M_B \setminus \{e\}$ is a maximum matching of $B-v$ that does not match the vertex u. But every maximum matching of $B-v$ can be extended to a maximum matching of $H-v$ by adding to it the edges of a maximum matching of $F-v$. Hence we have shown that there is a maximum matching of $H-v$ that does not match the neighbor u of v, contradicting Proposition 3. Hence, $\alpha'(B) = \alpha'(B-v)$. Similarly, $\alpha'(F) = \alpha'(F-v)$. Thus since the graph H is $\gamma_{\rm pr}$-vertex-critical, $\alpha'(H) = \alpha'(H-v) = \alpha'(B-v) + \alpha'(F-v) = \alpha'(B) + \alpha'(F)$, as claimed. □

Claim 2. $\operatorname{diam}(F) \leqslant \frac{1}{2}(|V(F)|-1)$.

P r o o f. Let $w \in V(F)$. Then, by Claim 1, $\alpha'(B)+\alpha'(F) = \alpha'(H) = \alpha'(H-w) \leqslant \alpha'(B)+\alpha'(F-w)$, and so $\alpha'(F) \leqslant \alpha'(F-w)$. Consequently, F is a connected graph with $b(F) = b$ such that $\alpha'(F) = \alpha'(F-w)$ for every vertex $w \in V(F)$. Applying the inductive hypothesis to F, we conclude that $\operatorname{diam}(F) \leqslant \frac{1}{2}(|V(F)|-1)$. □

The proof of the following claim is similar to the proof of Claim 2 and is omitted.

Claim 3. $\operatorname{diam}(B) \leqslant \frac{1}{2}(|V(B)|-1)$.

The desired upper bound on the diameter of H now follows readily from Claims 2 and 3 and the observations that $\operatorname{diam}(H) \leqslant \operatorname{diam}(B) + \operatorname{diam}(F)$ and $|V(B)| + |V(F)| = n+1$. This completes the proof of Proposition 4. □

As a consequence of Theorem 2 and Propositions 3 and 4, we have the following results.

Theorem 5. *No tree is $\gamma_{\rm pr}$-vertex-critical.*

Theorem 6. *If G is a connected $\gamma_{\rm pr}$-vertex-critical graph with at least one end-vertex, then $\operatorname{diam}(G) \leqslant \frac{1}{2}(\gamma_{\rm pr}(G)+2)$, and this bound is sharp.*

Proof. By Theorem 2, $G = \operatorname{cor}(H)$ for some connected graph H satisfying $\alpha'(H) = \alpha'(H-v)$ for every $v \in V(H)$. Hence, $\operatorname{diam}(G) = 2 + \operatorname{diam}(H)$. Suppose $\gamma_{\rm pr}(G) = k$. Since H does not have a perfect matching, $|V(H)| \leqslant k-1$. By Proposition 4, $\operatorname{diam}(H) \leqslant \frac{1}{2}(|V(H)|-1) \leqslant \frac{1}{2}(k-2)$. Hence, $\operatorname{diam}(G) = 2 + \operatorname{diam}(H) \leqslant 2 + \frac{1}{2}(k-2) = \frac{1}{2}(k+2)$. To see that this bound is sharp, take $H = C_{k-1}$. □

3. $\gamma_{\rm pr}$-VERTEX-CRITICAL GRAPHS WITH LARGE DIAMETER

In this section we provide a construction of $\gamma_{\rm pr}$-vertex-critical graphs with large diameter. First we give a way of constructing a $\gamma_{\rm pr}$-vertex-critical graph from two smaller $\gamma_{\rm pr}$-vertex-critical graphs.

