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EXTREME PRESERVERS OF MAXIMAL COLUMN RANK INEQUALITIES OF MATRIX SUMS OVER SEMIRINGS

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Abstract. We characterize linear operators that preserve sets of matrix ordered pairs which satisfy extreme properties with respect to maximal column rank inequalities of matrix sums over semirings.

Keywords: linear operator, rank inequality, maximal column rank.

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1. Introduction

During the past century a lot of literature had been devoted to the problems of determining the linear operators on the $m \times n$ matrix algebra $M_{m\times n}(F)$ over a field F that leave certain matrix subsets invariant, see [8]. These problems have been extended to the $m \times n$ matrices over various semirings, see [1], [2].

Marsaglia and Styan [7] studied inequalities for the rank of matrices. Beasley and Guterman [1] investigated the rank inequalities of matrices over semirings, and characterized the linear operators that preserve inequalities [2]. The structure of matrix varieties which arise as extremal cases in the inequalities is far from being understood over fields as well as over semirings. The investigation of linear preserver problems of extreme cases for rank inequalities of matrices over fields was obtained in [4]. Song studied the linear operators that preserve maximal column ranks of nonnegative integer matrices in [9].

In this paper we characterize linear operators that preserve the sets of matrix pairs which satisfy equality in the maximal column rank inequalities over semirings.

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2. Preliminaries

Definition 2.1. A *semiring* $\mathbb{S}$ consists of a set and two binary operations, addition and multiplication, such that

- $\mathbb{S}$ is an Abelian monoid under addition (identity denoted by 0);
- $\mathbb{S}$ is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathbb{S}$.

In this paper we will always assume that there is a multiplicative identity 1 in $\mathbb{S}$ which is different from 0.

In particular, a semiring $\mathbb{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

Throughout this paper, we will assume that all semirings are antinegative and have no zero divisors.

Definition 2.2. The *Boolean algebra* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

Let $\mathbb{M}_{m,n}(\mathbb{S})$ denote the set of $m \times n$ matrices with entries from the semiring $\mathbb{S}$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{S})$ instead of $\mathbb{M}_{n,n}(\mathbb{S})$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J, and O, respectively. Let R_i denote the matrix whose ith row is all ones and all other rows are zero, and C_j the matrix whose jth column is all ones and all other columns are zero.

The matrix $E_{i,j}$, called a *cell*, is the matrix with 1 in (i, j) position and zero elsewhere. A *weighted cell* is any nonzero scalar multiple of a cell, that is, $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{S}$.

A *line* of a matrix A is a row or a column of A. We let $\mathcal{Z}(\mathbb{S})$ denote the *center* of the semiring $\mathbb{S}$, $|A|$ the number of nonzero entries in the matrix A, and $A[i_1, \ldots, i_k | j_1, \ldots, j_l]$ the $k \times l$-submatrix of A which lies in the intersection of the $i_1, \ldots, i_k$ rows and $j_1, \ldots, j_l$ columns.

Let $\Delta_{m,n} = \{(i, j)\colon\ i = 1, \ldots, m;\ j = 1, \ldots, n\}$. If $m = n$, we use the notation Δ_n instead of $\Delta_{n,n}$.

Definition 2.3. An element in $\mathbb{M}_{n,1}(\mathbb{S})$ is called a *vector* over $\mathbb{S}$.

A set of vectors with entries from a semiring is called *linearly independent* if there is no vector in this set that can be expressed as a nontrivial linear combination of the others.

A nonzero matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *maximal column rank* k ($\mathrm{mc}(A) = k$) if k is the maximal number of the columns of A which are linearly independent.

A nonzero matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *maximal row rank* k ($\mathrm{mr}(A) = k$) if k is the maximal number of the rows of A which are linearly independent.

A matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$ is said to be of *factor rank* k ($\mathrm{rank}(A) = k$) if there exist matrices $B \in \mathbb{M}_{m,k}(\mathbb{S})$ and $C \in \mathbb{M}_{k,n}(\mathbb{S})$ such that $A = BC$ and k is the smallest positive integer for which such factorization exists. By definition, the only matrix with factor rank 0 is the zero matrix, O.

