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ON CONGRUENCES AND IDEALS OF PARTIALLY ORDERED QUASIGROUPS

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Abstract. Some results concerning congruence relations on partially ordered quasigroups (especially, Riesz quasigroups) and ideals of partially ordered loops are presented. These results generalize the assertions which were proved by Fuchs in [5] for partially ordered groups and Riesz groups.

Keywords: partially ordered quasigroup, partially ordered loop, Riesz quasigroup, congruence relation, ideal

MSC 2010: 20N05, 06F99

Introduction

In this paper the congruence relations and ideals of partially ordered quasigroups (and especially of Riesz quasigroups) are studied. It is shown that the convex directed congruence relations (see 2.1. Definition) of a Riesz quasigroup Q form a distributive sublattice in the lattice of all congruence relations of Q. Further, some results valid to partially ordered groups (which were proved by Fuchs [5]) are generalized to partially ordered loops and Riesz quasigroups. Namely, it is shown that there exists a one-to-one correspondence between the o-ideals of a partially ordered loop Q and all normal convex subgroupoids S of Q such that $S \subseteq Q^+$, $1 \in S$ (see 4.9. Theorem) and also, that o-ideals of a Riesz loop Q form a distributive sublattice in the lattice of all normal subloops of Q (see 4.11. Theorem).

The l-ideals of lattice ordered commutative loops were investigated by Naik, Swammy, and Misra [8].

The notion of a Riesz quasigroup generalizes the notion of a Riesz group. The foundations of the theory of Riesz groups were laid by Fuchs [5]. Properties of some types of Riesz quasigroups have been studied by Testov in [10].

1. Preliminaries

A quasigroup is an algebra $(Q, \cdot, \backslash, /)$ with three binary operations satisfying the identities

$$\text{(Q1)} \qquad y \backslash (y \cdot x) \approx x; \qquad (x \cdot y)/y \approx x,$$

$$\text{(Q2)} \qquad y \cdot (y \backslash x) \approx x; \qquad (x/y) \cdot y \approx x.$$

The identities (Q1), (Q2) imply that, given $x, y \in Q$, the equations $y \cdot z = x$ and $z \cdot y = x$ have unique solutions $z = y \backslash x$ and $z = x/y$, respectively (see e.g. [3]). A loop is a quasigroup $(Q, \cdot, \backslash, /)$ with an identity element for $(Q, \cdot)$, i.e., an algebra $(Q, \cdot, \backslash, /, 1)$ that satisfies (Q1), (Q2) and $x \cdot 1 \approx 1 \cdot x \approx x$. Every group is a loop, where $x/y = x \cdot y^{-1}$ and $y \backslash x = y^{-1} \cdot x$. For the basic notions concerning quasigroups cf. Belousov [1].

Let $(Q, \cdot, \backslash, /)$ be a quasigroup. For any $a \in Q$ we denote by L_a and R_a the mappings of Q onto itself defined by the rules $L_a(x) = ax, R_a(x) = xa$. Clearly L_a and R_a are bijections of Q to itself and $L_a^{-1}(x) = a \backslash x$, $R_a^{-1}(x) = x/a$. To make the notation easier we will often use $L_a x, R_a x$ instead of $L_a(x), R_a(x)$.

Let $(Q, \cdot, \backslash, /)$ be a quasigroup. Any congruence relation θ on $(Q, \cdot, \backslash, /)$ satisfies

$$\text{(1)} \qquad x\theta y \Leftrightarrow xz\theta yz \Leftrightarrow zx\theta zy.$$

Conversely, if an equivalence relation θ on Q satisfies (1), then θ is a congruence relation on $(Q, \cdot, \backslash, /)$. A quotient-quasigroup of a quasigroup $(Q, \cdot, \backslash, /)$ over its congruence relation θ will be denoted by Q/θ and its elements by $[a]\theta$ (i.e., $[a]\theta = \{x \in Q\colon\ x\theta a\}$).

Let α, β be two congruence relations on a quasigroup $(Q, \cdot, \backslash, /)$. Similarly to equivalence relations we define $\alpha \leqslant \beta$ if $a\alpha b$ implies $a\beta b$; $a(\alpha \wedge \beta)b$ if $a\alpha b$ and $a\beta b$; $a(\alpha \vee \beta)b$ if there exist $z_1, z_2, \ldots, z_n$ such that $a\alpha z_1\beta z_2\alpha z_3 \ldots z_n\beta b$. Then $\alpha \wedge \beta$ and $\alpha \vee \beta$ are congruence relations on $(Q, \cdot, \backslash, /)$. Kiokemeister [6; Theorem 1.3] showed that the lattice of all congruences on a quasigroup is modular. However, there is a stronger result: The variety of quasigroups is congruence-permutable (see [3], p. 79). Hence

$$\alpha \vee \beta = \alpha\beta = \beta\alpha$$

for any congruences α and β on a quasigroup.

From [6; Theorem 1.2] we have

1.1. Lemma. *Let $(Q, \cdot, \backslash, /)$ be a quasigroup, $a, b \in Q$. Let α, β be congruence relations on $(Q, \cdot, \backslash, /)$. Then $[ab](\alpha\beta) = [a]\alpha \cdot [b]\beta$.*

A quasigroup $(Q, \cdot, \backslash, /)$ with a binary relation $\leqslant$ will be called a partially ordered quasigroup (po-quasigroup) if (cf. e.g. [2])

(i) $(Q, \leqslant)$ is a partially ordered set;
(ii) for all $x, y, a \in Q$, $x \leqslant y \Leftrightarrow ax \leqslant ay \Leftrightarrow xa \leqslant ya$.

