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ON LIPSCHITZ AND D.C. SURFACES OF FINITE CODIMENSION IN A BANACH SPACE

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Abstract. Properties of Lipschitz and d.c. surfaces of finite codimension in a Banach space and properties of generated σ-ideals are studied. These σ-ideals naturally appear in the differentiation theory and in the abstract approximation theory. Using these properties, we improve an unpublished result of M. Heisler which gives an alternative proof of a result of D. Preiss on singular points of convex functions.

Keywords: Banach space, Lipschitz surface, d.c. surface, multiplicity points of monotone operators, singular points of convex functions, Aronszajn null sets

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1. Introduction

Let X be a real separable Banach space. A number of σ-ideals of subsets of X have been considered in literature. Besides the most classical system of first category sets let us mention the σ-ideals of Haar null sets, Aronszajn (equivalently Gaussian) null sets (see [2]), Γ-null sets (see [12], [11]) and σ-(lower or upper) porous sets (see e.g. [21]). In some questions of the differentiability theory and of the abstract approximation theory, the σ-ideals $\mathcal{L}^1(X)$ and $\mathcal{DC}^1(X)$ generated by Lipschitz and d.c. Lipschitz hypersurfaces (i.e., "graphs" of Lipschitz and of d.c. Lipschitz functions), respectively, naturally appear. These σ-ideals are proper subsystems of all σ-ideals mentioned above. The sets from $\mathcal{L}^1(X)$ were used in $\mathbb{R}^2$ (under a different but equivalent definition) by W. H. Young (under the name "ensemble ridée") and by H. Blumberg (under the name "sparse set"); cf. [20, p. 294]. These sets were used in $\mathbb{R}^n$ e.g. (implicitly) by P. Erdös [4], and in infinite-dimensional spaces (possibly for the first time) in [18] and [17]. The sets from $\mathcal{DC}^1(X)$ were probably first applied in [19] (cf. [2, p. 93]). In some articles (e.g., [18], [19], [20], [15]), also sets from smaller

σ-ideals $\mathcal{L}^n(X)$ and $\mathcal{DC}^n(X)$ generated by Lipschitz and d.c. Lipschitz surfaces of codimension $n > 1$ were used.

In the present article we prove some properties of Lipschitz and Lipschitz locally d.c. surfaces of finite codimension (Section 3; Proposition 3.6 and Proposition 3.7).

Using these properties, we study in Section 4 sets which are projections of sets from $\mathcal{L}^n(X)$ on a closed space $Y \subset X$ of codimension $d < n$. The study of such projections was suggested by D. Preiss in connection with a result of [13] (see Remark 4.7(i)). M. Heisler [7] proved that any such projection is a first category set in Y, which provides (together with a result of [19]) an alternative proof of a result of [13]. We prove that each such projection is also a subset of an Aronszajn null set in Y (and even a subset of a set from a smaller class $\mathcal{C}_n^*$). As a consequence, we obtain a result on projections of sets of multiplicity of monotone operators (Theorem 4.9) which improves both [13, Theorem 1.3] and the corresponding result of [7].

Our proof is more transparent than that in [7] and gives stronger results, since it uses "perturbation" Proposition 3.7. To prove (and apply) it, we need some results on perturbations of finite-dimensional subspaces. These results are collected in Preliminaries, where also needful results on d.c. mappings are recalled.

2. Preliminaries

We consider only real Banach spaces. By $\mathrm{sp}\{M\}$ we denote the linear span of the set M. A mapping is called K-Lipschitz if it is Lipschitz with a (not necessarily minimal) constant K. A bijection f is called bilipschitz (K-bilipschitz) if both f and f^{-1} are Lipschitz (K-Lipschitz).

A real function on an open convex subset of a Banach space is called d.c. (delta-convex) if it is a difference of two continuous convex functions. Hartman's notion of d.c. mappings between Euclidean spaces [6] was generalized and studied in [16].

Definition 2.1. Let X, Y be Banach spaces, $C \subset X$ an open convex set, and let $F\colon C \to Y$ be a continuous mapping. We say that F is d.c. if there exists a continuous convex function $f\colon C \to \mathbb{R}$ such that $y^* \circ F + f$ is convex whenever $y^* \in Y^*$, $\|y^*\| \leqslant 1$.

It is easy to see (cf. [16, Corollary 1.8.]) that, if Y is finite dimensional, then F is d.c. if and only if $y^* \circ F$ is d.c. for each $y^* \in Y^*$ (or for each y^* from a fixed basis of Y^*). Note also that each d.c. mapping is locally Lipschitz ([16, p. 10]). If X is finite-dimensional, then each locally d.c. mapping is d.c. (see [16, p. 14]) but it is not true (see [9]) if X is infinite dimensional. We will need also the following well-known facts on d.c. mappings.

Lemma 2.2. *Let X, X_1, Y, Y_1, Y_2, Z be Banach spaces.*

(i) *Let $f\colon X \to Y$ be d.c. and let $g\colon X_1 \to X$, $h\colon Y \to Y_1$ be linear and continuous. Then both $f \circ g$ and $h \circ f$ are d.c.*

(ii) *A mapping $f = (f_1, f_2)\colon X \to Y_1 \times Y_2$ is d.c. if and only if both f_1 and f_2 are d.c.*

(iii) *If $g\colon X \to Y$, $h\colon X \to Y$ are d.c. and $a, b \in \mathbb{R}$, then $ag + bh$ is d.c.*

(iv) *If $f\colon X \to Y$ is locally d.c. and $g\colon Y \to Z$ is locally d.c., then $g \circ f$ is locally d.c.*

(v) *Suppose that $G\colon X \to Y$ is a linear isomorphism, $g\colon X \to Y$ is a locally d.c. bilipschitz bijection, and the range of $g - G$ is contained in a finite dimensional space. Then g^{-1} is locally d.c.*

Proof. The statements (i) and (ii) are very easy (cf. [16, Lemma 1.5 and Lemma 1.7]) and (iii) follows from (i) and (ii). The statement (iv) is a special case of [16, Theorem 4.2] and (v) is a special case of [3, Theorem 2.1]. □

We will need some notions and results concerning distances of two subspaces of a Banach space, which are well-known from the perturbation theory of linear operators ([5], [8], [1]). Let X be a Banach space and $S(X)$ the unit sphere of X. Let Y and Z be closed non-trivial subspaces of X. Then the *gap between Y and Z* (called also the opening or the deviation of Y and Z) is defined by

$$\gamma(Y, Z) = \max\Big\{ \sup_{y \in Y \cap S(X)} \operatorname{dist}(y, Z), \sup_{z \in Z \cap S(X)} \operatorname{dist}(z, Y) \Big\}.$$

We set $\gamma(\{0\}, \{0\}) := 0$ and $\gamma(Y, Z) = 1$ if one and only one of Y, Z is $\{0\}$. The gap need not be a metric on the set of all non-trivial subspaces of X; this property is possessed by the distance $\varrho(Y, Z)$ between Y and Z defined as the Hausdorff distance between $Y \cap S(X)$ and $Z \cap S(X)$.