Remark 2.4. It follows that

$$1 \leqslant \mathrm{rank}(A) \leqslant \mathrm{mc}(A) \leqslant n \tag{1.1}$$

for every nonzero matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$.

If $\mathbb{S}$ is a subsemiring of a real field then there is a real rank function $\varrho(A)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{S})$, which is considered as a matrix over the real field. Easy examples show that over semirings these functions are not equal in general. However, the inequality $\mathrm{mc}(A) \geqslant \varrho(A)$ always holds.

Theorem 2.5 ([1])**.** *Let $\mathbb{S}$ be an antinegative semiring without zero divisors. If $A, B \in \mathbb{M}_{m,n}(\mathbb{S})$ with $A \neq O$, $B \neq O$, then*

1. $1 \leqslant \mathrm{mc}(A + B) \leqslant n$.

If $\mathbb{S}$ is a subsemiring of $\mathbb{R}^+$, the nonnegative reals, then

2. $\mathrm{mc}(A + B) \geqslant |\varrho(A) - \varrho(B)|$.

If $A \in \mathbb{M}_{m,n}(\mathbb{S})$, $B \in \mathbb{M}_{n,k}(\mathbb{S})$ with $A \neq O$, $B \neq O$, then

3. $\mathrm{mc}(AB) \leqslant \mathrm{mc}(B)$.

As was proved in [1], these inequalities are sharp and the best possible.

The following example shows that $\mathrm{mc}(A+B) \nleqslant \mathrm{mc}(A)+\mathrm{mc}(B)$, which is different from the rank inequality for matrices over a real field.

Example 2.6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_3(\mathbb{Z}^+)$, $B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_3(\mathbb{Z}^+)$, where $\mathbb{Z}^+$ is the semiring of nonnegative integers. Then $\mathrm{mc}(A) = 1$, $\mathrm{mc}(B) = 1$, and $\mathrm{mc}(A + B) = 3$ over $\mathbb{Z}^+$.

Definition 2.7. For matrices $X = [x_{i,j}]$ and $Y = [y_{i,j}]$ in $\mathbb{M}_{m,n}(\mathbb{S})$, the matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i,j)th entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

We say that a matrix A *dominates* a matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geqslant B$ or $B \leqslant A$ in this case.

Definition 2.8. Let $\mathbb{S}$ be a semiring, not necessary commutative. An operator $T\colon \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ is called *linear* if $T(\alpha X) = \alpha T(X)$, $T(X\beta) = T(X)\beta$, and $T(X+Y) = T(X) + T(Y)$ for all $X, Y \in \mathbb{M}_{m,n}(\mathbb{S})$, $\alpha, \beta \in \mathbb{S}$.

We say that an operator T *preserves* a set $\mathscr{P}$ if $X \in \mathscr{P}$ implies that $T(X) \in \mathscr{P}$, or, if $\mathscr{P}$ is a set of ordered pairs, that $(X, Y) \in \mathscr{P}$ implies $(T(X), T(Y)) \in \mathscr{P}$.

An operator T on $\mathbb{M}_{m,n}(\mathbb{S})$ is called a *(P,Q,B)-operator* if there exist permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$ and a matrix $B \in \mathbb{M}_{m,n}(\mathbb{S})$ with $B \geqslant J$ such that

$$T(X) = P(X \circ B)Q \tag{2.1}$$

for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$, or $m = n$ and

$$T(X) = P(X \circ B)^t Q \tag{2.2}$$

for all $X \in \mathbb{M}_n(\mathbb{S})$, where X^t denotes the transpose of X. Operators of the form (2.1) are called *non-transposing (P,Q,B)-operators*; operators of the form (2.2) are *transposing (P,Q,B)-operators.*

