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ON THE MEAN VALUE OF A SUM ANALOGOUS TO
CHARACTER SUMS OVER SHORT INTERVALS

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Abstract. The main purpose of this paper is to study the mean value properties of a sum analogous to character sums over short intervals by using the mean value theorems for the Dirichlet L-functions, and to give some interesting asymptotic formulae.

Keywords: character sums, short intervals, even power mean, asymptotic formula

MSC 2010: 11L40

1. Introduction

Let $q \geqslant 3$ be an integer, and χ a Dirichlet character modulo q. Over the past several decades, many authors investigated various arithmetical properties of the character sums

$$\sum_{a=N+1}^{N+H} \chi(a).$$

Pólya [1] and Vinogradov [2] studied the character sums when the modulus q is equal to a prime p and obtained the following inequality

$$\left| \sum_{a=1}^{x} \chi(a) \right| \leqslant c\sqrt{p} \ln p,$$

where c is a constant. Actually, one can establish the above inequality with the constant $c = 1$. If χ is a primitive character modulo q, A. V. Sokolovskiĭ [3] proved the existence of N with

$$\left| \sum_{N \leqslant n \leqslant N+[q/2]} \chi(n) \right| > \sqrt{1 - \frac{8\ln q}{q}} \cdot \frac{1}{2\sqrt{2}} \cdot \sqrt{q},$$

where $[y]$ denotes the greatest integer which less than or equal to y. For general non-principal character, D. A. Burgess [4] obtained the mean value estimate of character sums

$$\sum_{n=1}^{k} \left| \sum_{m=1}^{h} \chi(n+m) \right|^2 < k \cdot h,$$

where h is any positive integer. Xu Zhefeng and Zhang Wenpeng [5] gave the $2k$-th power mean of the character sums over the interval $\left[1, \frac{1}{4}q\right)$

$$\sideset{}{^*}\sum_{\chi(-1)=1} \left| \sum_{a<q/4} \chi(a) \right|^{2k} = \frac{J(q)q^k}{16} \left(\frac{\pi}{8}\right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1}$$
$$\prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O\left(q^{k+\varepsilon}\right).$$

In this paper, we study the even power mean of analogous character sums over the short intervals

$$\sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{a<q/2} (-1)^a \chi(a) \right|^{2k} \quad \text{and} \quad \sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{a<q/4} (-1)^a \chi(a) \right|^{2k}.$$

First we transform the sum to L-functions. Then using the mean value theorems for Dirichlet L-functions, we study the mean value properties of the sums over short intervals, and obtain a sharper asymptotic formula for them. That is, we shall prove the following:

Theorem 1. *Let $q > 8$ be an odd integer. Then we have the following asymptotic formulae:*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \left| \sum_{a<q/2} (-1)^a \chi(a) \right|^{2k}$$
$$= \frac{J(q)q^k}{4} \left(\frac{\pi}{4}\right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O\left(q^{k+\varepsilon}\right),$$

and

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} \left| \sum_{a<\frac{q}{4}} (-1)^a \chi(a) \right|^{2k}$$
$$= \frac{J(q)q^k}{2^{k+1}} \left(\frac{\pi}{4}\right)^{2k-2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O\left(q^{k+\varepsilon}\right),$$

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where $\sum\limits_{\substack{\chi(-1)=1}}^{*}$ denotes the summation over all primitive characters modulo q such that $\chi(-1) = 1$, ε is any fixed positive number, $J(q)$ denotes the number of all primitive characters modulo q, $\prod\limits_{p|q}$ denotes the product over all prime divisors p of q and $C_m^n = m!/n!(m-n)!$.

Theorem 2. Let $q > 8$ be an odd integer. Then we have the two asymptotic formulae:

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \left| \sum_{a<q/2} (-1)^a \chi(a) \right|^4 = \frac{3J(q)q^2}{16} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(q^{2+\varepsilon}\right),$$

and

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \left| \sum_{a<q/4} (-1)^a \chi(a) \right|^4 = \frac{15J(q)q^2}{64} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)}$$

$$- \frac{3\sqrt{2}J(q)q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv1,7(\bmod 8)}} \left(1+\frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv3,5(\bmod 8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1-\frac{1}{p^2}\right)^2$$

$$+ \frac{J(q)q^2}{4} \prod_{\substack{p\nmid q \\ p\equiv1,7(\bmod 8)}} \left(1+\frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv3,5(\bmod 8)}} \frac{p^2-1}{p^2+1} \prod_{p|q} \left(1-\frac{1}{p^2}\right)^2$$

$$+ \frac{J(q)q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv1,7(\bmod 8)}} \left(1+\frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv3,5(\bmod 8)}} \left(\frac{p^2-1}{p^2+1}\right)^3 \prod_{p|q} \left(1-\frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}).$$

Taking $k = 2$ in Theorem 1 and noting that

$$\prod_{p|q} \left(1-\frac{1}{p^2}\right)^3 \prod_{p\nmid 2q} \left(1+\frac{1}{p^2}\right) = \prod_{p} \left(1+\frac{1}{p^2}\right) \prod_{p|q} \left(1-\frac{1}{p^2}\right)^3 \prod_{p|2q} \frac{1}{1+1/p^2}$$

$$= \frac{4}{5}\frac{\zeta(2)}{\zeta(4)} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)},$$

and

$$\sum_{\chi \bmod q}^{*} = \sum_{\chi(-1)=1}^{*} + \sum_{\chi(-1)=-1}^{*}$$

we may immediately get

Corollary 1. Let $q > 8$ be an odd integer. Then we have the following asymptotic formulas

$$\sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{a < q/2} (-1)^a \chi(a) \right|^4 = \frac{3}{8} J(q) q^2 \prod_{p \mid q} \frac{(p^2 - 1)^3}{p^4 (p^2 + 1)} + O\left(q^{2+\varepsilon}\right)$$