The po-quasigroup Q is called a partially ordered loop (po-loop) if the set Q with respect to the quasigroup operations is a loop. In the case Q is a po-loop we use the notation $Q^+ = \{x \in Q\colon\ x \geqslant 1\}$, $Q^- = \{x \in Q :\ x \leqslant 1\}$.

Let Q be a po-quasigroup and let $x, y, a \in Q$. Using (ii) we obtain (see [4; Lemma 3.1])

(P) $$x \leqslant y \Rightarrow x/a \leqslant y/a,\ a \backslash x \leqslant a \backslash y,\ a/y \leqslant a/x,\ y \backslash a \leqslant x \backslash a.$$

A partially ordered quasigroup (loop) Q is called a lattice ordered quasigroup (loop), if it has a lattice-order.

A partially ordered quasigroup (loop) Q is said to be a directed quasigroup (loop) if Q is a directed set (i.e., for each $a, b \in Q$ there exist $c, d \in Q$ such that $c \leqslant a, b$ and $a, b \leqslant d$).

1.2. Lemma. *Let Q be a po-quasigroup, $h \in Q$. Then Q is directed if for each $a \in Q$ there exists $d \in Q$ such that $a \leqslant d$ and $h \leqslant d$.*

Proof. Let $a, b \in Q$. There exist $x, y \in Q$ such that $a = hx$, $b = yx$. By assumption there is d such that $d \geqslant h, y$. Hence $dx \geqslant a, b$. Further, take elements $u, v \in Q$ such that $a = h/u$ and $b = h/v$ (since Q is a quasigroup, such elements exist). There is $z \in Q$, $z \geqslant u, v$. Hence, according to (P), $h/z \leqslant a,\ b$. □

1.3. Lemma. *Let Q be a partially ordered loop. Then the following assertions are equivalent:*

(i) *Q is a directed loop;*
(ii) *for each $x \in Q$ there exist $u \in Q^+, v \in Q^-$ such that $x = uv$;*
(iii) *for each $x \in Q$ there exist $p, q \in Q^+$ such that $x = p/q$.*

Proof. (i)$\Leftrightarrow$ (ii) This equivalence immediately follows from [4; Lemma 3.4].

(i)$\Leftrightarrow$ (iii) Assume that Q is a directed loop, $x \in Q$. There exists $q \in Q$ such that $1 \leqslant q$ and $x \backslash 1 \leqslant q$. Then $xq \in Q^+$, and thus $x = p/q$, where $p, q \in Q^+$. Conversely, let $x, y \in Q$. By assumption, $x = p/q$, $y = r/s$, where $p, q, r, s \in Q^+$. Obviously, $p \leqslant (pr)q$ and $r \leqslant (pr)s$. Hence $x, y \leqslant pr$ and thus, by 1.2, we can conclude that Q is a directed loop. □

2. Congruence relations on partially ordered quasigroups

By a congruence relation on a partially ordered quasigroup Q such an equivalence relation θ on the set Q will be meant for which the conditions (1) hold (clearly, θ is a congruence on $(Q, \cdot, \backslash, /)$).

2.1. Definition. Let Q be a po-quasigroup. A congruence relation θ on Q is said to be

(i) a directed congruence relation on Q if there exists $a \in Q$ such that $[a]\theta$ is a directed subset of Q;

(ii) a convex congruence relation on Q if there exists $a \in Q$ such that $[a]\theta$ is a convex subset of Q.

2.2. Lemma. *Let Q be a po-quasigroup. If θ is a directed (or convex) congruence relation on Q, then $[x]\theta$ is a directed (or convex, respectively) subset of Q for each $x \in Q$.*

Proof. Assume that θ is a directed congruence relation on Q. Then there exists $a \in Q$ such that $[a]\theta$ is a directed subset of Q. Take any $x \in Q$ and $x_1, x_2 \in [x]\theta$. There exist $c, b \in Q$ such that $x_1 = ca$ and $x_2 = cb$. From $x_1 \theta x_2$ it follows that $a\theta b$. Since $[a]\theta$ is a directed set, there are $b_1, b_2 \in [a]\theta$ such that $b_1 \leqslant a, b$ and $a, b \leqslant b_2$. Hence $cb_1 \leqslant ca, cb \leqslant cb_2$. Obviously, $cb_1, cb_2 \in [x_1]\theta = [x]\theta$, thus we can conclude that $[x]\theta$ is a directed subset of Q.