We will work with the gap $\gamma(Y, Z)$. However, since it is easy to prove (see e.g., [8]) that (for nontrivial Y, Z) always

$$\tfrac{1}{2}\varrho(Y, Z) \leqslant \gamma(Y, Z) \leqslant \varrho(Y, Z), \tag{2.1}$$

we could work also with $\varrho(Y, Z)$. We will need the following well-known facts.

Lemma 2.3. *Let X be a Banach space and let F, $\tilde{F}$, K be finite dimensional subspaces of X. Then:*

(i) *If $\gamma(F, \tilde{F}) < 1$, then $\dim F = \dim \tilde{F}$.*

(ii) *If $F \cap K = \{0\}$, then there exists $\omega > 0$ such that $\gamma(F, \tilde{F}) < \omega$ implies $\tilde{F} \cap K = \{0\}$.*

(iii) *If* $E \oplus F = X$, *then there exists* $\omega > 0$ *such that* $\gamma(F, \tilde{F}) < \omega$ *implies* $E \oplus \tilde{F} = X$.

Proof. The statement (i) is proved in [5] (see [1, Theorem 2.1]) and (ii) is an easy consequence of (2.1). (We can also apply [1, Theorem 5.2] with $Y := F$, $Z := K$ and $X := F \oplus K$.) The statement (iii) immediately follows from [1, Theorem 5.2]. □

The following simple lemma is also essentially well-known. Although it is not stated explicitly in [10], it follows from [10, Theorem 2.2] which works with complex Banach spaces. Since the formulation of [10, Theorem 2.2] is rather complicated and we work with real spaces, for the sake of completeness we give a proof.

Lemma 2.4. *Let* X *be a Banach space, let* $(v_1, \ldots, v_n)$ *be a basis of a space* $V \subset X$ *and* $\varepsilon > 0$. *Then there exists* $\delta > 0$ *such that the inequalities* $\|w_1 - v_1\| < \delta, \ldots, \|w_n - v_n\| < \delta$ *imply that* $W := \mathrm{sp}\{w_1, \ldots, w_n\}$ *is* n*-dimensional and* $\gamma(V, W) < \varepsilon$.

Proof. First we will show that there exists $\eta > 0$ and $\delta^* > 0$ such that the inequality

$$\left\| \sum_{i=1}^{n} c_i w_i \right\| \geqslant \eta \|c\|_\infty \tag{2.2}$$

holds whenever $\|w_1 - v_1\| < \delta^*, \ldots, \|w_n - v_n\| < \delta^*$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ is arbitrary. To this end observe that there exists $\eta^* > 0$ such that (2.2) holds for $\eta = \eta^*$, $w_i = v_i$ and arbitrary c. Put $\eta := \eta^*/2$ and $\delta^* := \eta^*/2n$. Then the inequalities $\|w_1 - v_1\| < \delta^*, \ldots, \|w_n - v_n\| < \delta^*$ imply that, for each $0 \neq c \in \mathbb{R}^n$,

$$\left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_\infty} v_i \right\| - \left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_\infty} w_i \right\| \leqslant \left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_\infty} (v_i - w_i) \right\| < n\delta^* = \eta^*/2.$$

Consequently, using the definition of η^*, we obtain

$$\left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_\infty} w_i \right\| \geqslant \left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_\infty} v_i \right\| - \eta^*/2 \geqslant \eta^* - \eta^*/2 = \eta,$$

which implies (2.2).

Now set $\delta := \min\{\delta^*, \varepsilon\eta/2n\}$ and suppose that the inequalities $\|w_1 - v_1\| < \delta, \ldots, \|w_n - v_n\| < \delta$ hold. Let $w = \sum\limits_{i=1}^{n} c_i w_i$ with $\|w\| = 1$ be given. Set $v = \sum\limits_{i=1}^{n} c_i v_i$. Since $\|c\|_\infty \leqslant 1/\eta$ by (2.2), we obtain

$$\|v - w\| \leqslant \sum_{i=1}^{n} |c_i| \delta \leqslant n(1/\eta)\delta \leqslant \varepsilon/2.$$

Consequently, $\sup\limits_{w\in W\cap S(X)} \operatorname{dist}(w,V) < \varepsilon$. In a quite symmetrical way we obtain $\sup\limits_{v\in V\cap S(X)} \operatorname{dist}(v,W) < \varepsilon$, so $\gamma(V,W) < \varepsilon$. Since we can suppose $\varepsilon < 1$, we know that W is n-dimensional by Lemma 2.3(i). □

Lemma 2.5. *Let X, Y be Banach spaces and $F\colon X \to Y$ a linear isomorphism. Then there exists $C > 0$ such that*

$$C^{-1}\gamma(F(V),F(W)) \leqslant \gamma(V,W) \leqslant C\gamma(F(V),F(W))$$

whenever V and W are subspaces of X.

Proof. We can clearly suppose that V and W are non-trivial. Since F^{-1} is also a linear isomorphism, it is clearly sufficient to find $D > 0$ such that $\gamma(V,W) \leqslant D\gamma(F(V),F(W))$ always holds. Choose $K > 0$ such that F is K-bilipschitz and consider $v \in V$ with $\|v\| = 1$. We can clearly find $\tilde{w} \in F(W)$ for which $\|\tilde{w} - \|F(v)\|^{-1}\cdot F(v)\| \leqslant 2\gamma(F(V),F(W))$. Since $\|F(v)\| \leqslant K$, we have $\|F(v) - \|F(v)\|\cdot\tilde{w}\| \leqslant 2K\gamma(F(V),F(W))$, and therefore $\|v - \|F(v)\|\cdot F^{-1}(\tilde{w})\| \leqslant 2K^2\gamma(F(V),F(W))$. Since the roles of V and W are symmetric, we can clearly set $D := 2K^2$. □

Lemma 2.6. *Let X be an infinite dimensional Banach space, $V,W \subset X$ non-trivial finite dimensional spaces, and $\delta > 0$. Then there exists a space $\tilde{V} \subset X$ with $\gamma(V,\tilde{V}) < \delta$ and $\tilde{V} \cap W = \{0\}$.*