and

$$\sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{a < \frac{q}{4}} (-1)^a \chi(a) \right|^4 = \frac{21 J(q) q^2}{64} \prod_{p \mid q} \frac{(p^2 - 1)^3}{p^4 (p^2 + 1)}$$

$$- \frac{3\sqrt{2} J(q) q^2}{8} \prod_{\substack{p \nmid q \\ p \equiv 1,7 (\bmod 8)}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3,5 (\bmod 8)}} \frac{p^4 + 1}{(p^2 + 1)^2} \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right)^2$$

$$+ \frac{J(q) q^2}{4} \prod_{\substack{p \nmid q \\ p \equiv 1,7 (\bmod 8)}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3,5 (\bmod 8)}} \frac{p^2 - 1}{p^2 + 1} \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right)^2$$

$$+ \frac{J(q) q^2}{8} \prod_{\substack{p \nmid q \\ p \equiv 1,7 (\bmod 8)}} \left(1 + \frac{2}{p^2 - 1}\right) \prod_{\substack{p \nmid q \\ p \equiv 3,5 (\bmod 8)}} \left(\frac{p^2 - 1}{p^2 + 1}\right)^3 \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}).$$

2. Some lemmas

To prove the theorem, we need the following lemmas.

Lemma 1. Let χ be a primitive Dirichlet character modulo q with $q \geqslant 3$, then for any real number $\lambda \in [0, 1]$ with $\lambda \neq r/q$, we have

$$\sum_{0 < n \leqslant \lambda q} \chi(n) = \begin{cases} \dfrac{\tau(\chi)}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{\bar{\chi}(n) \sin(2\pi n \lambda)}{n}, & \text{if } \chi(-1) = 1; \\[4mm] \dfrac{\tau(\chi)}{i\pi} \left(L(1, \bar{\chi}) - \displaystyle\sum_{n=1}^{\infty} \bar{\chi}(n) \dfrac{\cos 2\pi n \lambda}{n} \right), & \text{if } \chi(-1) = -1, \end{cases}$$

where $\tau(\chi) = \sum\limits_{a=1}^{q} \chi(a) e(a/q)$ is the Gauss sum, $e(y) = e^{2\pi i y}$, and $L(s, \chi)$ denotes the Dirichlet L-function corresponding to χ.

P r o o f.   See Section 3.1 of [6]. □

Lemma 2. Let $q > 8$ be an odd integer and χ a primitive Dirichlet character modulo q. Then we have

(1) $\displaystyle\sum_{a=1}^{[q/2]} \chi(a) = 0, \qquad$ if $\chi(-1) = 1,$

(2) $\displaystyle\sum_{a=1}^{[q/2]} \chi(a) = \frac{(2 - \bar{\chi}(2))}{i\pi} \tau(\chi) L(1, \bar{\chi}), \qquad$ if $\chi(-1) = -1,$

(3) $\displaystyle\sum_{a=1}^{[q/4]} \chi(a) = \frac{\tau(\chi)}{\pi} L(1, \bar{\chi}\chi_4), \qquad$ if $\chi(-1) = 1,$

(4) $\displaystyle\sum_{a=1}^{[q/4]} \chi(a) = \frac{2 + \bar{\chi}(2) - \bar{\chi}(4)}{2\pi i} \tau(\chi) L(1, \bar{\chi}), \qquad$ if $\chi(-1) = -1,$

(5) $\displaystyle\sum_{a=1}^{[q/8]} \chi(a) = \frac{\tau(\chi)}{\pi} \left[\frac{\bar{\chi}(2)}{2} L(1, \bar{\chi}\chi_4) + \frac{1}{\sqrt{2}} L(1, \bar{\chi}\chi_{8_1}) \right], \qquad$ if $\chi(-1) = 1,$

and

(6) $\displaystyle\sum_{a=1}^{[\frac{q}{8}]} \chi(a) = \frac{\tau(\chi)}{i\pi} \left[\frac{4 + \bar{\chi}(4) - \bar{\chi}(8)}{4} L(1, \bar{\chi}) - \frac{1}{\sqrt{2}} L(1, \bar{\chi}\chi_{8_2}) \right], \quad$ if $\chi(-1) = -1,$

where χ_4 is the primitive Dirichlet character modulo 4, and χ_{8_1} and χ_{8_2} are the two primitive Dirichlet characters modulo 8 with $\chi_{8_1}(3) = 1$ and $\chi_{8_2}(3) = -1$ respectively.

P r o o f. We only prove formula (5), the others can be obtained in the same way. Since q is an odd number, it is easily seen that $r/q \neq \frac{1}{8}$, and from Lemma 1 we can have

$$
\begin{aligned}
\sum_{a=1}^{[q/8]} \chi(a) &= \frac{\tau(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \sin(\frac{1}{4}\pi n)}{n} \\
&= \frac{\tau(\chi)}{\pi} \left(\sum_{n=1}^{\infty} \frac{\bar{\chi}(2n) \sin(\frac{1}{2}\pi n)}{2n} + \sum_{n=1}^{\infty} \frac{\bar{\chi}(2n - 1) \sin(\frac{1}{4}(2n - 1)\pi)}{2n - 1} \right) \\
&= \frac{\tau(\chi)}{\pi} \left(\frac{\bar{\chi}(2)}{2} \sum_{n=1}^{\infty} \frac{\bar{\chi}\chi_4(n)}{n} + \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\chi_2(n)(-1)^{[(n-1)/4]}}{\sqrt{2}n} \right) \\
&= \frac{\tau(\chi)}{\pi} \left(\frac{\bar{\chi}(2)}{2} L(1, \bar{\chi}\chi_4) + \frac{1}{\sqrt{2}} L(1, \bar{\chi}\chi_{8_1}) \right).
\end{aligned}
$$

This proves formula (5). Similarly we can deduce the other formulas. $\qquad\square$

Lemma 3. Let q and r be integers with $q \geqslant 2$ and $(r,q) = 1$, and χ a Dirichlet character modulo q. Then we have the identities

$$\sideset{}{^*}\sum_{\chi \bmod q} \chi(r) = \sum_{d \mid (q,r-1)} \mu\left(\frac{q}{d}\right) \varphi(d)$$

and

$$J(q) = \sum_{d \mid q} \mu(d) \varphi\left(\frac{q}{d}\right),$$

where $\sideset{}{^*}\sum\limits_{\chi \bmod q}$ denotes the summation over all primitive characters modulo q, and $J(q)$ denotes the number of all primitive characters modulo q.