Now, suppose that θ is a convex congruence relation on Q. Thus there exists $a \in Q$ such that $[a]\theta$ is a convex subset of Q. Let $x, z \in Q, x_1, x_2 \in [x]\theta$, $x_1 \leqslant z \leqslant x_2$. There exist $c, b \in Q$ such that $x_1 = ca, x_2 = cb$. From $x_1 \theta x_2$ we get $a\theta b$. According to (P), $ca \leqslant z \leqslant cb$ implies $a \leqslant c \backslash z \leqslant b$. Therefore, from convexity of $[a]\theta$, $c \backslash z \in [a]\theta$. Thus $z\theta ca$, i.e., $z \in [x]\theta$. □

Let Q be a po-quasigroup, $h \in Q$. Let α be a congruence relation on Q. For the proof that α is a convex congruence relation on Q it is sufficient to prove that for all $x \in [h]\alpha$ and $z \in Q$ the condition $h \leqslant z \leqslant x$ implies $z \in [h]\alpha$. Indeed, assume that $a, b \in [h]\alpha$, $z \in Q$ and $a \leqslant z \leqslant b$. There exist $c, d \in Q$ such that $a = hc$ and $b = dc$. Then $h \leqslant z/c \leqslant d$. Since $dc \in [hc]\alpha$, we have $d \in [h]\alpha$. Thus $z/c \in [h]\alpha$, which yields $z \in [hc]\alpha = [a]\alpha$.

Let Q be a po-quasigroup, $h \in Q$ and let α be a congruence relation on Q. By $(\alpha, h)^+$ we denote the set $\{x \in [h]\alpha \colon\ x \geqslant h\}$.

2.3. Lemma. *Let α, β be two directed congruence relations on a po-quasigroup Q and let $h \in Q$. If $(\alpha, h)^+ \subseteq (\beta, h)^+$, then $[h]\alpha \subseteq [h]\beta$.*

Proof. Take some elements $a, b \in Q$ such that $ba = h$ (it is obvious that such elements exist) and define $x \circ y = R_a^{-1}x \cdot L_b^{-1}y$. For all $x, y \in Q$ we denote by $z = x \nearrow y$ (or $z = y \searrow x$) the solution of the equation $x = z \circ y$ (or $x = y \circ z$, respectively). It is known that Q with respect to the operations $\circ$, $\searrow$ and $\nearrow$ is a loop with the identity element $h = ba$ (see e.g. [1]). Further, it is a routine to verify that $(Q, \circ, \searrow, \nearrow)$ with respect to the relation $\leqslant$, which is defined for the po-quasigroup Q, is a partially ordered loop. Let us denote this po-loop by Q_0. Clearly, α, β are directed congruence relations on Q_0. Since h is the identity element for Q_0, $[h]\alpha$ and $[h]\beta$ are directed subloops of Q_0. By 1.3, each element x from $[h]\alpha$ (or $[h]\beta$) can be written in the form of $x = p/q$, where $p, q \in (\alpha, h)^+$ (or $p, q \in (\beta, h)^+$, respectively). This yields that for Q_0 the inclusion $(\alpha, h)^+ \subseteq (\beta, h)^+$ implies $[h]\alpha \subseteq [h]\beta$. Clearly, it is also true for a po-quasigroup Q. $\square$

We immediately have

2.4. Lemma. *Let α, β be two convex congruence relations on a partially ordered quasigroup Q. Then $\alpha \wedge \beta$ is a convex congruence relation on Q, too.*

2.5. Lemma. *Let Q be a po-quasigroup and let α, β be two directed congruence relations on Q. Then $\alpha \vee \beta$ is a directed congruence relation on Q.*

Proof. Since $\alpha \vee \beta = \alpha\beta$, it suffices to show that $[x](\alpha\beta)$ is a directed subset of Q for some element $x \in Q$. Let $c, d \in [x](\alpha\beta)$. There exist $a, b \in Q$ such that $x = ab$. Then, by 1.1, $c, d \in [a]\alpha \cdot [b]\beta$. Thus $c = a_1b_1$, $d = a_2b_2$, where $a_i \in [a]\alpha$, $b_i \in [b]\beta$ $(i = 1, 2)$. Since α, β are directed, there exist $a_0 \in [a]\alpha$, $b_0 \in [b]\beta$ such that $a_0 \geqslant a_i$, $b_0 \geqslant b_i$ for each $i = 1, 2$. Hence $a_0b_0 \geqslant c, d$ and evidently $a_0b_0 \in [ab](\alpha\beta)$. Analogously, there exists $z \in [ab](\alpha\beta)$ such that $z \leqslant c, d$. $\square$

Let Q be a po-quasigroup, θ a congruence relation on Q. Let us define

(2) $[x]\theta \leqslant [y]\theta$ if and only if there exist $x_0 \in [x]\theta, y_0 \in [y]\theta$ such that $x_0 \leqslant y_0$.

2.6. Theorem. *Let Q be a partially ordered quasigroup. Let θ be a congruence relation on Q. A quotient-quasigroup Q/θ with the relation defined by* (2) *is a partially ordered quasigroup if and only if θ is a convex congruence relation on Q.*

Proof. If the quotient-quasigroup Q/θ with the relation defined by (2) is a po-quasigroup, then we immediately obtain that θ is a convex congruence relation on Q. To prove the converse, assume that θ is a convex congruence relation on Q.

Since θ is a congruence relation on $(Q, \cdot, \backslash, /)$, it is clear that Q/θ is a quasigroup. It only remains to show that Q/θ is a po-quasigroup. To prove this we claim first the following statement:

(3) $\quad [x]\theta \leqslant [y]\theta$ iff for each $x' \in [x]\theta$ there exists $y' \in [y]\theta$ such that $x' \leqslant y'$.