Lemma 7. *Let F and H be a j-$\gamma_{\rm pr}$-vertex-critical and a k-$\gamma_{\rm pr}$-vertex-critical graph, respectively, with minimum degrees at least two, and let G be a graph formed by identifying a vertex of F with a vertex of H. If $\gamma_{\rm pr}(G) = j+k-2$, then G is $\gamma_{\rm pr}$-vertex-critical.*

Proof. Note that since $\delta(F) \geqslant 2$ and $\delta(H) \geqslant 2$, $S(G) = \emptyset$. Label the identified vertex v. Let $u \in V(G)$. Without loss of generality, $u \in V(F)$. Since F is j-$\gamma_{\rm pr}$-vertex-critical, $\gamma_{\rm pr}(F-u) = j-2$. If $u \neq v$, then every $\gamma_{\rm pr}(F-u)$-set dominates v and can be extended to a PDS of $G-u$ by adding to it $\gamma_{\rm pr}(H-v) = k-2$ vertices from $H-v$. Hence, $\gamma_{\rm pr}(G-u) \leqslant j-2+k-2 = \gamma_{\rm pr}(G)-2$. If $u = v$, then $\gamma_{\rm pr}(G-v) = \gamma_{\rm pr}(F-v) + \gamma_{\rm pr}(H-v) = j-2+k-2 = \gamma_{\rm pr}(G)-2$. Thus, $\gamma_{\rm pr}(G-u) < \gamma_{\rm pr}(G)$ and G is $\gamma_{\rm pr}$-vertex-critical. □

Next we establish a lower bound on the maximum diameter of a k-$\gamma_{\rm pr}$-vertex-critical graph. For this purpose, following the notation of Goddard et al. [10] we define a graph as *pointed* if there are two designated diametrical vertices called LEFT and RIGHT. Then, for two pointed graphs G and H, we define $G \circ H$ as the pointed graph obtained by identifying and undesignating the RIGHT-vertex from G and the LEFT-vertex from H. Note that the operator $\circ$ is associative.

For a graph $G = (V,E)$ with $\operatorname{diam}(G) = d$ we also define the following subsets of V, and use this notation throughout the rest of the paper. Fix a diametrical vertex v of G. For $i = 0, 1, \ldots, d$, define

$$(3) \qquad V_i = \{u \in V\colon\ d(u, v) = i\}, \quad V_{\leqslant i} = \bigcup_{j=0}^{i} V_j \quad \text{and} \quad V_{\geqslant i} = \bigcup_{j=i}^{d} V_j.$$

Note that $V_0 = \{v\}$ and $V_1 = N(v)$.

Theorem 8. *For every even integer $k \geqslant 4$ there exists a connected k-$\gamma_{\rm pr}$-vertex-critical graph of diameter $\frac{3}{2}(k-2)$.*

P r o o f. We begin by constructing a 4-$\gamma_{\rm pr}$-vertex-critical graph with diameter 3. Let H_1 be a copy of P_4 and let H_2 be a copy of $\overline{H_1}$. Let F be the pointed graph obtained from $H_1 \cup H_2$ by adding all edges between H_1 and H_2 except for a perfect matching between the corresponding vertices of H_1 and H_2, and then adding two new vertices, LEFT and RIGHT, such that LEFT is joined to every vertex in H_1 and RIGHT is joined to every vertex in H_2. The graph F is shown in Fig. 1 where for clarity we omit the edges between H_1 and H_2. Then F is 4-$\gamma_{\rm pr}$-vertex-critical with diameter 3.

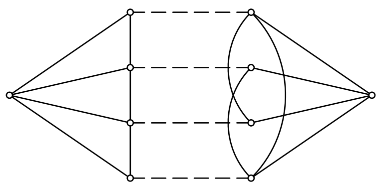

Figure 1. The 4-$\gamma_{\rm pr}$-vertex-critical graph F of diameter 3

For $q \geqslant 1$ define the pointed graph $G_q = F \circ F \circ \ldots \circ F$ for q copies of F. Then $\operatorname{diam}(G_q) = 3q$. We show that G_q is a $2(q+1)$-$\gamma_{\rm pr}$-vertex-critical graph. We proceed by induction on q. When $q = 1$, then $G_q = F$ which is a 4-$\gamma_{\rm pr}$-vertex-critical graph. This establishes the base case. Assume then that $q \geqslant 2$ and that $G_{q'}$ is a $2(q'+1)$-$\gamma_{\rm pr}$-vertex-critical graph for $1 \leqslant q' < q$. We now consider the graph G_q.