P r o o f.   See Lemma 4 in the reference [7]. □

Lemma 4. Let q be any integer with $q > 2$, let $\tau_k(n)$ denote the k-th divisor function (i.e., the number of solutions of the equation $n_1 n_2 \ldots n_k = n$ in positive integers $n_1, n_2, \ldots, n_k$) and let $\tau(n)$ be the divisor function. Then we have the identity

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\tau_k^2(n)}{n^2} = \zeta^{2k-1}(2) \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right),$$

where $\zeta(s)$ is the Riemann zeta function.

P r o o f.   See Lemma 3 in the reference [8]. □

Lemma 5. Let $q > 8$ be an odd integer, χ be a Dirichlet character modulo q, χ_4 be the primitive character modulo 4, χ_{8_1} and χ_{8_2} be the two primitive characters modulo 8. Then we have the identities

$$\sideset{}{^*}\sum_{\chi(-1)=1} |L(1, \bar{\chi}\chi_4)|^{2k}$$

$$= \frac{J(q)}{2} \left(\frac{\pi^2}{8}\right)^{2k-1} \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q} \left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O(q^{\varepsilon}),$$

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$$\sum_{\chi(-1)=1}^{*} |L(1, \bar{\chi}\chi_{8_1})|^{2k}$$

$$= J(q)\left(\frac{\pi^2}{8}\right)^{2k-1} \prod_{p|q}\left(1 - \frac{1}{p^2}\right)^{2k-1} \prod_{p \nmid 2q}\left(1 - \frac{1 - C_{2k-2}^{k-1}}{p^2}\right) + O(q^\varepsilon),$$

$$\sum_{\chi(-1)=-1}^{*} |L(1, \bar{\chi})|^4 = \frac{5\pi^4 J(q)}{144} \prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O(q^\varepsilon),$$

$$\sum_{\chi(-1)=-1}^{*} |L(1, \bar{\chi}\chi_{8_2})|^4 = \frac{3\pi^4 J(q)}{256} \prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O(q^\varepsilon),$$

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)|L(1, \bar{\chi})|^4 = \frac{(3m + 5)\pi^4 J(q)}{72 \cdot 2^{m+1}} \prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O(q^\varepsilon),$$

and

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m)|L(1, \bar{\chi})|^4 = \frac{(3m + 5)\pi^4 J(q)}{72 \cdot 2^{m+1}} \prod_{p|q} \frac{(p^2 - 1)^3}{p^4(p^2 + 1)} + O(q^\varepsilon),$$

where $\sum_{\chi \bmod q}^{*}$ denotes the summation over all primitive characters modulo q, and $J(q)$ denotes the number of all primitive characters modulo q.

P r o o f.   Using the method from the proof of Lemma 5 in the reference [5], we can deduce the results.   □

Lemma 6.   Let $q > 8$ be any odd integer, and χ the Dirichlet character modulo q, $m \geqslant 0$. Then we have the following asymptotic formulae:

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1, \bar{\chi})L(1, \chi)L(1, \chi\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon),$$

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1, \chi)L(1, \bar{\chi})L(1, \bar{\chi}\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^\varepsilon),$$

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1, \bar{\chi})L^2(1, \chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' \frac{\tau(2^m n)\chi_{8_2}(n)\tau(n)}{n^2} + O(q^\varepsilon),$$

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)|L(1,\bar{\chi})|^2|L(1,\bar{\chi}\chi_{8_2})|^2$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2} \sum_{t|2^m n} \chi_{8_2}(t) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L^2(1,\chi)L^2(1,\bar{\chi}\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' \frac{\tau(n)\chi_{8_2}(2^m n)\tau(2^m n)}{n^2} + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m)L^2(1,\bar{\chi})L(1,\chi)L(1,\chi\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m)L^2(1,\chi)L(1,\bar{\chi})L(1,\bar{\chi}\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m)L(1,\chi)L(1,\chi\chi_{8_2})L^2(1,\bar{\chi}\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(n)\tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m)L^2(1,\bar{\chi})L^2(1,\chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' \frac{\tau(n)\chi_{8_2}(2^m n)\tau(2^m n)}{n^2} + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m)|L(1,\bar{\chi})|^2|L(1,\bar{\chi}\chi_{8_2})|^2$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2} \sum_{t|2^m n} \chi_{8_2}(t) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m)L^2(1,\chi)L^2(1,\bar{\chi}\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' \frac{\tau(2^m n)\chi_{8_2}(n)\tau(n)}{n^2} + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m)L(1,\bar{\chi})L(1,\bar{\chi}\chi_{8_2})L^2(1,\chi\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(n)\tau(n) \sum_{t|2^m n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

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$$\sum_{\chi(-1)=-1}^{*} \chi(2^m) L(1,\chi) L(1,\chi\chi_{8_2}) L^2(1,\bar{\chi}\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2} \chi_{8_2}(2^m n) \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

$$\sum_{\chi(-1)=-1}^{*} \bar{\chi}(2^m) L(1,\bar{\chi}) L(1,\bar{\chi}\chi_{8_2}) L^2(1,\chi\chi_{8_2})$$

$$= \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2} \chi_{8_2}(2^m n) \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}),$$

where $\sum_{n=1}^{\infty}{}'$ denotes the summation over all positive integers n with $(n,q)=1$.

P r o o f.　We only prove the first formula, the others can be obtained by the same method.