We take any element $a \in Q$. Let $[x]\theta \leqslant [y]\theta$. Then there exist $x_0 \in [x]\theta$ and $y_0 \in [y]\theta$ such that $x_0 \leqslant y_0$. Let $x' \in [x]\theta$. There is $b \in Q$ such that $x' = R_a^{-1}b \cdot L_a^{-1}x_0$. Since $x'\theta x_0$, we have $R_a^{-1}b \cdot L_a^{-1}x_0 \ \theta \ R_a^{-1}a^2 \cdot L_a^{-1}x_0$. Therefore $b\theta a^2$, and hence $R_a^{-1}b \cdot L_a^{-1}y_0 \ \theta \ y_0$. Denote $y' = R_a^{-1}b \cdot L_a^{-1}y_0$. Obviously $y' \in [y]\theta$ and from $x_0 \leqslant y_0$ we obtain $x' \leqslant y'$. Thus (3) is valid. Further, we are going to show that $\leqslant$ is a partial order on Q/θ. Evidently $\leqslant$ is reflexive over Q/θ. Let $[x]\theta \leqslant [y]\theta$ and at the same time $[y]\theta \leqslant [x]\theta$. Then, by (3), there exist $y' \in [y]\theta$ and $x' \in [x]\theta$ such that $x \leqslant y' \leqslant x'$. Hence, from convexity of $[x]\theta$, we get $y' \in [x]\theta$, and thus $[x]\theta = [y]\theta$. The relation $\leqslant$ is antisymmetric. To prove that $\leqslant$ is transitive over Q/θ we proceed similarly.

Now, let $[x]\theta, [y]\theta, [z]\theta \in Q/\theta$. It is trivial to prove that $[x]\theta \leqslant [y]\theta$ implies $[x]\theta \cdot [z]\theta \leqslant [y]\theta \cdot [z]\theta$ (and also, $[z]\theta \cdot [x]\theta \leqslant [z]\theta \cdot [y]\theta$). Let $[x]\theta \cdot [z]\theta \leqslant [y]\theta \cdot [z]\theta$. Then there exists $b \in [yz]\theta$ such that $xz \leqslant b$. Take $c \in Q$ such that $b = cz$. Apparently, $c \in [y]\theta$ and $x \leqslant c$. Hence $[x]\theta \leqslant [y]\theta$. Analogously, $[z]\theta \cdot [x]\theta \leqslant [z]\theta \cdot [y]\theta$ implies $[x]\theta \leqslant [y]\theta$. Thus we can conclude that Q/θ is a po-quasigroup. $\square$

The relationships between convex directed congruence relations on a lattice ordered quasigroup Q and congruence relations on Q with Substitution Property

(SP) $\quad x\theta y \Rightarrow (x \vee z) \ \theta \ (y \vee z)$ and $(x \wedge z) \ \theta \ (y \wedge z)$ for each $z \in Q$

are established by the following

2.7. Lemma. *A congruence relation θ on a lattice ordered quasigroup Q is directed and convex if and only if it has the Substitution Property* (SP).

Proof. Let θ be a convex directed congruence relation on a lattice ordered quasigroup Q. By 2.6, Q/θ is a po-quasigroup. Clearly, $[x]\theta,\ [y]\theta \leqslant [x \vee y]\theta$. Assume that $[x]\theta,\ [y]\theta \leqslant [z]\theta \leqslant [x \vee y]\theta$. Then there exist $z_1, z_2 \in [z]\theta$ such that $x \leqslant z_1$, $y \leqslant z_2$. Since θ is directed, there is $z_0 \in [z]\theta$ such that $z_1, z_2 \leqslant z_0$. Hence $x, y \leqslant z_0$, which yields that $x \vee y \leqslant z_0$. Therefore $[x \vee y]\theta \leqslant [z]\theta$. However, also $[z]\theta \leqslant [x \vee y]\theta$. Therefore $[z]\theta = [x \vee y]\theta$. Hence $[x]\theta \vee [y]\theta = [x \vee y]\theta$. Analogously, $[x]\theta \wedge [y]\theta = [x \wedge y]\theta$. Now, it is easy to verify that $[x]\theta = [y]\theta$ implies $[x \vee z]\theta = [y \vee z]\theta$, $[x \wedge z]\theta = [y \wedge z]\theta$ for each $z \in Q$.

Conversely, let θ be a congruence relation on a lattice ordered quasigroup Q. Suppose that (SP) holds and take any element $h \in Q$. Clearly, if $x, y \in [h]\theta$, then $x \vee y$, $x \wedge y \in [h]\theta$. Thus $[h]\theta$ is a directed subset of Q. Now, assume that $x \leqslant z \leqslant y$, $x, y \in [h]\theta$, $z \in Q$. Then $z = (x \vee z)\ \theta\ (y \vee z) = y$, which yields that $z \in [h]\theta$. Thus θ is a convex congruence relation on Q. □

From 2.6 and 2.7 we obtain

2.8. Theorem. *Let Q be a lattice ordered quasigroup, θ a convex directed congruence relation on Q. Then the quotient-quasigroup Q/θ with the relation defined by (2) is a lattice ordered quasigroup.*

3. Congruence relations on a Riesz quasigroup

The notion of a Riesz quasigroup generalizes the notion of a Riesz group, which was studied by Fuchs [5]. Further, some results on varietes, radical classes and torsion classes of Riesz groups have been established by Lihová [7]. Properties of some types of Riesz quasigroups have been studied by Testov in [10]. This section deals with congruence relations on Riesz quasigroups.