The graph G_q is the pointed graph obtained from the pointed graphs F and G_{q-1}; that is, $G_q = F \circ G_{q-1}$, where F is a 4-$\gamma_{\rm pr}$-vertex-critical graph and, by induction, G_{q-1} is a $2q$-$\gamma_{\rm pr}$-vertex-critical graph. Let F_1 denote the first copy of F in G_q, and let v and w denote the LEFT-vertex and RIGHT-vertex from F_1.

The vertex v is a diametrical vertex of G_q. Let $d = \operatorname{diam}(G_q) = 3q$. As in (3), $V_0 = \{v\}$ and $V_1 = N(v)$. Further, $V_3 = \{w\}$, while V_2 is the neighborhood of w in F_1 and V_4 is the neighborhood of w in G_{q-1}.

Among all $\gamma_{\rm pr}(G_q)$-sets, let S be one which contains as few vertices of $V_{\leqslant 2}$ as possible. To dominate V_0, we have that $|S \cap V_{\leqslant 2}| \geqslant 2$. Suppose that $|S \cap V_{\leqslant 2}| \geqslant 3$.

Then $|S \cap V_{\leqslant 3}| \geqslant 4$. Note that $V_4 \not\subseteq S$, otherwise, if $x, x' \in V_4$ are partners in S, then $S \setminus \{x, x'\}$ is a PDS of G_q of smaller cardinality than S, which is impossible. Replacing the vertices in $S \cap V_{\leqslant 3}$ by the two central vertices of the P_4 in $G_q[V_1]$ and the vertex w, and then adding to the resulting set a neighbor of w from V_4 (to serve as a partner of w) produces a new $\gamma_{\rm pr}(G_q)$-set that contains fewer vertices from $V_{\leqslant 2}$ than does S, contradicting our choice of S. Hence, $|S \cap V_{\leqslant 2}| = 2$. If follows that $S \cap V_{\geqslant 3}$ is a PDS of G_{q-1} and that $|S \cap V_{\geqslant 3}| = \gamma_{\rm pr}(G_q) - 2$. Hence, $\gamma_{\rm pr}(G_{q-1}) \leqslant \gamma_{\rm pr}(G_q) - 2$. Every $\gamma_{\rm pr}(G_{q-1})$-set can easily be extended to a PDS of G_q by adding to it two vertices (namely, the two central vertices of the P_4 in $G_q[V_1]$), and so $\gamma_{\rm pr}(G_q) \leqslant \gamma_{\rm pr}(G_{q-1}) + 2$. Consequently, $\gamma_{\rm pr}(G_q) = \gamma_{\rm pr}(G_{q-1}) + 2 = \gamma_{\rm pr}(F) + \gamma_{\rm pr}(G_{q-1}) - 2$. Hence, by Lemma 7, G_q is $\gamma_{\rm pr}$-vertex-critical. By induction, $\gamma_{\rm pr}(G_{q-1}) = 2q$, and so G_q is a k-$\gamma_{\rm pr}$-vertex-critical graph where $k = 2(q+1)$ with $\operatorname{diam}(G_q) = 3q = \frac{3}{2}(k-2)$. □

4. Bounds on the Diameter

In this section we establish bounds on the diameter of a connected k-$\gamma_{\rm pr}$-vertex-critical graph. First we mention a sufficient condition for a graph not to be $\gamma_{\rm pr}$-vertex-critical. (We assume in what follows that G has no end-vertex, for otherwise we have the upper bound given in Theorem 6.)

Proposition 9 ([5, Proposition 5.4]). *If a graph G has nonadjacent vertices u and v with $N(u) \subseteq N(v)$, then G is not a $\gamma_{\rm pr}$-vertex-critical graph.*

We provide next a trivial upper bound on the diameter of a k-$\gamma_{\rm pr}$-vertex-critical graph. Throughout this section, for a graph $G = (V, E)$ and a vertex $x \in V$, we let S_x denote a $\gamma_{\rm pr}(G - x)$-set.