For convenience, we put

$$A(y,\bar{\chi}) = \sum_{2^m N < n \leqslant y} \bar{\chi}(n)\tau(n), \qquad \text{and} \qquad B(y,\chi) = \sum_{N < n \leqslant y} \chi(n) \sum_{t|n} \chi_{8_2}(t),$$

where N is a parameter with $q \leqslant N < q^3$. Then from Abel's identity we have

$$L^2(1,\bar{\chi}) = \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\tau(n)}{n} = \sum_{1 \leqslant n \leqslant 2^m N} \frac{\bar{\chi}(n)\tau(n)}{n} + \int_{2^m N}^{\infty} \frac{A(y,\bar{\chi})}{y^2} \, dy$$

and

$$L(1,\chi) L(1,\chi\chi_{8_2}) = \sum_{n=1}^{\infty} n^{-1} \chi(n) \sum_{t|n} \chi_{8_2}(t)$$

$$= \sum_{1 \leqslant n \leqslant N} n^{-1} \chi(n) \sum_{t|n} \chi_{8_2}(t) + \int_{N}^{\infty} \frac{B(y,\chi)}{y^2} \, dy.$$

Hence, we can have

$$\sum_{\chi(-1)=-1}^{*} \chi(2^m) L^2(1,\bar{\chi}) L(1,\chi) L(1,\chi\chi_{8_2})$$

$$= \sum_{\chi(-1)=-1}^{*} \chi(2^m) \left(\sum_{1 \leqslant n_1 \leqslant 2^m N} \frac{\bar{\chi}(n_1)\tau(n_1)}{n_1} + \int_{2^m N}^{\infty} \frac{A(y,\bar{\chi})}{y^2} \, dy \right)$$

$$\times \left(\sum_{1 \leqslant n_2 \leqslant N} n_2^{-1} \chi(n_2) \sum_{t|n_2} \chi_{8_2}(t) + \int_{N}^{\infty} \frac{B(y,\chi)}{y^2} \, dy \right)$$

$$= \sum_{\chi(-1)=-1}^{*} \chi(2^m) \left(\sum_{1 \leqslant n_1 \leqslant 2^m N} \frac{\bar{\chi}(n_1)\tau(n_1)}{n_1} \right) \left(\sum_{1 \leqslant n_2 \leqslant N} n_2^{-1}\chi(n_2) \sum_{t|n_2} \chi_{8_2}(t) \right)$$

$$+ \sum_{\chi(-1)=-1}^{*} \chi(2^m) \left(\sum_{1 \leqslant n_1 \leqslant 2^m N} \frac{\bar{\chi}(n_1)\tau(n_1)}{n_1} \right) \left(\int_N^\infty \frac{B(y,\chi)}{y^2} \, dy \right)$$

$$+ \sum_{\chi(-1)=-1}^{*} \chi(2^m) \left(\sum_{1 \leqslant n_2 \leqslant N} n_2^{-1}\chi(n_2) \sum_{t|n_2} \chi_{8_2}(t) \right) \left(\int_{2^m N}^\infty \frac{A(y,\bar{\chi})}{y^2} \, dy \right)$$

$$+ \sum_{\chi(-1)=-1}^{*} \chi(2^m) \left(\int_{2^m N}^\infty \frac{A(y,\bar{\chi})}{y^2} \, dy \right) \left(\int_N^\infty \frac{B(y,\chi)}{y^2} \, dy \right)$$

$$(7) \quad = M_1 + M_2 + M_3 + M_4.$$

Now we shall calculate each term in the expression (7).

(i) From Lemma 3 we have

$$M_1 = \sum_{\chi(-1)=-1}^{*} \chi(2^m) \left(\sum_{1 \leqslant n_1 \leqslant 2^m N} \frac{\bar{\chi}(n_1)\tau(n_1)}{n_1} \right) \left(\sum_{1 \leqslant n_2 \leqslant N} n_2^{-1}\chi(n_2) \sum_{t|n_2} \chi_{8_2}(t) \right)$$

$$= \frac{1}{2} \sum_{1 \leqslant n_1 \leqslant 2^m N}' \sum_{1 \leqslant n_2 \leqslant N}' (n_1 n_2)^{-1}\tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \sum_{d|(q, 2^m \bar{n}_1 n_2 - 1)} \mu\left(\frac{q}{d}\right)\varphi(d)$$

$$- \frac{1}{2} \sum_{1 \leqslant n_1 \leqslant 2^m N}' \sum_{1 \leqslant n_2 \leqslant N}' (n_1 n_2)^{-1}\tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \sum_{d|(q, 2^m \bar{n}_1 n_2 + 1)} \mu\left(\frac{q}{d}\right)\varphi(d)$$

$$(8) \quad = \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right)\varphi(d) \sum_{\substack{1 \leqslant n_1 \leqslant 2^m N \\ 2^m n_2 \equiv n_1 (\mathrm{mod}\, d)}}' \sum_{1 \leqslant n_2 \leqslant N}' (n_1 n_2)^{-1}\tau(n_1) \sum_{t|n_2} \chi_{8_2}(t)$$

$$- \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right)\varphi(d) \sum_{\substack{1 \leqslant n_1 \leqslant 2^m N \\ 2^m n_2 \equiv -n_1 (\mathrm{mod}\, d)}}' \sum_{1 \leqslant n_2 \leqslant N}' (n_1 n_2)^{-1}\tau(n_1) \sum_{t|n_2} \chi_{8_2}(t),$$

where $\displaystyle\sum_{1 \leqslant n \leqslant N}'$ denotes the summation over n from 1 to N such that $(n, q) = 1$.