An operator T is called a *(U,V)-operator* if there exist invertible matrices $U \in \mathbb{M}_m(\mathbb{S})$ and $V \in \mathbb{M}_n(\mathbb{S})$ such that

$$T(X) = UXV \tag{2.3}$$

for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$, or $m = n$ and

$$T(X) = UX^tV \tag{2.4}$$

for all $X \in \mathbb{M}_n(\mathbb{S})$. Operators of the form (2.3) are called *non-transposing (U,V)-operators*; operators of the form (2.4) are *transposing (U,V)-operators.*

Lemma 2.9. *Let T be a (P,Q,B)-operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $\operatorname{mc}(B) = 1$ and all entries of B are units in $\mathscr{Z}(\mathbb{S})$. If $\mathbb{S}$ is commutative, then T is a (U,V)-operator.*

P r o o f. Since T is a (P,Q,B)-operator, so there exist permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$ such that $T(X) = P(X \circ B)Q$, or $m = n$ and $T(X) = P(X \circ B)^tQ$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. Since $\operatorname{mc}(B) = 1$, so it follows from (1.1) that $\operatorname{rank}(B) = 1$, or equivalently, there exist vectors $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{S}^m$ and $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{S}^n$ such that $B = \mathbf{d}^t\mathbf{e}$. Since $b_{i,j}$ are units, d_i and e_j are invertible elements in $\mathbb{S}$ for all $(i,j) \in \Delta_{m,n}$. Let $D = \operatorname{diag}(d_1, \ldots, d_m) \in \mathbb{M}_m(\mathbb{S})$ and $E = \operatorname{diag}(e_1, \ldots, e_n) \in \mathbb{M}_n(\mathbb{S})$ be diagonal matrices. Since $\mathbb{S}$ is commutative, it is straightforward to check that $X \circ B = DXE$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. For the case of $T(X) = P(X \circ B)Q$, if we let $U = PD$ and $V = EQ$, then $T(X) = UXV$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. If T is of the form $T(X) = P(X \circ B)^tQ$, then $U = PE$ and $V = DQ$ shows that $T(X) = UX^tV$ for all $X \in \mathbb{M}_{m,n}(\mathbb{S})$. Thus the lemma follows. $\square$

If A and B are matrices and $A \geqslant B$ we denote by $A \setminus B$ the matrix $C = [c_{i,j}]$ where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0; \\ a_{i,j} & \text{otherwise.} \end{cases}$$

We recall some results proved in [2] for later use.

Theorem 2.10 ([2, Theorem 2.14]). *Let $\mathbb{S}$ be an antinegative semiring without zero divisors and $T\colon \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ a linear operator. Then the following assertions are equivalent:*

(1) *T is bijective.*

(2) *T is surjective.*

(3) *There exists a permutation σ on $\Delta_{m,n}$ and units $b_{i,j} \in \mathscr{Z}(\mathbb{S})$ such that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i,j) \in \Delta_{m,n}$.*

Lemma 2.11 ([2, Lemma 2.16]). *Let $\mathbb{S}$ be an antinegative semiring without zero divisors, $T\colon \mathbb{M}_{m,n}(\mathbb{S}) \to \mathbb{M}_{m,n}(\mathbb{S})$ an operator which maps lines to lines and is defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, where σ is a permutation on $\Delta_{m,n}$ and $b_{i,j} \in \mathscr{Z}(\mathbb{S})$ are nonzero entries. Then T is a (P,Q,B)-operator.*

Remark 2.12. One can easily check that if $m = 1$ or $n = 1$ then all operators under consideration are (P,Q,B)-operators, if $m = n = 1$ then all operators under consideration are (P,P^t,B)-operators.

Henceforth we will always assume that $m,n \geqslant 2$.