Proof. Denote $n := \dim V$, choose an n-dimensional space $Y \subset X$ with $Y \cap (V + W) = \{0\}$ and a linear bijection $L\colon V \to Y$. For $t > 0$, set $\tilde{V}_t := \{v + tL(v)\colon v \in V\}$. It is easy to check that each $\tilde{V}_t$ is an n-dimensional space with $\tilde{V}_t \cap W = \{0\}$. Applying Lemma 2.4 to a basis $v_1,\dots,v_k$ of V and $w_i := v_i + tL(v_i)$, it is easy to see that $\gamma(V,\tilde{V}_t) \to 0$ $(t \to 0+)$, which implies our assertion. □

Lemma 2.7. *Let X be a Banach space, $1 \leqslant n < \dim X$ and $K \geqslant 1$. Let $X = E \oplus F$, where F is an n-dimensional space. Suppose that the canonical mapping $\mu\colon E \oplus F \to E \times F$ (where $E \times F$ is equipped with the maximum norm) is K-bilipschitz. Then there exists $\omega > 0$ such that if $\tilde{F} \subset X$ is a closed space with $\gamma(F,\tilde{F}) < \omega$, then $X = E \oplus \tilde{F}$ and the canonical mapping $\tilde{\mu}\colon E \oplus \tilde{F} \to E \times \tilde{F}$ is $2K$-bilipschitz.*

Proof. Distinguishing the cases $\lambda < 1$, $\lambda = 1$ and $\lambda > 1$, it is easy to check that there exists $1 > \omega_0 > 0$ such that the inequalities

$$\begin{gathered} K\max(1+\omega,\lambda) + \omega \leqslant 2K\max(1,\lambda), \\ K^{-1}\max(1-\omega,\lambda) - \omega \geqslant (2K)^{-1}\max(1,\lambda) \end{gathered} \tag{2.3}$$

hold for each $\lambda \geqslant 0$ and $0 < \omega < \omega_0$. By Lemma 2.3(iii), we can choose $0 < \omega < \omega_0$ such that $X = E \oplus \tilde{F}$ whenever $\gamma(F, \tilde{F}) < \omega$. Let $\tilde{F}$ with $\gamma(F, \tilde{F}) < \omega$ be given and consider arbitrary $\tilde{f} \in \tilde{F}$ and $e \in E$. We will prove

$$(2.4) \qquad (2K)^{-1} \max(\|\tilde{f}\|, \|e\|) \leqslant \|\tilde{f} + e\| \leqslant 2K \max(\|\tilde{f}\|, \|e\|).$$

Since the case $\tilde{f} = 0$ is trivial, by homogeneity of the norm we can suppose $\|\tilde{f}\| = 1$ and find $f \in F$ with $\|f - \tilde{f}\| < \omega$. Applying (2.3) to $\lambda := \|e\|$, we obtain

$$\begin{aligned} \|\tilde{f} + e\| \leqslant \|f + e\| + \omega &\leqslant K \max(\|f\|, \|e\|) + \omega \\ &\leqslant K \max(1 + \omega, \|e\|) + \omega \leqslant 2K \max(1, \|e\|) \end{aligned}$$

and

$$\|\tilde{f} + e\| \geqslant \|f + e\| - \omega \geqslant K^{-1} \max(1 - \omega, \|e\|) - \omega \geqslant (2K)^{-1} \max(1, \|e\|).$$

Thus, (2.4) holds, and $\tilde{\mu}$ is $(2K)$-bilipschitz. □

3. Properties of Lipschitz surfaces of finite codimension

If X is a Banach space and $X = E \oplus F$, then we denote by $\pi_{E,F}$ the projection of X to E along the space F.

Definition 3.1. Let X be a Banach space and $A \subset X$.

(i) Let F be a closed subspace of X. We say that A is an *F-Lipschitz surface* if there exists a topological complement E of F and a Lipschitz mapping $\varphi\colon E \to F$ such that $A = \{x + \varphi(x)\colon x \in E\}$.

(ii) Let $1 \leqslant n < \dim X$ be a natural number. We say that A is a *Lipschitz surface of codimension n* if A is an F-Lipschitz surface for some n-dimensional space $F \subset X$.

(iii) If we consider in (i) mappings $\varphi\colon E \to F$ which are d.c. (Lipschitz d.c., locally d.c., Lipschitz locally d.c.), we obtain the notions of an F-d.c. surface, d.c. surface of codimension n (F-Lipschitz d.c. surface, etc.). A Lipschitz surface (d.c. surface, etc.) of codimension 1 is said to be a *Lipschitz hypersurface (d.c. hypersurface, etc.)*.

(iv) The σ-ideals of sets which can be covered by countably many Lipschitz surfaces or d.c. surfaces of codimension n will be denoted by $\mathcal{L}^n(X)$ or $\mathcal{DC}^n(X)$, respectively.

Lemma 3.2. *Let X be a Banach space, $F \subset X$ a space of dimension n ($1 \leqslant n < \dim X$), and $A \subset X$. Then the following properties are equivalent.*

(i) *A is an F-Lipschitz surface (an F-d.c. surface, an F-Lipschitz d.c. surface, an F-Lipschitz locally d.c. surface).*

(ii) *There exists a topological complement $\tilde{E}$ of F such that $\tilde{\pi}|_A \colon A \to \tilde{E}$ is a bijection and $(\tilde{\pi}|_A)^{-1}$ is Lipschitz (d.c., etc.), where $\tilde{\pi} := \pi_{\tilde{E},F}$.*

(iii) *If $X = F \oplus E$ and $\pi := \pi_{E,F}$, then $\pi|_A \colon A \to E$ is a bijection and $(\pi|_A)^{-1}$ is Lipschitz (d.c., etc.).*

(iv) *If $X = F \oplus E$, then there exists a Lipschitz mapping (a d.c. mapping, etc.) $\varphi \colon E \to F$ such that $A = \{x + \varphi(x) \colon x \in E\}$.*

P r o o f. In the proof we use Lemma 2.2(i)–(iii).

If (i) holds, then there exists a topological complement $\tilde{E}$ of F and a Lipschitz (d.c., etc.) mapping $\tilde{\varphi} \colon \tilde{E} \to F$ such that $A = \{x + \tilde{\varphi}(x) \colon x \in \tilde{E}\}$. Set $\tilde{\pi} := \pi_{\tilde{E},F}$. Then clearly $\tilde{\pi}|_A \colon A \to \tilde{E}$ is a bijection and $(\tilde{\pi}|_A)^{-1}$ is Lipschitz (d.c., etc.), since $(\tilde{\pi}|_A)^{-1}(\tilde{e}) = \tilde{e} + \tilde{\varphi}(\tilde{e})$.