For simplicity, we split the sum over n_1 or n_2 into four cases: i) $2^m d \leqslant n_1 \leqslant 2^m N$ and $d \leqslant n_2 \leqslant N$; ii) $2^m d \leqslant n_1 \leqslant 2^m N$ and $1 \leqslant n_2 \leqslant d - 1$; iii) $1 \leqslant n_1 \leqslant 2^m d - 1$ and $d \leqslant n_2 \leqslant N$; iv) $1 \leqslant n_1 \leqslant 2^m d - 1$ and $1 \leqslant n_2 \leqslant d - 1$. So we have

$$\sum_{d|q} \mu\left(\frac{q}{d}\right)\varphi(d) \sum_{\substack{2^m d \leqslant n_1 \leqslant 2^m N \\ 2^m n_2 \equiv n_1 (\mathrm{mod}\, d)}}' \sum_{d \leqslant n_2 \leqslant N}' (n_1 n_2)^{-1}\tau(n_1) \sum_{t|n_2} \chi_{8_2}(t)$$

$$\ll \sum_{d|q} \varphi(d) \sum_{2^m \leqslant r_1 \leqslant \frac{2^m N}{d}} \sum_{1 \leqslant r_2 \leqslant \frac{N}{d}} \sum_{l_1=1}^{d-1} \sum_{\substack{l_2=1 \\ l_2 \equiv l_1 (\mathrm{mod}\, d)}}^{d-1}{}' \frac{\tau(r_1 d + l_1)\tau(r_2 d + l_2)}{(r_1 d + l_1)(r_2 d + l_2)}$$

$$\ll \sum_{d|q} \varphi(d) \sum_{2^m \leqslant r_1 \leqslant \frac{2^m N}{d}} \sum_{1 \leqslant r_2 \leqslant \frac{N}{d}} \sideset{}{'}\sum_{l_1=1}^{d-1} \frac{[(r_1 d + l_1)(r_2 d + l_1)]^\varepsilon}{(r_1 d + l_1)(r_2 d + l_1)}$$

$$\ll \sum_{d|q} \frac{\varphi(d)}{d} \sum_{2^m \leqslant r_1 \leqslant \frac{2^m N}{d}} \sum_{1 \leqslant r_2 \leqslant \frac{N}{d}} \frac{[(r_1 d + 1)(r_2 d + 1)]^\varepsilon}{r_1 r_2} \ll q^\varepsilon,$$

$$\sum_{d|q} \mu\Big(\frac{q}{d}\Big)\varphi(d) \sideset{}{'}\sum_{\substack{2^m d \leqslant n_1 \leqslant 2^m N \\ 2^m n_2 \equiv n_1 (\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1 \leqslant n_2 \leqslant d-1} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t)$$

$$\ll \sum_{d|q} \varphi(d) \sum_{2^m \leqslant r_1 \leqslant \frac{2^m N}{d}} \sum_{1 \leqslant n_2 \leqslant d-1} (r_1 n_2 d)^{\varepsilon-1} \ll q^\varepsilon,$$

and

$$\sum_{d|q} \mu\left(\frac{q}{d}\right)\varphi(d) \sideset{}{'}\sum_{\substack{1 \leqslant n_1 \leqslant 2^m d-1 \\ n_2 \equiv n_1 (\mathrm{mod}\, d)}} \sideset{}{'}\sum_{d \leqslant n_2 \leqslant N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t)$$

$$\ll \sum_{d|q} \varphi(d) \sum_{1 \leqslant n_1 \leqslant 2^m d-1} \sum_{1 \leqslant r_2 \leqslant \frac{N}{d}} (n_1 r_2 d)^{\varepsilon-1} \ll q^\varepsilon,$$

where we have used the estimate $\tau(n) \ll n^\varepsilon$.

For the case $1 \leqslant n_1 \leqslant 2^m d - 1$, $1 \leqslant n_2 \leqslant d - 1$, the solution of the congruence $2^m n_2 \equiv n_1 (\mathrm{mod}\, d)$ is $2^m n_2 = n_1$. Hence,

$$\sum_{d|q} \mu\Big(\frac{q}{d}\Big)\varphi(d) \sideset{}{'}\sum_{\substack{1 \leqslant n_1 \leqslant 2^m d-1 \\ 2^m n_2 \equiv n_1 (\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1 \leqslant n_2 \leqslant d-1} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t)$$

$$= \sum_{d|q} \mu\Big(\frac{q}{d}\Big)\varphi(d) \sideset{}{'}\sum_{1 \leqslant n_2 \leqslant d-1} 2^{-m} n_2^{-2} \tau(2^m n_2) \sum_{t|n_2} \chi_{8_2}(t)$$

$$= \sum_{d|q} \mu\Big(\frac{q}{d}\Big)\varphi(d) \sum_{\substack{n_2=1 \\ (n_2,q)=1}}^{\infty} 2^{-m} n_2^{-2} \tau(2^m n_2) \sum_{t|n_2} \chi_{8_2}(t) + O(q^\varepsilon).$$

So we have

$$(9) \qquad \frac{1}{2} \sum_{d|q} \mu\Big(\frac{q}{d}\Big)\varphi(d) \sideset{}{'}\sum_{\substack{1 \leqslant n_1 \leqslant 2^m N \\ 2^m n_2 \equiv n_1 (\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1 \leqslant n_2 \leqslant N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t)$$

$$= \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon).$$

Similarly, we can also get the estimate

$$(10) \qquad \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \varphi(d) \sideset{}{'}\sum_{\substack{1 \leqslant n_1 \leqslant 2^m N \\ 2^m n_2 \equiv -n_1 (\mathrm{mod}\, d)}} \sideset{}{'}\sum_{1 \leqslant n_2 \leqslant N} (n_1 n_2)^{-1} \tau(n_1) \sum_{t|n_2} \chi_{8_2}(t) \ll q^{\varepsilon}.$$

Then from (8), (9) and (10), we have

$$(11) \qquad M_1 = \frac{J(q)}{2^{m+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^{\varepsilon}).$$