Lemma 2.13. *Let B be a matrix in $\mathbb{M}_{m,n}(\mathbb{S})$ with $\mathrm{mc}(B) = 1$. If all entries of B are units in $\mathscr{Z}(\mathbb{S})$, then $\mathrm{mc}(X) = \mathrm{mc}(P(X \circ B)Q)$ for all permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$.*

Proof. Let X be any matrix in $\mathbb{M}_{m,n}(\mathbb{S})$. Obviously, $\mathrm{mc}(X) = \mathrm{mc}(XQ)$ for all permutation matrices $Q \in \mathbb{M}_n(\mathbb{S})$. Let P be any permutation matrix in $\mathbb{M}_n(\mathbb{S})$. Then $\mathrm{mc}(X) = \mathrm{mc}((P)^tPXQ) \leqslant \mathrm{mc}(PXQ) \leqslant \mathrm{mc}(XQ) = \mathrm{mc}(X)$, and hence $\mathrm{mc}(X) = \mathrm{mc}(PXQ)$ for all permutation matrices $P \in \mathbb{M}_m(\mathbb{S})$ and $Q \in \mathbb{M}_n(\mathbb{S})$. Thus, it suffices to claim that $\mathrm{mc}(X) = \mathrm{mc}(X \circ B)$.

Since $\mathrm{mc}(B) = 1$, so there exists a column $\mathbf{b_k}$ of $B = [\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}]$ such that $B = \mathbf{b_k}[\alpha_1, \ldots, \alpha_{k-1}, 1, \alpha_{k+1}, \ldots, \alpha_n]$ where α_i are units. Thus, for any matrix $X = [\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}] \in \mathbb{M}_{\mathbf{m},\mathbf{n}}(\mathbb{S})$ we have $X \circ B = [\mathbf{x_1} \circ \mathbf{b_k}\alpha_1, \mathbf{x_2} \circ \mathbf{b_k}\alpha_2, \ldots, \mathbf{x_n} \circ \mathbf{b_k}\alpha_n] = [\mathbf{b_k}\alpha_1 \circ \mathbf{x_1}, \mathbf{b_k}\alpha_2 \circ \mathbf{x_2}, \ldots, \mathbf{b_k}\alpha_n \circ \mathbf{x_n}] = [\alpha_1(\mathbf{x_1} \circ \mathbf{b_k}), \alpha_2(\mathbf{x_2} \circ \mathbf{b_k}), \ldots, \alpha_n(\mathbf{x_n} \circ \mathbf{b_k})]$.

Thus the lemma follows. □

Let $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be a matrix in $\mathbb{M}_{2,1}(\mathbb{Z}^+)$. Then we have that $\operatorname{mc}(X) = 1$, but $\operatorname{mc}(X^t) = 2$. Thus, in general, it is not true that for a matrix $X \in \mathbb{M}_{m,n}(\mathbb{S})$, $\operatorname{mc}(X) = 1$ if and only if $\operatorname{mc}(X^t) = 1$. Nonetheless, the following lemma is obvious:

Lemma 2.14. *Let B be a matrix in $\mathbb{M}_{m,n}(\mathbb{S})$ all of whose entries are units in $\mathscr{Z}(\mathbb{S})$. Then $\operatorname{mc}(B) = 1$ if and only if $\operatorname{mc}(B^t) = 1$.*

Remark 2.15. Let

$$\Omega = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.5}$$

be a matrix in $\mathbb{M}_4(\mathbb{S})$. Then we can easily show that the first three rows (as well as the four columns) are linearly independent. Thus we have $\operatorname{mc}(\Omega) = 4$ and $\operatorname{mc}(\Omega^t) = 3$.

Now we consider the following sets of matrices that arise as extremal cases in the inequalities listed in Theorem 2.5.

$$\begin{aligned} \mathscr{A}_1(\mathbb{S}) &= \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2\colon\ \operatorname{mc}(X+Y) = n\}; \\ \mathscr{A}_{2N}(\mathbb{S}) &= \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2\colon\ \operatorname{mc}(X+Y) = 1\}; \\ \mathscr{A}_{2R}(\mathbb{S}) &= \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2\colon\ \operatorname{mc}(X+Y) = |\varrho(X) - \varrho(Y)|\}. \end{aligned}$$

In the next sections we characterize the linear operators that preserve the sets $\mathscr{A}_1$, $\mathscr{A}_{2N}$, $\mathscr{A}_{2R}$.