3.1. Definition. A partially ordered quasigroup Q is called a Riesz quasigroup if it is directed and satisfies the interpolation property

(IP) $$\text{for all } a_i, b_j \in Q \text{ where } a_i \leqslant b_j,\ i, j \in \{1,2\},$$
$$\text{there exists } c \in Q \text{ such that } a_i \leqslant c \leqslant b_j.$$

Every Riesz group is a Riesz quasigroup. To show an example of a Riesz quasigroup which is not a Riesz group we put (cf. [10]): $Q = R^2$ (R is the set of all real numbers) with the operation $(x, y) \cdot (u, v) = (x + u, \frac{1}{2}(y + v))$ and the relation $(x, y) < (u, v) \Leftrightarrow x < u$.

3.2. Remark. Take any element e from a partially ordered quasigroup Q. If we want to verify the validity of the condition (IP) in Q, it is sufficient to consider such elements $a_1, b_1, b_2 \in Q$ that $e \leqslant b_1, b_2$ and $a_1 \leqslant b_1, b_2$.

3.3. Lemma. *Let Q be a directed quasigroup. Let e be any element from Q. The following conditions are equivalent:*

(i) *Q is a Riesz quasigroup;*

(ii) *the intervals $[e, a]$ are multiplicative:*

$$[e, a] \cdot [e, b] = [e^2, ab];$$

(iii) *if $a \in Q$ satisfies*

$$e^2 \leqslant a \leqslant b_1 b_2 \quad \textit{with } b_i \geqslant e$$

then there exist $a_1, a_2 \in Q$ such that

$$a = a_1 a_2, \quad \textit{where } e \leqslant a_i \leqslant b_i \ (i = 1, 2).$$

P r o o f. (i) $\Rightarrow$ (ii). Let Q be a Riesz quasigroup, $a, b \in Q$, $e \leqslant a, b$. Clearly $[e, a] \cdot [e, b] \subseteq [e^2, ab]$. To prove the converse, let $x \in [e^2, ab]$. There exist $u, v \in Q$ such that $a = R_e^{-1} u$ and $b = L_e^{-1} v$. Then $e^2 \leqslant x \leqslant R_e^{-1} u \cdot L_e^{-1} v$. This yields that $x / L_e^{-1} v \leqslant R_e^{-1} u$ and $e \leqslant R_e^{-1} x$. From $e \leqslant b = L_e^{-1} v$ we get $R_e^{-1} x \cdot e \leqslant R_e^{-1} x \cdot L_e^{-1} v$, and hence $x / L_e^{-1} v \leqslant R_e^{-1} x$. Thus $e,\ x / L_e^{-1} v \leqslant R_e^{-1} u$ and $e,\ x / L_e^{-1} v \leqslant R_e^{-1} x$. By (IP) there is $y \in Q$ such that $e, x / L_e^{-1} v \leqslant R_e^{-1} y \leqslant R_e^{-1} u, R_e^{-1} x$. Since $R_e^{-1} y \in [e, R_e^{-1} u]$ and $x = R_e^{-1} y \cdot R_e^{-1} y \setminus x$, to complete the proof it suffices to verify that $R_e^{-1} y \setminus x \in [e, L_e^{-1} v]$. From $x / L_e^{-1} v \leqslant R_e^{-1} y \leqslant R_e^{-1} x$ it follows that $x \leqslant R_e^{-1} y \cdot L_e^{-1} v$ and $y \leqslant x$. The former relation yields $R_e^{-1} y \setminus x \leqslant L_e^{-1} v$ and from the latter relation we have $R_e^{-1} y \cdot e \leqslant x$, which implies $e \leqslant R_e^{-1} y \setminus x$. Thus $R_e^{-1} y \setminus x \in [e, L_e^{-1} v]$ and the proof is complete.

(ii) $\Rightarrow$ (iii). The proof is trivial.

(iii) $\Rightarrow$ (i). Assume that $e, a \leqslant b_1$ and $e, a \leqslant b_2$. From $e \leqslant b_1$ it follows $e^2 \leqslant R_e b_1$ and from $a \leqslant b_2$ we obtain $R_e b_1 \leqslant b_2 \cdot a \setminus R_e b_1$. Thus

$$e^2 \leqslant R_e b_1 \leqslant b_2 \cdot a \setminus R_e b_1. \tag{$*$}$$

By assumption we have $e \leqslant b_2$ and from $a \leqslant b_1$ we obtain $e \leqslant a \setminus R_e b_1$. Thus, by (iii), $(*)$ yields that there exist $c_1, c_2 \in Q$ such that $R_e b_1 = c_1 c_2$, where $e \leqslant c_1 \leqslant b_2$ and $e \leqslant c_2 \leqslant a \setminus R_e b_1$. Using that $c_2 = c_1 \setminus R_e b_1$ we get $e \leqslant c_1 \setminus R_e b_1 \leqslant a \setminus R_e b_1$. Therefore $a \leqslant c_1 \leqslant b_1$. At the same time $e \leqslant c_1 \leqslant b_2$ and thus we can conclude that c_1 is an element which satisfies our requirements. $\square$