Now let $\tilde{E}$ be as in (ii), and let E and π be as in (iii). Since $\pi|_{\tilde{E}} \colon \tilde{E} \to E$ is clearly a linear isomorphism, $(\pi|_{\tilde{E}})^{-1} = \tilde{\pi}|_E$, $\pi|_A = (\pi|_{\tilde{E}}) \circ (\tilde{\pi}|_A)$ and $(\pi|_A)^{-1} = (\tilde{\pi}|_A)^{-1} \circ (\tilde{\pi}|_E)$, we easily obtain (iii).

Letting $\varphi(x) := (\pi|_A)^{-1}(x) - x$ for $x \in E$, we easily see that (iii) implies (iv). The implication (iv) $\Rightarrow$ (i) is trivial. □

Remark 3.3.

(i) Every Lipschitz surface of codimension n in X is clearly a closed subset of X.

(ii) If $S \subset X$ is a Lipschitz (d.c., etc.) surface of codimension $n \geqslant 2$, then S is a subset of a Lipschitz (d.c., etc.) surface of codimension $n-1$. Indeed, suppose that $S = \{x + \varphi(x) \colon x \in E\}$, where $\varphi \colon E \to F$, $X = E \oplus F$, and F is n-dimensional. Choose $0 \neq v \in F$ and write $F = \mathrm{sp}\{v\} \oplus \tilde{F}$. Set $\tilde{E} := E + \mathrm{sp}\{v\}$ and, for $x \in \tilde{E}$, define $\tilde{\varphi}(x) := \pi_{\tilde{F},\tilde{E}}(\varphi(\pi_{E,F}(x)))$. Set $\tilde{S} := \{y + \tilde{\varphi}(y) \colon y \in \tilde{E}\}$. It is easy to see that $S \subset \tilde{S}$ and $\tilde{\varphi} \colon \tilde{E} \to F$ is Lipschitz (d.c., etc.) if φ is Lipschitz (d.c., etc.).

Consequently, if $\dim X > n \geqslant 2$, then $\mathcal{L}^n(X) \subset \mathcal{L}^{n-1}(X)$. If X is separable, then this inclusion is proper, see Remark 4.8, which shows that no Lipschitz surface of codimension $n-1$ belongs to $\mathcal{L}^n(X)$ (if $\dim X < \infty$, it is sufficient to use in the obvious way the basic properties of Hausdorff dimension).

(iii) If X is separable, then the σ-ideal $\mathcal{DC}^n(X)$ coincides with the σ-ideal generated by Lipschitz d.c. surfaces (or Lipschitz locally d.c. surfaces, or locally d.c. surfaces). This easily follows from local lipschitzness of d.c. functions, from the well-known fact that each Lipschitz convex function defined on an open ball in a space E can be extended to a Lipschitz convex function on E, and from separability of X.

(iv) It is not difficult to show that $\mathcal{DC}^n(X)$ is a proper subset of $\mathcal{L}^n(X)$ (if $\dim X > n \geqslant 1$); see [17, p. 295] for $n = 1$.

Remark 3.4. Suppose that $X = E \oplus F$ and F is finite dimensional. An easy argument using local compactness of F shows that $\pi_{E,F}(A)$ is closed in E whenever A is closed and bounded in X. Consequently, $\pi_{E,F}(A)$ is an F_σ subset of E whenever A is closed in X.

We will need the following well-known easy consequence of Brouwer's Invariance of Domain Theorem. Because of the lack of a suitable reference, we present a short proof.

Lemma 3.5. *Let C, $\tilde{C}$ be Banach spaces with $0 < \dim C = \dim \tilde{C} < \infty$ and let $f\colon \tilde{C} \to C$ be an injective continuous mapping such that $f^{-1}\colon f(\tilde{C}) \to \tilde{C}$ is Lipschitz. Then $f(\tilde{C}) = C$.*

Proof. We can clearly suppose that $C = \tilde{C}$ and $X := C = \tilde{C}$ is a Euclidean space. Brouwer's Invariance of Domain Theorem implies that $f(X)$ is open in X. Let $y_n \to y$, where $y_n \in f(X)$. Then (y_n) is bounded and, since f^{-1} is Lipschitz, $(x_n) := (f^{-1}(y_n))$ is bounded as well. Choose a subsequence $x_{n_k} \to x \in X$. Then $f(x_{n_k}) = y_{n_k} \to f(x) = y$. Thus, we have proved that $f(X)$ is closed; the connectedness of X implies $f(X) = X$. □

Proposition 3.6. *Let X be a Banach space, $S \subset X$ a Lipschitz surface of codimension n, and let $X = D \oplus F$ with $\dim F = n$. Let $\psi = \pi_{D,F}|_S\colon S \to D$ be injective and let $\psi^{-1}\colon \psi(S) \to S$ be Lipschitz. Then S is an F-Lipschitz surface. Moreover, if S is a Lipschitz locally d.c. surface of codimension n, then S is an F-Lipschitz locally d.c. surface.*

Proof. Choose an n-dimensional space $\tilde{F}$ such that S is an $\tilde{F}$-Lipschitz surface. Since the case $F = \tilde{F}$ is obvious by Lemma 3.2, we suppose $F \neq \tilde{F}$. Put $K := F \cap \tilde{F}$ and choose spaces C, $\tilde{C}$ such that $F = K \oplus \tilde{C}$ and $\tilde{F} = K \oplus C$. Then clearly $1 \leqslant \dim C = \dim \tilde{C} < \infty$. Choose a topological complement Z of the (finite dimensional) space $F + \tilde{F} = K \oplus C \oplus \tilde{C}$ and denote $E := Z \oplus C$, $\tilde{E} := Z \oplus \tilde{C}$. Clearly $X = F \oplus E = \tilde{F} \oplus \tilde{E}$.

By Lemma 3.2, $\tilde{\varphi} := \pi_{\tilde{E},\tilde{F}}|_S\colon S \to \tilde{E}$ is a bilipschitz bijection. It is easy to see (proceeding similarly to the proof of Lemma 3.2) that $\varphi := \pi_{E,F}|_S\colon S \to E$ is injective and $\varphi^{-1}\colon \varphi(S) \to S$ is Lipschitz. So Lemma 3.2 implies that, to prove the first part of the assertion, it is sufficient to verify $\varphi(S) = E$.