3. Linear operators that preserve the extreme set $\mathscr{A}_1(\mathbb{S})$

In this section we investigate the linear operators that preserve the set $\mathscr{A}_1(\mathbb{S})$.

Definition 3.1. We say that $\mathbb{M}_{m,n}(\mathbb{S})$ is *fully maximal* if for each $k \leqslant \min\{m,n\}$, $\mathbb{M}_{m-k,n-k}(\mathbb{S})$ contains a matrix of maximal column rank $n-k$.

If $m \geqslant n$, then we can easily show that $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal. However, for $m < n$, $\mathbb{M}_{m,n}(\mathbb{S})$ may be fully maximal or not depending on the given semiring $\mathbb{S}$. For example, $\mathbb{M}_{2,3}(\mathbb{Z}^+)$ is fully maximal, while $\mathbb{M}_{2,3}(\mathbb{B})$ is not.

Recall that

$$\mathscr{A}_1(\mathbb{S}) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2\colon\ \operatorname{mc}(X+Y) = n\}.$$

Lemma 3.2. *Let $\mathbb{M}_{m,n}(\mathbb{S})$ be fully maximal, let σ be a permutation of $\Delta_{m,n}$, and let T be defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all $(i,j) \in \Delta_{m,n}$, where all $b_{i,j}$ are units in $\mathscr{Z}(\mathbb{S})$. If T preserves $\mathscr{A}_1$, then T preserves lines.*

P r o o f. Suppose that T^{-1} does not map lines to lines. Then there are two non collinear cells which are mapped to a line. There are two cases, either they are mapped to two weighted cells in a column or to two weighted cells in a row.

If two non collinear cells are mapped to two weighted cells in a column, we may assume without loss of generality that $T(E_{1,1} + E_{2,2}) = b_{1,1}E_{1,1} + b_{2,2}E_{2,1}$.

If $n \leqslant m$ it suffices to consider $A = E_{1,1} + E_{2,2} + \ldots + E_{n,n}$. In this case, $T(A)$ has the maximal column rank at most $n-1$, that is, $(O, A) \in \mathscr{A}_1$, $(O, T(A)) \notin \mathscr{A}_1$, a contradiction.

Let us consider the case $m \leqslant n$. Since $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal there exists a matrix $A' \in \mathbb{M}_{m-2,n-2}(\mathbb{S})$ such that $\operatorname{mc}(A') = n-2$. Let us choose A' with the minimal number of non-zero entries. Let $O_2 \oplus A' \in \mathbb{M}_{m,n}(\mathbb{S})$. Thus $\operatorname{mc}(A) = \operatorname{mc}(A') = n-2$. Hence $(E_{1,1} + E_{2,2}, A) \in \mathscr{A}_1$. Since T preserves $\mathscr{A}_1$, it follows that $(b_{1,1}E_{1,1} + b_{2,2}E_{2,1}, T(A)) \in \mathscr{A}_1$, that is, $\operatorname{mc}((b_{1,1}E_{1,1} + b_{2,2}E_{2,1} + T(A)) = n$. Therefore $\operatorname{mc}(T(A)[1,\ldots,m;3,\ldots,n]) \geqslant n-2$. Since the column rank of any matrix cannot exceed the number of columns, $\operatorname{mc}(T(A)[1,\ldots,m;3,\ldots,n]) = n-2$.

Further, $|T(A)[1,\ldots,m;3,\ldots,n]| < |A| = |A'|$ since T transforms cells to weighted cells and at least one cell has to be mapped in the second column. Thus we have an $(m-2)\times(n-2)$ submatrix of $T(A)[1,\ldots,m;3,\ldots,n]$ whose column rank is $n-2$ and the number of whose nonzero entries is less than that of A'. This contradicts the choice of A'.