3.4. Lemma. *Let α, β be two convex directed congruence relations on a Riesz quasigroup Q. Then also $\alpha \vee \beta$ and $\alpha \wedge \beta$ are both directed and convex.*

P r o o f. By 2.5, $\alpha \vee \beta$ is directed. Let $e \in Q$. We are going to show that $[e^2](\alpha\beta)$ is a convex subset of Q. We will do it by proving that $e^2 \leqslant z \leqslant x$, $x \in [e^2](\alpha\beta)$ implies $z \in [e^2](\alpha\beta)$. In view of 1.1, if $x \in [e^2](\alpha\beta)$, then $x = ab$, where $a \in [e]\alpha, b \in [e]\beta$. Since α, β are directed, there are $c \in [e]\alpha$ and $d \in [e]\beta$ such that $c \geqslant e, a$ and $d \geqslant e, b$. Obviously $cd \geqslant x$. Thus, if $e^2 \leqslant z \leqslant x$, $x \in [e^2](\alpha\beta)$, then $e^2 \leqslant z \leqslant cd$, where $c \in [e]\alpha$, $d \in [e]\beta$ and $c, d \geqslant e$. By 3.3 there exist c_1, d_1 such that $z = c_1 d_1$,

$e \leqslant c_1 \leqslant c$, $e \leqslant d_1 \leqslant d$. From convexity of α, β we get $c_1 \in [e]\alpha$, $d_1 \in [e]\beta$. Hence $z = c_1 d_1 \in [e^2](\alpha\beta)$. Thus $\alpha\beta$ (i.e. $\alpha \vee \beta$) is convex.

Now we turn our attention to $\alpha \wedge \beta$. By 2.4, $\alpha \wedge \beta$ is convex. We are going to prove that $\alpha \wedge \beta$ is directed. Let $e \in Q$, $x, y \in [e](\alpha \wedge \beta)$. Since α, β are directed, there exist $c \in [e]\alpha$ and $d \in [e]\beta$ such that $c \geqslant x, y$ and $d \geqslant x, y$. By (IP) there is $u \in Q$ such that $x \leqslant u \leqslant c$, $x \leqslant u \leqslant d$ and, from convexity of α and β, $u \in [e](\alpha \wedge \beta)$. Analogously, there exists $v \in [e](\alpha \wedge \beta)$ such that $v \leqslant x, y$. $\square$

3.5. Theorem. *The convex directed congruence relations of a Riesz quasigroup Q form a distributive sublattice in the lattice of all congruence relations of Q.*

Proof. In view of 3.4 it is sufficient to verify the distributive law

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

for convex directed congruence relations on Q. Let us denote $\delta = \alpha \wedge (\beta \vee \gamma)$ and $\omega = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$. It is easy to show that $\omega \leqslant \delta$. It only remains to prove $\delta \leqslant \omega$. In fact, by [6; Lemma 1.4], to complete the proof it is sufficient to establish the inclusion $\delta[x] \subseteq \omega[x]$ for some element x of Q.

Let $h \in Q$. We are going to show that $(\delta, h^2)^+ \subseteq (\omega, h^2)^+$. Then, by 2.3, $[h^2]\delta \subseteq [h^2]\omega$. Let $x \in (\delta, h^2)^+$. Then $x \in [h^2]\alpha$ and at the same time $x \in [h^2](\beta\gamma)$. Thus $x = bc$, where $b \in [h]\beta$, $c \in [h]\gamma$. Since α, β are directed, there exist $b_1 \in [h]\beta$ and $c_1 \in [h]\gamma$ such that $b_1 \geqslant h, b$ and $c_1 \geqslant h, c$. Applying 3.3 to $h^2 \leqslant x = bc \leqslant b_1 c_1$ we infer that $x = b_0 c_0$, where $h \leqslant b_0 \leqslant b_1$ and $h \leqslant c_0 \leqslant c_1$. From convexity of β, γ we have $b_0 \in [h]\beta$, $c_0 \in [h]\gamma$. Further, it is easy to see that $h \leqslant b_0 \leqslant x/h$ and $h \leqslant c_0 \leqslant h \setminus x$. Hence, from convexity of α, we have $b_0, c_0 \in [h]\alpha$. Summarizing, we conclude that $h^2 \leqslant x = b_0 c_0 \in [h](\alpha \wedge \beta) \cdot [h](\alpha \wedge \gamma)$ and therefore $x \in (\omega, h^2)^+$, as desired. $\square$

3.6. Corollary. *The congruence relations of a lattice ordered quasigroup Q with Substitution Property* (SP) *form a distributive sublattice in the lattice of all congruence relations of Q.*

If Q is a Riesz quasigroup and θ is a convex congruence relation on Q, then the quotient-quasigroup Q/θ with the relation (2) need not be a Riesz quasigroup. Nonetheless, a proof similar to that for Riesz groups (see [5; Proposition 5.3]) can be applied to obtain the following

3.7. Theorem. *Let Q be a Riesz quasigroup, θ a convex directed congruence relation on Q. Then the quotient-quasigroup Q/θ with the relation* (2) *is a Riesz quasigroup.*

4. Ideals of partially ordered loops

Let $(Q, \cdot, \backslash, /, 1)$ be a loop. A subalgebra of $(Q, \cdot, \backslash, /, 1)$ will be called a subloop of $(Q, \cdot, \backslash, /, 1)$. Let $S \subseteq Q$. If S, with $\cdot$ restricted to S, is a groupoid, we say that S is a subgroupoid of Q. A subgroupoid (a subloop) S of Q will be called a normal subgroupoid (a normal subloop) of Q if for all $x, y \in Q$ the following assertions are valid:

$$xS = Sx,\ xy \cdot S = x \cdot yS \text{ and } S \cdot xy = Sx \cdot y.$$

Let M be a subset of Q. By $\langle M \rangle$ we denote the subloop generated by M (i.e., the intersection of all subloops of Q which contain M).