To this end choose an arbitrary $e \in E$ and write $e = z + c$, where $z \in Z$ and $c \in C$. For each $x \in \tilde{C}$, put $f(x) := \varphi \circ (\tilde{\varphi})^{-1}(x+z) - z$. Clearly $f(x) \in (F + \tilde{F}) \cap E = C$; so $f\colon \tilde{C} \to C$. It is easy to see that f is continuous injective and $f^{-1}(y) = \tilde{\varphi} \circ \varphi^{-1}(y+z) - z$ for each $y \in f(\tilde{C})$. Consequently, f^{-1} is Lipschitz, and so $f(\tilde{C}) = C$ by Lemma 3.5. For $\tilde{c} := f^{-1}(c)$ we have $\varphi((\tilde{\varphi})^{-1}(\tilde{c} + z)) = c + z = e$; so $\varphi(S) = E$.

To prove the second part of the assertion, we suppose that $(\tilde{\varphi})^{-1}\colon \tilde{E} \to X$ is moreover locally d.c. Then $g := \varphi \circ (\tilde{\varphi})^{-1} = \pi_{E,F} \circ (\tilde{\varphi})^{-1}$ is clearly Lipschitz and it is locally d.c. by Lemma 2.2 (i). Since $\varphi(S) = E$, we have that $g\colon \tilde{E} \to E$ is a bijection and $g^{-1} = \tilde{\varphi} \circ \varphi^{-1}$ is Lipschitz. Choose a linear bijection $L\colon \tilde{C} \to C$, and let $G\colon \tilde{E} \to E$ be the mapping which assigns to a point $\tilde{e} = \tilde{c} + z$ ($\tilde{c} \in \tilde{C}$, $z \in Z$) the point $G(\tilde{e}) := L(\tilde{c}) + z$. Then clearly G is a linear isomorphism. Since $G(\tilde{e}) - \tilde{e} \in C + \tilde{C}$ and $g(\tilde{e}) - \tilde{e} \in F + \tilde{F}$, we obtain that $g - G$ has a finite dimensional range. Consequently, Lemma 2.2(v) implies that g^{-1} is locally d.c. Thus, Lemma 2.2(iv) implies that $\varphi^{-1} = (\tilde{\varphi})^{-1} \circ g^{-1}$ is locally d.c. So, Lemma 3.2 implies that S is an F-Lipschitz locally d.c. surface. □

Proposition 3.7. *Let X be a Banach space, $F \subset X$ an n-dimensional space, and $A \subset X$ an F-Lipschitz or F-Lipschitz locally d.c. surface. Then there exists $\varepsilon > 0$ such that if $\tilde{F} \subset X$ is respectively an n-dimensional space with $\gamma(F, \tilde{F}) < \varepsilon$, then A is an $\tilde{F}$-Lipschitz or $\tilde{F}$-Lipschitz locally d.c. surface.*

Proof. Choose E such that $X = E \oplus F$ and $K \geqslant 1$ such that the canonical mapping $\gamma\colon E \oplus F \to E \times F$ is K-bilipschitz. Choose a corresponding $\omega > 0$ by Lemma 2.7. Denote $\pi := \pi_{E,F}$ and choose $L \geqslant 1$ such that $(\pi|_A)^{-1}$ is Lipschitz with the constant L. Choose $\varepsilon > 0$ such that $\varepsilon < \omega$ and

$$2K^2 L\varepsilon < 1/2. \tag{3.1}$$

Now suppose that an n-dimensional space $\tilde{F}$ with $\gamma(F, \tilde{F}) < \varepsilon$ is given. Since $\varepsilon < \omega$, we have that $X = E \oplus \tilde{F}$ and the canonical mapping $\tilde{\gamma}\colon E \oplus \tilde{F} \to E \times \tilde{F}$ is $2K$-bilipschitz. By Proposition 3.6, it is sufficient to prove that, putting $\tilde{\pi} := \pi_{E,\tilde{F}}$, the mapping $(\tilde{\pi}|_A)^{-1}$ is Lipschitz with the constant $2L$; i.e., that

$$\|x - y\| \leqslant 2L\|\tilde{\pi}(x) - \tilde{\pi}(y)\| = 2L\|\tilde{\pi}(x - y)\|, \quad x, y \in A. \tag{3.2}$$

Thus, consider $x, y \in A$, $x \neq y$, and write $x - y = e_1 + f = e_2 + \tilde{f}$, where $e_1 = \pi(x - y) \in E$, $e_2 = \tilde{\pi}(x - y) \in E$, $f \in F$ and $\tilde{f} \in \tilde{F}$. We know that $\|x - y\| \leqslant L\|e_1\|$ and so $\|\tilde{f}\| \leqslant 2K\|x - y\| \leqslant 2KL\|e_1\|$.

If $\tilde{f} = 0$, then (3.2) is obvious. If $\tilde{f} \neq 0$, put $\tilde{z} := \|\tilde{f}\|^{-1}\tilde{f}$ and find $z \in F$ such that $\|\tilde{z} - z\| \leqslant \varepsilon$. Then $f_2 := \|\tilde{f}\|z$ satisfies $\|\tilde{f} - f_2\| \leqslant \varepsilon\|\tilde{f}\|$, and so

$$K^{-1}\|e_1 - e_2\| \leqslant \|e_1 - e_2 + f - f_2\| = \|\tilde{f} - f_2\| \leqslant \varepsilon\|\tilde{f}\| \leqslant 2KL\varepsilon\|e_1\|.$$

Thus, by (3.1), we obtain $\|e_1 - e_2\| \leqslant \|e_1\|/2$, and so $\|e_2\| \geqslant \|e_1\|/2$. Therefore $\|x - y\| \leqslant L\|e_1\| \leqslant 2L\|e_2\|$, which proves (3.2) and completes the proof. □

Remark 3.8. I do not know whether analogues of Proposition 3.6 and Proposition 3.7 hold for Lipschitz d.c. surfaces.

4. Projections of Lipschitz surfaces of finite codimension

Definition 4.1. Let X be a separable Banach space and let a finite-dimensional space $V \subset X$ be given. We define the following classes of sets:

(i) $\mathcal{A}(V)$ is the system of all Borel sets $B \subset X$ such that $V \cap (B + a)$ is Lebesgue null (in V) for each $a \in X$. For $0 \neq v \in X$ we put $\mathcal{A}(v) := \mathcal{A}(\mathrm{sp}\{v\})$.