If two non-collinear cells are mapped to two cells in a row, we may assume without loss of generality that $T(E_{1,1} + E_{2,2}) = b_{1,1}E_{1,1} + b_{2,2}E_{1,2}$. In this case, by considering the matrices $b_{1,1}^{-1}E_{1,1} + b_{2,2}^{-1}E_{2,2}$ and A chosen above, the result follows. Thus, T preserves lines. $\square$

Theorem 3.3. *Let T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $m \neq n$ or $m = n \geqslant 4$. If $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal, then T preserves $\mathscr{A}_1$ if and only if T is a non-transposing (P,Q,B)-operator, where $\operatorname{mc}(B) = 1$ and all entries of B are units in $\mathscr{Z}(\mathbb{S})$.*

P r o o f. By Lemma 2.13 we have that all non-transposing (P,Q,B)-operators with $\operatorname{mc}(B) = 1$ preserve $\mathscr{A}_1$.

Suppose that T preserves $\mathscr{A}_1$. By Lemma 3.2 we have that T preserves lines and applying Theorem 2.10 to Lemma 2.11 we obtain that T is a (P,Q,B)-operator.

Suppose that $\operatorname{mc}(B) \geqslant 2$. Without loss of generality we may assume that the first two rows as well as the first two columns of B are linearly independent. Since

$\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal, there exists a matrix $Y' \in \mathbb{M}_{m-2,n-2}(\mathbb{S})$ such that $\mathrm{mc}(Y') = n-2$. Consider matrices $X = \sum\limits_{i=1}^{m} b_{i,1}^{-1}E_{i,1} + b_{i,2}^{-1}E_{i,2}$ and $Y = O_2 \oplus Y'$ in $\mathbb{M}_{m,n}(\mathbb{S})$. Then all columns of $X+Y$ are linearly independent and hence $(X,Y) \in \mathscr{A}_1$. But the first two columns of $T(X)+T(Y)$ are equal and hence $\mathrm{mc}(T(X),T(Y)) \leqslant n-1$, a contradiction. Thus $\mathrm{mc}(B) = 1$.

Since all non-transposing (P,Q,B)-operators with $\mathrm{mc}(B) = 1$ preserve $\mathscr{A}_1$, it only remains to show that if $m = n$ then transposition does not preserve $\mathscr{A}_1$. Let $A = \begin{bmatrix} \Omega & O \\ O & I_{n-4} \end{bmatrix}$. Then, by Remark 2.15, we have that $\mathrm{mc}(A) = n$ and $\mathrm{mc}(A^t) = n-1$, so that $(A,O) \in \mathscr{A}_1$ while $(A^t,O) \notin \mathscr{A}_1$. Thus T is a non-transposing (P,Q,B)-operator with $\mathrm{mc}(B) = 1$. □

Corollary 3.4. *Let T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $m \neq n$ or $m = n \geqslant 4$, and $\mathbb{M}_{m,n}(\mathbb{S})$ is fully maximal. If $\mathbb{S}$ is commutative, then T preserves $\mathscr{A}_1$ (if and) only if T is a non-transposing (U,V)-operator.*

Proof. Suppose T preserves $\mathscr{A}_1$. By Theorem 3.3, T is a non-transposing (P,Q,B)-operator on $\mathbb{M}_{m,n}(\mathbb{S})$, where $\mathrm{mc}(B) = 1$ and all entries of B are units in $\mathscr{Z}(\mathbb{S})$. Since $\mathbb{S}$ is commutative, it follows from Lemma 2.9 that T is a non-transposing (U,V)-operator. □

4. Linear operators that preserve the extreme set $\mathscr{A}_{2N}(\mathbb{S})$

In this section we investigate the linear operators that preserve the set $\mathscr{A}_{2N}(\mathbb{S})$. Recall that

$$\mathscr{A}_{2N}(\mathbb{S}) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \colon \mathrm{mc}(X+Y) = 1\}.$$

Theorem 4.1. *If T is a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$ that preserves $\mathscr{A}_{2N}$, then T is a (P,Q,B)-operator, where $\mathrm{mc}(B) = 1$ and all entries of B are units in $\mathscr{Z}(\mathbb{S})$. In particular, if $\mathbb{S}$ is commutative, then T is a (U,V)-operator.*

Proof. Suppose that T does not preserve lines. Then, without loss of generality, we may assume that either $T(E_{1,1}+E_{1,2}) = b_{1,1}E_{1,1} + b_{1,2}E_{2,2}$ or $T(E_{1,1}+E_{2,1}) = b_{1,1}E_{1,1}+b_{2,1}E_{2,2}$. In either case, let $Y = O$ and X be either $E_{1,1}+E_{1,2}$ or $E_{1,1}+E_{2,1}$, so that $(X,Y) \in \mathscr{A}_{2N}$ while $(T(X),T(Y)) \notin \mathscr{A}_{2N}$, a contradiction. Thus T preserves lines.