(ii) From Lemma 2 of [9], we have the estimate

$$\sum_{\chi \neq \chi_0} |A(y, \bar{\chi})|^2 \ll y^{1+\varepsilon} \varphi^2(q) \qquad \text{and} \qquad \sum_{\chi \neq \chi_0} |B(y, \chi)|^2 \ll y^{1+\varepsilon} \varphi^2(q),$$

where χ_0 denotes the principal character modulo q. Then from the Cauchy inequality we can easily get

$$\sum_{\chi(-1)=-1} |A(y, \bar{\chi})| \ll \sum_{\chi \neq \chi_0} |A(y, \bar{\chi})| \ll y^{1/2+\varepsilon} q^{3/2}$$

and

$$\sum_{\chi(-1)=-1} |B(y, \chi)| \ll \sum_{\chi \neq \chi_0} |B(y, \chi)| \ll y^{1/2+\varepsilon} q^{3/2}.$$

Using these estimates we have

$$(12) \qquad M_2 = \sideset{}{^*}\sum_{\chi(-1)=-1} \left(\sum_{1 \leqslant n_1 \leqslant 2^m N} \frac{\bar{\chi}(n_1) \tau(n_1)}{n_1} \right) \left(\int_N^{\infty} \frac{B(y, \chi)}{y^2} \, \mathrm{d}y \right)$$

$$\ll \sum_{1 \leqslant n_1 \leqslant 2^m N} n_1^{\varepsilon-1} \int_N^{\infty} \frac{1}{y^2} \left(\sum_{\chi(-1)=-1} |B(y, \chi)| \right) \mathrm{d}y$$

$$\ll N^{\varepsilon} \int_N^{\infty} \frac{q^{3/2} y^{1/2+\varepsilon_1}}{y^2} \, \mathrm{d}y \ll \frac{q^{3/2}}{N^{\frac{1}{2}-\varepsilon}}.$$

(iii) Similar by to (ii), we can also get

$$(13) \qquad M_3 \ll \frac{q^{3/2}}{N^{\frac{1}{2}-\varepsilon}}.$$

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(iv) By the same argument as in (ii), and noting the absolute convergence of the integrals, we can write

$$(14) \quad M_4 = \sum_{\substack{\chi(-1)=-1}}^{*} \left(\int_{2^m N}^{\infty} \frac{A(y,\bar{\chi})}{y^2} \, \mathrm{d}y \right) \left(\int_{N}^{\infty} \frac{B(y,\chi)}{y^2} \, \mathrm{d}y \right)$$

$$\leqslant \int_{2^m N}^{\infty} \int_{N}^{\infty} \frac{1}{y^2 z^2} \sum_{\substack{\chi(-1)=-1}}^{*} |A(y,\bar{\chi})||B(z,\chi)| \, \mathrm{d}y \, \mathrm{d}z$$

$$\ll \int_{2^m N}^{\infty} \frac{1}{y^2} \int_{N}^{\infty} \frac{1}{z^2} \left(\sum_{\chi \neq \chi_0} |A(y,\bar{\chi})|^2 \right)^{\frac{1}{2}} \left(\sum_{\chi \neq \chi_0} |B(z,\chi)|^2 \right)^{\frac{1}{2}} \mathrm{d}y \, \mathrm{d}z$$

$$\ll \int_{2^m N}^{\infty} \frac{1}{y^2} \left(\sum_{\chi \neq \chi_0} |A(y,\bar{\chi})|^2 \right)^{\frac{1}{2}} \mathrm{d}y \times \int_{N}^{\infty} \frac{1}{z^2} \left(\sum_{\chi \neq \chi_0} |B(z,\chi)|^2 \right)^{\frac{1}{2}} \mathrm{d}z$$

$$\ll \int_{2^m N}^{\infty} \frac{\varphi(q)}{y^{3/2-\varepsilon}} \, \mathrm{d}y \times \int_{N}^{\infty} \frac{\varphi(q)}{z^{3/2-\varepsilon}} \mathrm{d}z \ll \frac{\varphi^2(q)}{N^{1-\varepsilon}}.$$

Now taking $N = q^4$ and $\varepsilon < \frac{1}{2}$, combining (7) and (11)–(14) we obtain the asymptotic formula

$$\sum_{\substack{\chi(-1)=-1}}^{*} \chi(2^m) L^2(1,\bar{\chi}) L(1,\chi) L(1,\chi\chi_{8_2}) = \frac{J(q)}{2^{m+1}} \sum_{n=1}^{\infty}{}' n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t) + O(q^\varepsilon).$$

This completes the proof of the first formula of Lemma 6. $\qquad\square$

Lemma 7. *Let $q > 8$ be any odd integer, and χ the Dirichlet character modulo $q, m \geqslant 0$. Then we have the following asymptotic formulae:*

$$\sum_{n=1}^{\infty}{}' n^{-2} \tau(2^m n) \sum_{t|n} \chi_{8_2}(t)$$

$$= \frac{(3m+4)\pi^4}{144} \prod_{\substack{p \nmid q \\ p \equiv 1,7(\mathrm{mod}\,8)}} \left(1 + \frac{2}{p^2-1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3,5(\mathrm{mod}\,8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^2,$$

$$\sum_{n=1}^{\infty}{}' \frac{\tau(n) \sum_{t|2^m n} \chi_{8_2}(t)}{n^2}$$

$$= \frac{\pi^4}{36} \prod_{\substack{p \nmid q \\ p \equiv 1,7(\mathrm{mod}\,8)}} \left(1 + \frac{2}{p^2-1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3,5(\mathrm{mod}\,8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2} \right)^2,$$

$$\sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(n)\tau(n)\sum_{t|2^m n}\chi_{8_2}(t)$$

$$=\frac{\pi^4}{64}\prod_{\substack{p\nmid q\\ p\equiv 1,7(\mathrm{mod}\,8)}}\left(1+\frac{2}{p^2-1}\right)\prod_{\substack{p\nmid q\\ p\equiv 3,5(\mathrm{mod}\,8)}}\frac{p^4+1}{(p^2+1)^2}\prod_{p|q}\left(1-\frac{1}{p^2}\right)^2,$$