(ii) $\mathcal{A}^*(V, \varepsilon)$ (where $0 < \varepsilon < 1$) is the system of all Borel sets $B \subset X$ such that $B \in \mathcal{A}(W)$ for each space W with $\gamma(V, W) < \varepsilon$, and $\mathcal{A}^*(V)$ is the system of all sets B such that $B = \bigcup_{k=1}^{\infty} B_k$, where $B_k \in \mathcal{A}^*(V, \varepsilon_k)$ for some $0 < \varepsilon_k < 1$.

(iii) $\mathcal{C}_d^*$ (where $d \in \mathbb{N}$) is the system of those $B \subset X$ that can be written as $B = \bigcup_{k=1}^{\infty} B_k$, where each B_k belongs to $\mathcal{A}^*(V_k)$ for some V_k with $\dim V_k = d$.

(iv) $\mathcal{A}$ is the system of those $B \subset X$ that can be, for every complete sequence (v_k) in X, written as $B = \bigcup_{k=1}^{\infty} B_k$, where each B_k belongs to $\mathcal{A}(v_k)$.

Note that $\mathcal{C}_1^*$ coincides with $\mathcal{C}^*$ from [14] and $\mathcal{A}$ is the system of all Aronszajn null sets. For basic properties of sets from $\mathcal{A}$ see [2]. Lemma 2.5 easily implies the following fact.

Lemma 4.2. *Let X, Y be Banach spaces and $F\colon X \to Y$ a linear isomorphism. Let $S \subset X$ belong to $\mathcal{A}^*(V, \delta)$. Then there exists $\varepsilon > 0$ such that $F(S) \in \mathcal{A}^*(F(V), \varepsilon)$ (in the space Y).*

Proposition 4.3. *Let X be a separable infinite dimensional Banach space. Then $\mathcal{C}_1^* \subset \mathcal{C}_2^* \subset \ldots \subset \mathcal{A}$ and all inclusions are proper.*

Proof. To prove the inclusions $\mathcal{C}_d^* \subset \mathcal{A}$, it is sufficient to show that $\mathcal{A}^*(V, \varepsilon) \subset \mathcal{A}$ whenever $V \subset X$ is a d-dimensional space and $0 < \varepsilon < 1$. Let V, ε and $B \in \mathcal{A}^*(V, \varepsilon)$ be given. Choose a basis $(v_1, \ldots, v_d)$ of V and consider an arbitrary complete sequence (u_i) in X. Choose a $\delta > 0$ that corresponds to $(v_1, \ldots, v_d)$ and ε by Lemma 2.4. Clearly we can choose $n \in \mathbb{N}$ and vectors $w_1, \ldots w_d$ in $U := \mathrm{sp}\{u_1, \ldots, u_n\}$ such that $\|w_i - v_i\| < \delta$, $i = 1, \ldots, d$. Then, denoting $W := \mathrm{sp}\{w_1, \ldots, w_d\}$, we have $\gamma(V, W) < \varepsilon$, and so $B \in \mathcal{A}(W)$. Consequently, by the Fubini theorem, $B \in \mathcal{A}(U)$.

Using [2, Proposition 6.29], we easily obtain that B can be decomposed as $B = \bigcup_{i=1}^{n} B_i$, where $B_i \in \mathcal{A}(u_i)$. So, $B \in \mathcal{A}$, and $\mathcal{C}_d^* \subset \mathcal{A}$ is proved.

To prove $\mathcal{C}_d^* \subset \mathcal{C}_{d+1}^*$, consider a $B \in \mathcal{A}^*(V, \varepsilon)$ where $\dim V = d$ and $1 > \varepsilon > 0$. Choose a basis $v_1, \dots, v_d$ of V with $\|v_i\| = 1$ and find a corresponding $\delta > 0$ by Lemma 2.4. Now choose an arbitrary $Z \supset V$ with $\dim Z = d + 1$. To prove $B \in \mathcal{A}^*(Z, \delta)$, consider an arbitrary $(d+1)$-dimensional W with $\gamma(W, Z) < \delta$. By the definition of γ, find $w_1, \dots, w_d \in W$ with $\|w_1 - v_1\| < \delta, \dots, \|w_d - v_d\| < \delta$ and set $\tilde{W} := \mathrm{sp}\{w_1, \dots, w_d\}$. The choice of δ implies that $\gamma(\tilde{W}, V) < \varepsilon$, and so $\tilde{W} \cap (B + a)$ is Lebesgue null in $\tilde{W}$ for each $a \in X$. Consequently, by the Fubini theorem, $W \cap (B + a)$ is Lebesgue null in W for each $a \in X$. So $B \in \mathcal{A}^*(Z, \delta)$, and $\mathcal{C}_d^* \subset \mathcal{C}_{d+1}^*$ follows.

A construction of a set in $\mathcal{A} \setminus \mathcal{C}_1^*$ is presented in the proof of [14, Proposition 13]. Moreover, it is shown in [14] that this set $(F_2(I))$ meets any 2-dimensional affine space in a 2-dimensional Lebesgue null set, which shows that even $\mathcal{C}_2^* \setminus \mathcal{C}_1^* \neq \emptyset$. It is not difficult to modify that construction and obtain a set in $\mathcal{C}_{d+1}^* \setminus \mathcal{C}_d^*$ for each d (see Remark 4.4). However, since the notation is somewhat complicated in the general case, we will give a detailed proof for $d = 2$ only.

Our construction starts quite similarly to the construction of a set from $\mathcal{A} \setminus \mathcal{C}^*$ on p. 20 of [14]. Namely, by the same procedure as in [14] we can define positive numbers $c_0, c_1, c_2, \dots$ and nonzero vectors $u_0, u_1, u_2, \dots$ in X such that both $\{u_{6n-3} \colon n \in \mathbb{N}\}$ and $\{u_{6n} \colon n \in \mathbb{N}\}$ are dense in X, and the formula $F(x) = \sum_{k=0}^{\infty} c_k x_{k+1} u_k$ (where $x = (x_1, x_2, \dots)$) defines a continuous linear injective mapping of ℓ_∞ to X, which is continuous on $I := \{x \in \ell_\infty : \ 1 \leqslant x_k \leqslant 2\}$.

As in [14], we equip I with the topology of pointwise convergence (so it is a compact metrizable space) and with the measure μ defined as the product of countably many copies of the Lebesgue measure on $[1, 2]$.