Applying Theorem 2.10 to Lemma 2.11 we obtain that T is a (P,Q,B)-operator.

Suppose that $\mathrm{mc}(B) \geqslant 2$, T preserves $\mathscr{A}_{2N}$. Since $\mathrm{mc}(T(J)) = \mathrm{mc}(B)$, we have $(J,O) \in \mathscr{A}_{2N}$ while $(T(J),T(O)) \notin \mathscr{A}_{2N}$, a contradiction.

By Lemma 2.9, T is a (U,V)-operator since $\mathbb{S}$ is commutative. □

In general, the converse of Theorem 4.1 may be true or not depending on the given semiring $\mathbb{S}$. Obviously, by Lemma 2.13, all non-transposing (P, Q, B)-operators with $\mathrm{mc}(B) = 1$ (all entries of B are units in $(Z\mathbb{S})$ preserve $\mathscr{A}_{2N}$. However, the following example shows that transposing (P, Q, B)-operators may preserve $\mathscr{A}_{2N}$ or not depending on the given semirings $\mathbb{S}$.

Example 4.2. (1) Consider the semiring $\mathbb{Z}^+$ of all nonnegative integers. Let $X = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \oplus O_{n-2} \in \mathbb{M}_n(\mathbb{Z}^+)$. Then we can easily show that $(X, O) \in \mathscr{A}_{2N}$ while $(X^t, O^t) \notin \mathscr{A}_{2N}$. So, the converse of Theorem 4.1 is not true in this case.

(2) Consider the binary Boolean semiring $\mathbb{B} = \{0, 1\}$. Then it is straightforward that for a matrix $A \in \mathbb{M}_n(\mathbb{B})$, $\mathrm{mc}(A) = 1$ if and only if all non-zero columns of A are the same. Thus all non-zero rows of A are the same and $\mathrm{mc}(A^t) = 1$. That is, for any permutation matrices $P, Q \in \mathbb{M}_n(\mathbb{B})$ we have that $\mathrm{mc}(A) = 1$ if and only if $\mathrm{mc}(PA^tQ) = 1$. This shows that the converse of Theorem 4.1 is true in this case.

5. Linear operators that preserve the extreme set $\mathscr{A}_{2R}(\mathbb{S})$

In this section we investigate the linear operators that preserve the set $\mathscr{A}_{2R}(\mathbb{S})$. Recall that for $\mathbb{S} \subseteq \mathbb{R}^+$

$$\mathscr{A}_{2R}(\mathbb{S}) = \{(X, Y) \in \mathbb{M}_{m,n}(\mathbb{S})^2 \colon \mathrm{mc}(X + Y) = |\varrho(X) - \varrho(Y)|\}.$$

Lemma 5.1. *Let $\mathbb{S}$ be any subsemiring of $\mathbb{R}^+$, let σ be a permutation of $\Delta_{m,n}$, and let T be defined by $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ for all $(i, j) \in \Delta_{m,n}$, where all $b_{i,j}$ are units and $\min\{m, n\} \geqslant 3$. If T preserves $\mathscr{A}_{2R}$, then T preserves lines.*

Proof. The sum of three distinct weighted cells has maximal column rank at most 3. Thus $T(E_{1,1} + E_{1,2} + E_{2,1})$ is either the sum of 3 collinear cells, and hence has real rank 1, or is contained in two lines, and hence has real rank 2, or is the sum of three cells of maximal column rank 3, and hence has real rank 3.