4.1. Lemma. *Let S be a normal subgroupoid of a loop Q. Then $\langle S \rangle = \{p/q\colon\ p, q \in S\}$.*

P r o o f. Let us denote $M = \{p/q\colon\ p, q \in S\}$. Clearly $M \subseteq \langle S \rangle$. Conversely, since $S \subseteq M$ (each element $p \in S$ can be written in the form pp/p), it is sufficient to prove that M is a subloop of Q. Clearly $1 \in M$. Let $u, v \in M$. Then $u = a/b$, $v = c/d$, where $a, b, c, d \in S$. We are going to show that $uv \in M$. There exist $s, r \in Q$ such that

$$v \cdot ds = vd \cdot b \tag{3}$$

and

$$uv \cdot r = u(v \cdot ds). \tag{4}$$

The element s (or r) is positively determined by (3) (or (4), respectively). Since $b \in S$ and since S is a normal subgroupoid of Q, there exists $b' \in S$ such that $vd \cdot b = v \cdot db'$. From (3) we obtain $v \cdot ds = v \cdot db'$. Thus $s \in S$, and hence $ds \in S$. Also, since S is normal in Q, from (4) we get $r \in S$. Let us denote

$$p = uv \cdot r. \tag{5}$$

By (4) and (3) we can write

$$p = u(v \cdot ds) = u(vd \cdot b) = u \cdot cb.$$

Since $c \in S$ and S is a normal subgroupoid of Q, there is $c' \in S$ such that $u{\cdot}cb = ub{\cdot}c'$. Thus $p = ub \cdot c' = (a/b \cdot b)c' = ac'$, and therefore $p \in S$. Now, using (5), we can conclude that $uv \in M$.

Let $z = u \setminus v$, $u, v \in M$. There exist $a, b, c, d \in S$ such that $u = a/b$, $v = c/d$. From $uz = v = c/d$ we have $uz \cdot d = c$. Since S is normal in Q and since $d \in S$, there is $d' \in S$ such that $ud' \cdot z = c$. The elements u, d' belong to M, therefore $ud' \in M$. Thus $ud' = p/q$ where $p, q \in S$ and we can write $c = p/q \cdot z$. Further, there exists $y \in Q$ with $cy = pz$. Then $(p/q \cdot z)y = (p/q \cdot q)z$. Since $q \in S$, there is $q' \in S$ such that $(p/q \cdot q)z = (p/q \cdot z)q'$. Thus $(p/q \cdot z)y = (p/q \cdot z)q'$, which yields $y = q'$, and therefore $y \in S$. Hence $pz = cy \in S$. Then there exists $s \in S$ such that $z = p \setminus s$. Since S is normal in Q, there is $p' \in S$ such that $p \cdot p \setminus s = p \setminus s \cdot p'$, which yields $s/p' = p \setminus s$. Therefore $z \in M$, i.e., $u \setminus v \in M$. Analogously, $u/v \in M$ for all $u, v \in M$. $\square$

Now, let Q be a partially ordered loop with an identity 1. We recall that by Q^+ (or Q^-) we have denoted the set $\{x \in Q\colon\ x \geqslant 1\}$ (or $\{x \in Q\colon\ x \leqslant 1\}$, respectively). Clearly, Q^+ and Q^- are normal subgroupoids of Q, but, if Q is a non-trivially ordered loop, they are not subloops of Q.

A subgroupoid (a subloop) H of a po-loop Q is said to be convex, if $a \leqslant z \leqslant b$, $a, b \in H$, implies $z \in H$. If H is a subloop of Q, then to prove that H is a convex subloop of Q it is sufficient to show that $1 \leqslant z \leqslant b$, $b \in H$, implies $z \in H$.

4.2. Lemma. *If S is a normal convex subgroupoid of Q, then $\langle S\rangle$ is a convex subloop of Q.*

P r o o f. Suppose that $1 \leqslant z \leqslant p/q$, where $p, q \in S$. Then $q \leqslant zq \leqslant p$ and therefore, from convexity of S, we have $zq \in S$. Thus $z \in \langle S\rangle$. $\square$

4.3. Lemma. *Let S be a normal convex subgroupoid of Q, $S \subseteq Q^+$ and $1 \in S$. Then $S = \langle S\rangle^+$.*

P r o o f. Clearly $S \subseteq \langle S\rangle^+$. To prove the converse inclusion, let $s \in \langle S\rangle^+$. Then $s = p/q$, where $p, q \in S \subseteq Q^+$. Thus $1 \leqslant s \leqslant p$ and therefore, by convexity of S, $s \in S$. $\square$