Choose two sequences $\xi_1^1, \xi_2^1, \dots$ and $\xi_1^2, \xi_2^2, \dots$ such that $0 < \xi_j^1 < 1/(j+1)!$, $0 < \xi_j^2 < 1/(j+1)!$ and

$$\lim_{k\to\infty} \sum_{j=k}^{\infty} c_{3j-2} \xi_j^1 2^j \|u_{3j-2}\| / c_{6k-3} = 0, \quad \lim_{k\to\infty} \sum_{j=k}^{\infty} c_{3j-1} \xi_j^2 2^j \|u_{3j-1}\| / c_{6k} = 0.$$

Now, for $x \in I$, set

$$G(x) = \sum_{k=1}^{\infty} c_{3k-2} \xi_k^1 x_1 x_3 \dots x_{2k-1} u_{3k-2} + \sum_{k=1}^{\infty} c_{3k-1} \xi_k^2 x_2 x_4 \dots x_{2k} u_{3k-1} + \sum_{k=1}^{\infty} x_k c_{3k} u_{3k}.$$

Then $G\colon I \to X$ is a continuous mapping. Indeed, we have $G = F \circ H$, where

$$H(x_1, x_2, \ldots) := (0, \xi_1^1 x_1, \xi_1^2 x_2, x_1, \xi_2^1 x_1 x_3, \xi_2^2 x_2 x_4, x_2, \xi_3^1 x_1 x_3 x_5, \xi_3^2 x_2 x_4 x_6, x_3, \ldots),$$

and $H\colon I \to \ell_\infty$ is clearly continuous. So, $G(I)$ is compact.

Let e_j be the j-th member of the canonical basis of ℓ_∞. Observe that if $x \in I$, k_1, $k_2 \in \mathbb{N}$, t, $\tau \in \mathbb{R}$ and $x + te_{2k_1-1} + \tau e_{2k_2} \in I$, then $G(x + te_{2k_1-1} + \tau e_{2k_2}) = G(x) + tv_{k_1}(x) + \tau w_{k_2}(x)$, where

$$v_k(x) := \sum_{j=k}^{\infty} c_{3j-2} \xi_j^1 (x_1 x_3 \ldots x_{2j-1}/x_{2k-1}) u_{3j-2} + c_{6k-3} u_{6k-3},$$
$$w_k(x) := \sum_{j=k}^{\infty} c_{3j-1} \xi_j^2 (x_2 x_4 \ldots x_{2j}/x_{2k}) u_{3j-1} + c_{6k} u_{6k}.$$

Now consider x, $y \in I$ such that $x \neq y$ and $tG(x) + (1-t)G(y) \in G(I)$ for infinitely many real t. Since F is a linear injection of ℓ_∞ to X, for any such t we have $tH(x) + (1-t)H(y) = H(z)$ for some $z \in I$. Considering, for each $k \in \mathbb{N}$, the $(3k+1)$-st coordinates of $H(z)$ we obtain $z = tx + (1-t)y$. Consequently, considering the $(3k-1)$-st and $3k$-th coordinates of $H(z)$, we obtain that, for each $k \in \mathbb{N}$,

$$tx_1 x_3 \ldots x_{2k-1} + (1-t) y_1 y_3 \ldots y_{2k-1} = (tx_1 + (1-t)y_1) \ldots (tx_{2k-1} + (1-t)y_{2k-1}),$$
$$tx_2 x_4 \ldots x_{2k} + (1-t) y_2 y_4 \ldots y_{2k} = (tx_2 + (1-t)y_2) \ldots (tx_{2k} + (1-t)y_{2k}).$$

Since the above equalities hold for infinitely many t, we infer that x and y differ at most in one odd coordinate and at most in one even coordinate (otherwise one of the right sides, for suficiently large k, would be a polynomial in t of degree greater than one, which is impossible). Consequently, there exist $k_1, k_2 \in \mathbb{N}$ such that $y \in x + \operatorname{sp}\{e_{2k_1-1}, e_{2k_2}\}$; so $G(y) \in G(x) + \operatorname{sp}\{v_{k_1}(x), w_{k_2}(x)\}$.

The above analysis shows that the set of lines which contain any fixed point $G(x)$, $x \in I$, and meet the set $G(I)$ in an infinite set, can be covered by countably many planes containing $G(x)$. Therefore $G(I)$ meets any 3-dimensional affine subspace of X in a set of 3-dimensional Lebesgue measure zero. Consequently, $G(I) \in \mathcal{C}_3^*$.

Now suppose that $G(I) \in \mathcal{C}_2^*$, hence $G(I) = \bigcup_{n=1}^{\infty} B_n$, where $B_n \in \mathcal{A}^*(V_n, \varepsilon_n)$ and V_n are 2-dimensional subspaces of X. Write $V_n = \operatorname{sp}\{p_n, q_n\}$ and choose $\delta_n > 0$ (by Lemma 2.4) such that $\gamma(V_n, \operatorname{sp}\{v, w\}) < \varepsilon_n$ whenever $\|v - p_n\| < \delta_n$ and $\|w - q_n\| <$

δ_n. For any given n find k_1, k_2 such that

$$\sum_{j=k_1}^{\infty} 2^j c_{3j-2}\xi_j^1 \|u_{3j-2}\| < c_{6k_1-3}\delta_n/2,$$
$$\|u_{6k_1-3} - p_n\| < \delta_n/2,$$
$$\sum_{j=k_2}^{\infty} 2^j c_{3j-1}\xi_j^2 \|u_{3j-1}\| < c_{6k_2}\delta_n/2,$$
$$\|u_{6k_2} - q_n\| < \delta_n/2.$$

For any $x \in I$ we have

$$\|v_{k_1}(x) - c_{6k_1-3}p_n\| \leqslant \sum_{j=k_1}^{\infty} 2^j c_{3j-2}\xi_j^1 \|u_{3j-2}\| + c_{6k_1-3}\|u_{6k_1-3} - p_n\| < c_{6k_1-3}\delta_n,$$
$$\|w_{k_2}(x) - c_{6k_2}q_n\| \leqslant \sum_{j=k_2}^{\infty} 2^j c_{3j-1}\xi_j^1 \|u_{3j-1}\| + c_{6k_2}\|u_{6k_2} - q_n\| < c_{6k_2}\delta_n.$$

So $\|v_{k_1}(x)/c_{6k_1-3} - p_n\| < \delta_n$ and $\|w_{k_2}(x)/c_{6k_2} - q_n\| < \delta_n$, which shows that the plane $G(x) + \mathrm{sp}\{v_{k_1}(x), w_{k_2}(x)\}$ meets B_n in a 2-dimensional Lebesgue null set. Hence the set

$$\{(t,\tau)\colon\ x + te_{2k_1-1} + \tau e_{2k_2} \in G^{-1}(B_n)\} = \{(t,\tau)\colon\ G(x) + tv_{k_1}(x) + \tau w_{k_2}(x) \in B_n\}$$

is Lebesgue null, and the Fubini theorem gives $\mu(G^{-1}(B_n)) = 0$. (Note that $G^{-1}(B_n)$ is Borel, since G is continuous.) But this contradicts $I = \bigcup_{n=1}^{\infty} G^{-1}(B_n)$, and we infer that $G(I) \notin \mathcal{C}_2^*$. □

Remark 4.4. For an arbitrary $d \in \mathbb{N}$ we obtain as above that $G_d(I) \in \mathcal{C}_{d+1}^* \setminus \mathcal{C}_d^*$, where $G_d = F \circ H_d$,

$$H_d(x) := (0, \xi_1^1 x_1, \ldots, \xi_1^d x_d, x_1, \xi_2^1 x_1 x_{d+1}, \ldots, \xi_2^d x_d x_{2d}, x_2, \xi_3^1 x_1 x_{d+1} x_{2d+1}, \ldots),$$

and $(\xi_i^1), \ldots, (\xi_i^d)$ are suitably chosen sequences.