Now, for $X = E_{1,1} + E_{1,2} + E_{2,1}$ and $Y = E_{2,2}$ we have that $(X, Y) \in \mathscr{A}_{2R}$ and the image of Y is a single weighted cell, and hence $\varrho(T(Y)) = 1$. Now, if $\varrho(T(X)) = 3$, then $T(X + Y)$ must have maximal column rank 3 or 4, and hence $(T(X), T(Y)) \notin \mathscr{A}_{2R}$, a contradiction. If $\varrho(T(X)) = 1$, then $(T(X), T(Y)) \notin \mathscr{A}_{2R}$ since $T(X + Y) \neq O$. Thus $\varrho(T(X)) = 2$ and $\mathrm{mc}(T(X + Y)) = 1$.

However, it is straightforward to see that the sum of four weighted cells has maximal column rank 1 if and only if they lie either in a line or in the intersection of two rows and two columns. The matrix $T(X + Y)$ is the sum of four weighted cells.

These cells do not lie in a line since $\varrho(T(X))=2$. Thus $T(X+Y)$ must be the sum of four weighted cells which lie in the intersection of two rows and two columns.

Similarly, for any i, j, k, l, $T(E_{i,j}+E_{i,l}+E_{k,j}+E_{k,l})$ lies in the intersection of two rows and two columns. It follows that any two rows must be mapped to two lines. By the bijectivity of T, if a pair of two rows is mapped to two rows (columns), then any pair of two rows is mapped to two rows (columns). Similarly, if a pair of two columns is mapped to two rows (columns), then any pair of two columns is mapped to two rows (columns).

Now, the image of three rows is contained in three lines, two of which are the image of two rows, thus, every row is mapped to a line. Similarly for columns. Thus T preserves lines. □

Theorem 5.2. *Let $\mathbb{S}$ be any subsemiring of $\mathbb{R}^+$, $m \neq n$ or $m = n \geqslant 4$, and let T be a surjective linear operator on $\mathbb{M}_{m,n}(\mathbb{S})$. Then T preserves $\mathscr{A}_{2R}$ if and only if T is a non-transposing (P,Q,B)-operator.*

Proof. By Lemma 2.13 we have that all non-transposing (P,Q,B)-operators with $\operatorname{mc}(B)=1$ preserve $\mathscr{A}_{2R}$.

Applying Lemma 5.1 and Theorem 2.10 to Lemma 2.11 we obtain that if T preserves $\mathscr{A}_{2R}$, then T is a (P,Q,B)-operator.

Suppose that $\operatorname{mc}(B)\geqslant 2$, $\mathbb{S}\subseteq\mathbb{R}^+$ and T preserves $\mathscr{A}_{2R}$. Without loss of generality assume that $n \leqslant m$. Consider

$$X=\Big(\sum_{1\leqslant j\leqslant i\leqslant n} E_{i,j}\Big)\oplus O_{m-n,n},\quad Y=\Big(\sum_{1\leqslant i<j\leqslant n} E_{i,j}\Big)\oplus O_{m-n,n}.$$

Then $\varrho(X)=n=\varrho(T(X))$, $\varrho(Y)=n-1=\varrho(T(Y))$ and $\operatorname{mc}(X+Y)=1=\varrho(X)-\varrho(Y)$. That is, $(X,Y)\in\mathscr{A}_{2R}$. But $\operatorname{mc}(T(X)+T(Y))=\operatorname{mc}(T(J))=\operatorname{mc}(B)\geqslant 2>1=\varrho(T(X))-\varrho(T(Y))$, a contradiction. Thus $\operatorname{mc}(B)=1$.

Since all non-transposing (P,Q,B)-operators with $\operatorname{mc}(B)=1$ preserve $\mathscr{A}_{2R}$ it remains to show that in the case $m=n$ the operator $X\to X^t$ does not preserve $\mathscr{A}_{2R}$. Let $X=\Omega\oplus O_{n-4}$ and $Y=O_n$. Then $(X,Y)\in\mathscr{A}_{2R}$ while $(X^t,Y^t)\notin\mathscr{A}_{2R}$. □

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