$$\sum_{n=1}^{\infty}{}'\frac{\tau(2^m n)\chi_{8_2}(n)\tau(n)}{n^2}$$

$$=\frac{(m+1)\pi^4}{64}\prod_{\substack{p\nmid q\\ p\equiv 1,7(\mathrm{mod}\,8)}}\left(1+\frac{2}{p^2-1}\right)\prod_{\substack{p\nmid q\\ p\equiv 3,5(\mathrm{mod}\,8)}}\left(\frac{p^2-1}{p^2+1}\right)^3\prod_{p|q}\left(1-\frac{1}{p^2}\right)^2,$$

$$\sum_{n=1}^{\infty}{}' n^{-2}\sum_{t|2^m n}\chi_{8_2}(t)\sum_{t|n}\chi_{8_2}(t)$$

$$=\frac{\pi^4}{48}\prod_{\substack{p\nmid q\\ p\equiv 1,7(\mathrm{mod}\,8)}}\left(1+\frac{2}{p^2-1}\right)\prod_{\substack{p\nmid q\\ p\equiv 3,5(\mathrm{mod}\,8)}}\frac{p^2-1}{p^2+1}\prod_{p|q}\left(1-\frac{1}{p^2}\right)^2,$$

$$\sum_{n=1}^{\infty}{}' n^{-2}\chi_{8_2}(n)\tau(2^m n)\sum_{t|n}\chi_{8_2}(t)$$

$$=\frac{(m+1)\pi^4}{64}\prod_{\substack{p\nmid q\\ p\equiv 1,7(\mathrm{mod}\,8)}}\left(1+\frac{2}{p^2-1}\right)\prod_{\substack{p\nmid q\\ p\equiv 3,5(\mathrm{mod}\,8)}}\frac{p^4+1}{(p^2+1)^2}\prod_{p|q}\left(1-\frac{1}{p^2}\right)^2.$$

P r o o f. Noting that $\tau(n)$ is a multiplicative function, we can write

$$(15)\quad \sum_{n=1}^{\infty}{}' n^{-2}\tau(2^m n)\sum_{t|n}\chi_{8_2}(t)$$

$$=\tau(2^m)\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}{}' n^{-2}\tau(n)\sum_{t|n}\chi_{8_2}(t)+\sum_{\substack{n=1\\ 2|n}}^{\infty}{}' n^{-2}\tau(2^m n)\sum_{t|n}\chi_{8_2}(t)$$

$$=(m+1)\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}{}' n^{-2}\tau(n)\sum_{t|n}\chi_{8_2}(t)+\sum_{\substack{r=1\\ 2\nmid r}}^{\infty}{}'\sum_{j=1}^{\infty}(r\cdot 2^j)^{-2}\tau(2^{m+j}r)\sum_{t|2^j r}\chi_{8_2}(t)$$

$$=\left(m+1+\sum_{j=1}^{\infty}\frac{m+j+1}{4^j}\right)\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}{}' n^{-2}\tau(n)\sum_{t|n}\chi_{8_2}(t).$$

By using the Euler product formula we can write

$$
\text{(16)} \quad \sideset{}{'}\sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} n^{-2}\tau(n) \sum_{t|n} \chi_{8_2}(t)
$$

$$
= \prod_{\substack{p \nmid q \\ p \neq 2}} \left(1 + \frac{1}{p^2}\tau(p)\sum_{t|p}\chi_{8_2}(t) + \frac{1}{p^4}\tau(p^2)\sum_{t|p^2}\chi_{8_2}(t) + \dots \right)
$$

$$
= \prod_{\substack{p \nmid q \\ p \equiv 1,7 (\mathrm{mod}\, 8)}} \left(1 + \frac{2^2}{p^2} + \frac{3^2}{p^4} + \dots \right) \prod_{\substack{p \nmid q \\ p \equiv 3,5 (\mathrm{mod}\, 8)}} \left(1 + \frac{3}{p^4} + \frac{5}{p^8} + \frac{7}{p^{12}} + \dots \right)
$$

$$
= \prod_{\substack{p \nmid q \\ p \equiv 1,7 (\mathrm{mod}\, 8)}} \left(1 - \frac{1}{p^2} \right)^{-2}\left(1 + \frac{2}{p^2-1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3,5 (\mathrm{mod}\, 8)}} \left(1 - \frac{1}{p^4} \right)^{-2}\left(1 + \frac{1}{p^4} \right)
$$

$$
= \frac{9}{16}\zeta^2(2) \prod_{\substack{p \nmid q \\ p \equiv 1,7 (\mathrm{mod}\, 8)}} \left(1 + \frac{2}{p^2-1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3,5 (\mathrm{mod}\, 8)}} \frac{1+1/p^4}{(1+1/p^2)^2} \prod_{p|q}\left(1 - \frac{1}{p^2} \right)^2 .
$$

So from (15), (16) we have

$$
\sideset{}{'}\sum_{n=1}^{\infty} n^{-2}\tau(2^m n) \sum_{t|n} \chi_{8_2}(t)
$$

$$
= \frac{(3m+4)\pi^4}{144} \prod_{\substack{p \nmid q \\ p \equiv 1,7 (\mathrm{mod}\, 8)}} \left(1 + \frac{2}{p^2-1} \right) \prod_{\substack{p \nmid q \\ p \equiv 3,5 (\mathrm{mod}\, 8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q}\left(1 - \frac{1}{p^2} \right)^2 .
$$

This proves the first formula. By the same method, one can obtain the other formulae. □