Proposition 10. *The diameter of a connected k-$\gamma_{\rm pr}$-vertex-critical G graph with $\operatorname{diam}(G) = d$ is at most $2k - 8 + (d \bmod 4)$.*

Proof. Let v be a diametrical vertex of G and let $d = \operatorname{diam}(G)$. As in (3), $V_0 = \{v\}$ and $V_1 = N(v)$. By Observation 1, $|S_v| = k - 2$ and $S_v \cap V_1 = \emptyset$. Hence to dominate V_1, $|S_v \cap V_2| \geqslant 1$. In fact, by Proposition 9, $|S_v \cap V_2| \geqslant 2$. Thus, $S = S_v \cup \{v, v_1\}$ is a $\gamma_{\rm pr}(G)$-set for any $v_1 \in V_1$ and $|S \cap (V_0 \cup V_1 \cup V_2)| \geqslant 4$. For any $i \geqslant 3$, $|S \cap (V_i \cup \ldots \cup V_{i+3})| \geqslant 2$. It follows that if $d = 2 + 4j + r$ where $0 \leqslant r \leqslant 3$, then $k = |S| \geqslant 4 + 2j$ if $r \in \{0, 1\}$ while $k \geqslant 4 + 2j + 2$ if $r \in \{2, 3\}$. The desired result now follows from simple algebra. □

Since $d \bmod 4 \in \{0, 1, 2, 3\}$, as an immediate consequence of Proposition 10 we have the following result.

Corollary 11. *The diameter of a connected k-$\gamma_{\rm pr}$-vertex-critical graph G is at most $2k-5$ with inequality if* $\operatorname{diam}(G) \not\equiv 3 \pmod 4$.

As an immediate consequence of Theorem 8, we have the following result.

Corollary 12. *The maximum diameter of a connected k-$\gamma_{\rm pr}$-vertex-critical graph is at least $\frac{3}{2}(k-2)$.*

Next we establish a sharp upper bound on the diameter of a connected k-$\gamma_{\rm pr}$-vertex-critical graph for small k. Recall that for a graph $G=(V,E)$ and sets $S,T \subseteq V$, we say that S *paired-dominates* T if S dominates T in G and $G[S]$ contains a perfect matching.

Theorem 13. *For $k \leqslant 8$, the diameter of a connected k-$\gamma_{\rm pr}$-vertex-critical graph is at most $\frac{3}{2}(k-2)$.*

Proof. Let $G=(V,E)$ be a connected k-$\gamma_{\rm pr}$-vertex-critical graph. If $\delta(G)=1$, then the upper bounds follow from Theorem 6. Hence we may assume in what follows that $\delta(G) \geqslant 2$. We will show that the diameter of G is at most the value given in Tab. 1.

k	4	6	8
$\operatorname{diam}(G)$	3	6	9

Table 1. The maximum value od $\operatorname{diam}(G)$ for $k \leqslant 8$.

If $k=4$, then the upper bound follows from Corollary 11. Hence we may assume $\delta(G) \geqslant 2$ and $k \geqslant 6$. Let v be a diametrical vertex of G and let $d = \operatorname{diam}(G)$. For $S,T \subseteq V$ we write $S \succ_{\rm pr} T$ if S paired-dominates T in G. Furthermore, we write $S \mapsto_{\rm pr} T$ if $S \cap T \succ_{\rm pr} T$. As before, for $x \in V$, let S_x be a $\gamma_{\rm pr}(G-x)$-set.

Suppose that $k=6$ and assume that $d \geqslant 7$. Again using the notation defined in (3), let $u \in V_1$; then $|S_u|=4$. To paired-dominate $V_0 \cup V_3 \cup V_4 \cup V_7$, it follows that $d=7$ and $|S_u \cap V_j| = 1$ for $j \in \{1,2,5,6\}$. Thus, $|S_u \cap V_{\geqslant 4}| = 2$ and $S_u \mapsto_{\rm pr} V_{\geqslant 4}$. By symmetry, for $w \in V_6$ it follows that $|S_w \cap V_{\leqslant 3}| = 2$ and $S_w \mapsto_{\rm pr} V_{\leqslant 3}$. Therefore, $(S_u \cap V_{\geqslant 4}) \cup (S_w \cap V_{\leqslant 3})$ is a PDS of G of cardinality 4, which contradicts $\gamma_{\rm pr}(G)=6$. Hence, if $k=6$, then $d \leqslant 6$, as desired.