Proposition 4.5. *Let X be a separable infinite dimensional Banach space, S a Lipschitz surface of codimension $n \geqslant 2$, and $P\colon X \to Y$ a continuous linear mapping onto a Banach space Y such that $\dim(\ker(P)) < n$. Then there exists an n-dimensional space $D \subset Y$ and $0 < \varepsilon < 1$ such that $P(S) \in \mathcal{C}^*(D, \varepsilon)$ in Y. Consequently, $P(S)$ is a first category subset of Y which is Aronszajn null in Y.*

P r o o f. Denote $K := \ker P$. Choose a space $F \subset X$ such that $\dim F = n$ and S is an F-Lipschitz surface. Using Lemma 3.7, Lemma 2.3 and Lemma 2.6, we can choose a space V with $\dim V = n$ such that S is an V-Lipschitz surface and $V \cap K = \{0\}$. Choose a closed space $H \subset X$ such that $X = H \oplus (K \oplus V)$. Denoting $Z := H \oplus V$, we have $X = Z \oplus K$. Set $\pi := \pi_{Z,K}$. Using Lemma 3.7 and Lemma 2.3, we find $0 < \delta < 1$ such that $\gamma(V, W) < \delta$ implies that S is a W-Lipschitz surface and $W \oplus (H \oplus K) = X$. Now consider an arbitrary $W \subset Z$ such that $\gamma(V, W) < \delta$. We can choose a Lipschitz mapping $\varphi\colon H \oplus K \to W$ such that $S = \{h+k+\varphi(h+k)\colon h \in H,\ k \in K\}$. Consequently, $\pi(S) = \{h + \varphi(h + k)\colon h \in H,\ k \in K\}$. Now consider an arbitrary $a = h_0 + w_0 \in Z$. Then $(\pi(S) + a) \cap W = \{w_0 + \varphi(-h_0 + k)\colon k \in K\}$. Since the mapping $\psi\colon K \to W$ defined by $\psi(k) := w_0 + \varphi(-h_0 + k)$ is Lipschitz and $\dim K < \dim W$, we obtain that $(\pi(S) + a) \cap W$ is Lebesgue null in W. Since $\pi(S)$ is an F_σ set by Remark 3.3(i) and Remark 3.4, we obtain that $\pi(S) \in \mathcal{C}^*(V, \delta)$ in Z. Since $F := P|_Z$ is a linear isomorphism with $F(\pi(S)) = P(S)$, Lemma 4.2 implies that $P(S) \in \mathcal{C}^*(D, \varepsilon)$ for $D := F(V)$ and some $\varepsilon > 0$. Consequently, $P(S)$ is Aronszajn null in Y by Lemma 4.3. Thus, $\operatorname{int} P(S) = \emptyset$. Since $P(S)$ is an F_σ set, we obtain that $P(S)$ is a first category set. $\square$

As an immediate consequence we obtain the following result.

Proposition 4.6. *Let X be a separable infinite dimensional Banach space, $n \geqslant 2$, $A \in \mathcal{L}^n(X)$, and let $P\colon X \to Y$ be a continuous linear mapping onto a Banach space Y such that $\dim(\ker(P)) < n$. Then $P(S)$ is a subset of a set from $\mathcal{C}^*_n$ in Y. Consequently, $P(S)$ is a first category subset of Y which is a subset of an Aronszajn null set in Y.*

Remark 4.7. Let X, Y, P and n be as in Proposition 4.6.

(i) Let f be a continuous convex function on X and $B_n := \{x \in X\colon \dim(\partial f(x)) \geqslant n\}$. Then [13, Theorem 1.3] states that $P(A)$ is a first category set. Using the results of [19], it is easy to see that [13, Theorem 1.3] is equivalent to the statement that $P(A)$ is a first category set for each $A \in \mathcal{DC}^n(X)$, but the proof of [13] is direct, it does not use [19].

(ii) The result that $P(A)$ is a first category set for each $A \in \mathcal{L}^n(X)$ is due to Heisler [7].

(iii) An example from [7] shows that there exists $A \in \mathcal{DC}^n(X)$ such that $P(A) \notin \mathcal{L}^1(Y)$.

(iv) It is not known whether $P(A)$ is σ-porous or Γ-null for each $A \in \mathcal{L}^n(X)$ (or $A \in \mathcal{DC}^n(X)$). The negative answer seems to be probable.

Remark 4.8. Let X be a separable infinite dimensional space. Proposition 4.6 easily implies that the inclusions $\mathcal{L}^n(X) \subset \mathcal{L}^{n-1}(X)$ $(n > 1)$ are proper. Indeed, no Lipschitz surface S of codimension $n-1$ can belong to $\mathcal{L}^n(X)$, since there is a surjective continuous linear projection of S to a space E of codimension $n-1$.

Proposition 4.6 implies the following result which improves both [13, Theorem 1.3] and [7, Theorem 5.6].

Theorem 4.9. *Let X be a separable infinite dimensional Banach space, $n \geqslant 2$, and let $T\colon X \to X^*$ be a monotone (mutivalued) operator. Denote by B_n the set of all $x \in X$ for which the convex hull of $T(x)$ is at least n-dimensional. Let $P\colon X \to Y$ be a continuous linear mapping onto a Banach space Y such that* $\dim(\ker(P)) < n$. *Then $P(B_n)$ is a subset of a set from $\mathcal{C}_n^*$ in Y. Consequently, $P(B_n)$ is a first category subset of Y which is a subset of an Aronszajn null set in Y.*

Proof. Since $B_n \in \mathcal{L}^n(X)$ by [18], the assertion follows from Proposition 4.6. $\square$

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