3. Proof of the theorem

In this section, we will complete the proof of the theorem. From Lemma 2, we can write

$$
\text{(17)} \quad \sum_{a=1}^{[\frac{q}{2}]} (-1)^a \chi(a) = 2\sum_{\substack{a=1,2|a}}^{[\frac{q}{2}]} \chi(a) - \sum_{a=1}^{[\frac{q}{2}]}\chi(a) = 2\chi(2)\sum_{a=1}^{[\frac{q}{4}]}\chi(a) - \sum_{a=1}^{[\frac{q}{2}]}\chi(a)
$$

$$
= \begin{cases} 2\chi(2)\dfrac{\tau(\chi)}{\pi}L(1,\bar\chi\chi_4) & \text{if } \chi(-1)=1 , \\[2mm] (2\chi(2)-1)\dfrac{\tau(\chi)}{\pi\mathrm{i}}L(1,\bar\chi) & \text{if } \chi(-1)=-1, \end{cases}
$$

and

$$(18) \quad \sum_{a=1}^{[\frac{q}{4}]}(-1)^a\chi(a) = 2\sum_{a=1,2|a}^{[\frac{q}{4}]}\chi(a) - \sum_{a=1}^{[\frac{q}{4}]}\chi(a)$$

$$= 2\chi(2)\sum_{a=1}^{[\frac{q}{8}]}\chi(a) - \sum_{a=1}^{[\frac{q}{4}]}\chi(a)$$

$$= \begin{cases} \sqrt{2}\chi(2)\dfrac{\tau(\chi)}{\pi}L(1,\bar\chi\chi_{8_1}) & \text{if } \chi(-1)=1, \\[2mm] \dfrac{\tau(\chi)}{\pi i}((2\chi(2)-1)L(1,\bar\chi) - \sqrt{2}\chi(2)L(1,\bar\chi\chi_{8_2})) & \text{if } \chi(-1)=-1. \end{cases}$$

First we prove Theorem 2. Noting that χ is a primitive character modulo q, we can get $|\tau(\chi)| = \sqrt{q}$. So from (17) and Lemma 5, we can obtain

$$\sum_{\chi(-1)=-1}^{*} \left| \sum_{a<\frac{q}{2}}(-1)^a\chi(a) \right|^4$$

$$= \sum_{\chi(-1)=-1}^{*} \left| (2\chi(2)-1)\frac{\tau(\chi)}{\pi}L(1,\bar\chi) \right|^4$$

$$= \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} |2\chi(2)-1|^4 |L(1,\bar\chi)|^4$$

$$= \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} (33 + 4\chi(4) + 4\bar\chi(4) - 20\chi(2) - 20\bar\chi(2)) |L(1,\bar\chi)|^4$$

$$= \frac{3J(q)q^2}{16} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)} + O\left(q^{2+\varepsilon}\right).$$

By formula (18) we can also write

$$\sum_{\chi(-1)=-1}^{*} \left| \sum_{a<\frac{q}{4}}(-1)^a\chi(a) \right|^4$$

$$= \sum_{\chi(-1)=-1}^{*} \left| \frac{\tau(\chi)}{\pi i}((2\chi(2)-1)L(1,\bar\chi) - \sqrt{2}\chi(2)L(1,\bar\chi\chi_{8_2})) \right|^4$$

$$= \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} (33 + 4\chi(4) + 4\bar{\chi}(4) - 20\chi(2) - 20\bar{\chi}(2)) |L(1,\bar{\chi})|^4$$

$$+ \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} (8 + 2\chi(4) - 8\chi(2))L^2(1,\chi)L^2(1,\bar{\chi}\chi_{8_2})$$

$$+ \frac{q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} (8 + 2\bar{\chi}(4) - 8\bar{\chi}(2))L^2(1,\bar{\chi})L^2(1,\chi\chi_{8_2})$$

$$- \frac{2\sqrt{2}q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} (12 - 9\chi(2) - 4\bar{\chi}(2) + 2\chi(4))L^2(1,\chi)L(1,\bar{\chi}\chi_{8_2})L(1,\bar{\chi})$$

$$- \frac{2\sqrt{2}q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} (12 - 9\bar{\chi}(2) - 4\chi(2) + 2\bar{\chi}(4))L^2(1,\bar{\chi})L(1,\chi\chi_{8_2})L(1,\chi)$$

$$+ \frac{8q^2}{\pi^4}(5 - 2\bar{\chi}(2) - 2\chi(2))|L(1,\bar{\chi})|^2|L(1,\chi\bar{\chi}_{8_2})|^2$$

$$- \frac{4\sqrt{2}q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} (2 - \bar{\chi}(2))L(1,\bar{\chi})L^2(1,\chi\chi_{8_2})L(1,\bar{\chi}\chi_{8_2})$$

$$- \frac{4\sqrt{2}q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} (2 - \chi(2))L(1,\chi)L^2(1,\bar{\chi}\chi_{8_2})L(1,\chi\chi_{8_2})$$

$$+ \frac{4q^2}{\pi^4} \sum_{\chi(-1)=-1}^{*} |L(1,\bar{\chi}\chi_{8_2})|^4 .$$

By Lemma 5, Lemma 6 and Lemma 7, we can obtain

$$\sum_{\chi(-1)=-1}^{*} \left| \sum_{a<\frac{q}{4}} (-1)^a \chi(a) \right|^4 = \frac{15J(q)q^2}{64} \prod_{p|q} \frac{(p^2-1)^3}{p^4(p^2+1)}$$

$$- \frac{3\sqrt{2}J(q)q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv 1,7(\mathrm{mod}\,8)}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5(\mathrm{mod}\,8)}} \frac{p^4+1}{(p^2+1)^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2$$

$$+ \frac{J(q)q^2}{4} \prod_{\substack{p\nmid q \\ p\equiv 1,7(\mathrm{mod}\,8)}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5(\mathrm{mod}\,8)}} \frac{p^2-1}{p^2+1} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2$$

$$+ \frac{J(q)q^2}{8} \prod_{\substack{p\nmid q \\ p\equiv 1,7(\mathrm{mod}\,8)}} \left(1 + \frac{2}{p^2-1}\right) \prod_{\substack{p\nmid q \\ p\equiv 3,5(\mathrm{mod}\,8)}} \left(\frac{p^2-1}{p^2+1}\right)^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^2 + O(q^{2+\varepsilon}).$$

Thus we complete the proof of Theorem 2. Similarly, combining (17), (18) and Lemma 5, we can use the same method to deduce Theorem 1.

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