Suppose that $k=8$ and assume that $d \geqslant 10$. Let $u \in V_1$; then $|S_u|=6$. To paired-dominate $V_{\leqslant 2}$, we must have $|S_u \cap V_{\leqslant 2}| \geqslant 2$, while to paired-dominate $V_{\geqslant 8}$, we must have $|S_u \cap V_{\geqslant 8}| \geqslant 2$. Hence, to paired-dominate $V_4 \cup V_5 \cup V_6$, we must have $|S_u \cap V_5| \geqslant 1$ and $|S_u \cap (V_4 \cup V_5 \cup V_6)| \geqslant 2$. Hence, $|S_u \cap V_{\leqslant 2}| = 2$, $|S_u \cap V_{\geqslant 8}| = 2$, $|S_u \cap V_5| \geqslant 1$ and $|S_u \cap (V_4 \cup V_5 \cup V_6)| = 2$. In particular, $|S_u \cap V_{\geqslant 4}| = 4$ and $S_u \mapsto_{\rm pr} V_{\geqslant 4}$.

Let $w \in V_9$. By symmetry, $|S_w \cap V_{\leqslant 2}| = 2$, $|S_w \cap V_{\geqslant 8}| = 2$, $|S_w \cap V_5| \geqslant 1$ and $|S_w \cap (V_4 \cup V_5 \cup V_6)| = 2$. In particular, $|S_w \cap V_{\leqslant 6}| = 4$ and $S_w \mapsto_{\rm pr} V_{\leqslant 6}$. If $S_u \cap V_6 = \emptyset$, then $S_u \mapsto_{\text{pr}} V_{\geqslant 7}$, and $(S_w \cap V_{\leqslant 6}) \cup (S_u \cap V_{\geqslant 8})$ is a PDS of G of cardinality 6, which contradicts $\gamma_{\text{pr}}(G) = 8$. Thus we may assume that $|S_u \cap V_6| = 1$, and so $|S_u \cap V_5| = 1$; similarly, $|S_w \cap V_4| = |S_w \cap V_5| = 1$.

Let $x \in V_5$. Then, as before, $|S_x \cap V_{\leqslant 2}| = 2$ and $|S_x \cap V_{\geqslant 8}| = 2$. Suppose there is another vertex in V_5. Then $|S_x \cap V_5| \geqslant 1$ and $|S_x \cap (V_4 \cup V_5 \cup V_6)| = 2$. Without loss of generality, $S_x \cap V_4 = \emptyset$, and so $S_x \mapsto_{\text{pr}} V_{\leqslant 3}$. Therefore $(S_x \cap V_{\leqslant 2}) \cup (S_u \cap V_{\geqslant 4}) \succ_{\text{pr}} V$, which contradicts $\gamma_{\text{pr}}(G) = 8$. Hence there is no other vertex in V_5. But then S_x contains at least one vertex in each of the sets $V_0 \cup V_1$ (to dominate V_0), $V_3 \cup V_4$ (to dominate V_4), $V_6 \cup V_7$ (to dominate V_6), and $V_9 \cup V_{10}$ (to dominate V_{10}). Thus, S_x contains four vertices that are pairwise nonadjacent, implying that $|S_x| \geqslant 8$, a contradiction. Hence, if $k = 8$, then $d \leqslant 9$, as desired. □

We close with the following question about the maximum diameter of a connected γ_{pr}-vertex-critical graph.

Question 1. If G is a connected γ_{pr}-vertex-critical graph, then is it true that

$$\operatorname{diam}(G) \leqslant \frac{3}{2}(\gamma_{\text{pr}}(G) - 2)?$$

Note that by Theorem 13, Question 1 is true for $\gamma_{\text{pr}}(G) \leqslant 8$. By Corollary 12, if Question 1 is true, then this bound is sharp.

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