From 4.3, 4.1 and 1.3 we immediately obtain

4.4. Lemma. *Let S be a normal convex subgroupoid of Q, $S \subseteq Q^+$ and $1 \in S$. Then $\langle S\rangle$ is a directed loop.*

4.5. Lemma. *Let S be a normal convex subgroupoid of Q, $S \subseteq Q^+$ and $1 \in S$. Then $\langle S \rangle$ is a normal subloop of Q.*

Proof. First, we are going to show that $\langle S \rangle^-$ is a normal subgroupoid of Q. To prove the inclusion $x\langle S \rangle^- \subseteq \langle S \rangle^- x$, let $a \in x\langle S \rangle^-$. There exist $u \in \langle S \rangle^-$ and $v \in Q$ such that $a = xu = vx$. Clearly, $u \setminus 1 \in \langle S \rangle^+$ and therefore, by 4.3, $u \setminus 1 \in S$. Since S is normal in Q, there is $p \in S$ such that $x(u \cdot u \setminus 1) = xu \cdot p$. Thus $x = vx \cdot p = p'v \cdot x$, where $p' \in S$. Hence $p'v = 1$ and, since $p' \in S \subseteq Q^+$, $v \in \langle S \rangle^-$. We have $x\langle S \rangle^- \subseteq \langle S \rangle^- x$. An analogous proof holds for the converse inclusion. Using methods similar to those above we can prove relations $\langle S \rangle^- \cdot xy = \langle S \rangle^- x \cdot y$ and $xy \cdot \langle S \rangle^- = x \cdot y \langle S \rangle^-$.

Now, we continue by proving the assertion of the lemma. Since $\langle S \rangle$ is directed, $\langle S \rangle = \{uv\colon u \in S, v \in \langle S \rangle^-\}$. Thus provided $z \in x\langle S \rangle$ we get $z = x \cdot uv$, where $u \in S, v \in \langle S \rangle^-$. Using the fact that S and $\langle S \rangle^-$ are normal in Q we obtain $z = u'v' \cdot x$, where $u' \in S$ and $v' \in \langle S \rangle^-$. Hence $z \in \langle S \rangle x$. Analogously, $z \in \langle S \rangle x$ implies $z \in x\langle S \rangle$. Similarly we obtain $xy \cdot \langle S \rangle = x \cdot y\langle S \rangle$ and $\langle S \rangle \cdot xy = \langle S \rangle x \cdot y$. □

Analogously to the case of partially ordered groups we introduce

4.6. Definition. An o-ideal of a partially ordered loop Q is any normal convex directed subloop A of Q.

From 4.2, 4.4 and 4.5 we obtain

4.7. Lemma. *Let S be a normal convex subgroupoid of Q, $S \subseteq Q^+$ and $1 \in S$. Then $\langle S \rangle$ is an o-ideal of Q.*

4.8. Lemma. *Let A be an o-ideal of Q. Let $S = A \cap Q^+$. Then S is a normal convex subgroupoid of Q and $\langle S \rangle = A$.*

Proof. Since A is a directed subloop of Q, $A = \langle A^+ \rangle = \langle S \rangle$. Further, A, Q^+ are convex, normal in Q, therefore $S = A \cap Q^+$ is convex and normal in Q, too. □

4.9. Theorem. *There exists a one-to-one correspondence between the o-ideals A of a partially ordered loop Q and all normal convex subgroupoids S of Q such that $S \subseteq Q^+$, $1 \in S$.*

Proof. Consider mappings given by

$$\varphi\colon\ A \mapsto Q^+ \cap A \quad \text{and} \quad \psi\colon\ S \mapsto \langle S \rangle,$$

where A is an o-ideal of Q and S is a normal convex subgroupoid of Q, $S \subseteq Q^+$, $1 \in S$. By 4.8, $\psi\varphi$ is the identity (the product rule for mappings is given by multiplying from right to left) and by 4.7, 4.3, $\varphi\psi$ is the identity. □

Let Q be a lattice ordered loop. An l-ideal of a lattice ordered loop Q is any normal convex subloop A of Q which is a sublattice of Q. It is routine to verify that an o-ideal of a lattice ordered loop is nothing else than an l-ideal. The l-ideals of lattice ordered commutative loops were investigated by Naik, Swammy, and Misra. They proved ([8,Theorem 9] or [9, Lemma 6]) that there is a one-to-one correspondence between congruence relations with the Substitution Property (SP) and l-ideals of a commutative lattice ordered loop.

Using the methods from [9] we can prove

4.10. Theorem. *There exists a one-to-one correspondence between the o-ideals of a partially ordered loop Q and all convex directed congruence relations on Q. This correspondence is given by the rules*

$$\varphi\colon\ A \mapsto \theta_A \ \ \textit{and} \ \ \psi\colon\ \theta \mapsto [1]\theta,$$

where A is an o-ideal of Q, $\theta_A = \{(x,y) \in Q \times Q\colon\ x/y \in A\}$.

It is not difficult to verify that the partial order on the set of all congruence relations and the partial order on the set of all o-ideals (ideals are ordered by inclusion) are preserved by the correspondence from 4.10. Thus, using 3.5 and 4.10 we have

4.11. Theorem. *The o-ideals of a Riesz loop Q form a distributive sublattice in the lattice of all normal subloops of Q.*

The last theorem generalizes the analogous result valid for Riesz groups (see [5; Theorem 5.6]).

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