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LOWER BOUNDS ON SIGNED EDGE TOTAL DOMINATION NUMBERS IN GRAPHS

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Abstract. The open neighborhood $N_G(e)$ of an edge e in a graph G is the set consisting of all edges having a common end-vertex with e. Let f be a function on $E(G)$, the edge set of G, into the set $\{-1, 1\}$. If $\sum\limits_{x\in N_G(e)} f(x) \geqslant 1$ for each $e \in E(G)$, then f is called a signed edge total dominating function of G. The minimum of the values $\sum\limits_{e\in E(G)} f(e)$, taken over all signed edge total dominating function f of G, is called the signed edge total domination number of G and is denoted by $\gamma'_{st}(G)$. Obviously, $\gamma'_{st}(G)$ is defined only for graphs G which have no connected components isomorphic to K_2. In this paper we present some lower bounds for $\gamma'_{st}(G)$. In particular, we prove that $\gamma'_{st}(T) \geqslant 2 - m/3$ for every tree T of size $m \geqslant 2$. We also classify all trees T with $\gamma'_{st}(T) = 2 - m/3$.

Keywords: signed edge domination, signed edge total dominating function, signed edge total domination number

MSC 2010: 05C69, 05C05

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [2] for terminology and notation which are not defined here and consider simple connected graphs only. Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common end-vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f\colon E(G) \longrightarrow \{-1, 1\}$ and a subset S of $E(G)$ we define $f(S) = \sum\limits_{e\in S} f(e)$. The *edge-neighborhood* $E_G(v)$ of a vertex $v \in V(G)$ is the set of all edges at the

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vertex v. For each vertex $v \in V(G)$ we also define $f(v) = \sum\limits_{e \in E_G(v)} f(e)$. A function $f\colon E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge total dominating function* (SETDF) of G, if $f(N_G(e)) \geqslant 1$ for each edge $e \in E(G)$. It is clear that there exists an SETDF only for graphs G which have no connected components isomorphic to K_2. Throughout this paper we assume G is a simple connected graph of order $n \geqslant 3$. The minimum of the values $f(E(G))$, taken over all signed edge total dominating functions f of G, is called the *signed edge total domination number* of G. The signed edge total domination number was introduced by B. Zelinka in [5] and denoted by $\gamma'_{st}(G)$. The signed edge total dominating function f of G with $f(E(G)) = \gamma'_{st}(G)$ is called the *$\gamma'_{st}(G)$-function*.

Similarly, a function $f\colon E(G) \longrightarrow \{-1, 1\}$ is called a *signed edge dominating function* (SEDF) of G, if $f(N_G[e]) \geqslant 1$ for each edge $e \in E(G)$. The minimum of the values $f(E(G))$, taken over all signed edge dominating functions f of G, is called the *signed edge domination number* of G. The signed edge domination number was introduced by B. Xu in [3] and denoted by $\gamma'_s(G)$.

Here are some well-known results on $\gamma'_s(G)$ and $\gamma'_{st}(G)$.

Theorem A [1], [4]**.** *For every tree T of order $n \geqslant 2$, $\gamma'_s(T) \geqslant 1$.*

Theorem B [5]**.** *Let G be a graph with m edges and with no K_2-components. Then $\gamma'_{st}(G) \equiv m \pmod 2$.*

Theorem C [5]**.** *Let P_m be a path of length $m \geqslant 2$. Then $\gamma'_{st}(P_m) = m$.*

Theorem D [5]**.** *Let C_m be a cycle of length $m \geqslant 3$. Then $\gamma'_{st}(C_m) = m$.*

Theorem E [5]**.** *Let T be a star with $m \geqslant 2$ edges. If m is odd, then $\gamma'_{st}(T) = 3$. If m is even, then $\gamma'_{st}(T) = 2$.*

The following terminology and notation are useful to prove our results. A graph G with an SETDF f of G, denoted by (G, f), is called a *signed total graph*. For simplicity, given a signed total graph (G, f), an edge e is said to be a $+1$ edge of (G, f) if $f(e) = 1$. Similarly, an edge e is said to be a -1 edge of (G, f) if $f(e) = -1$. We write $E^+(G, f) = \{e \in E(G);\ f(e) = 1\}$ and $E^-(G, f) = \{e \in E(G);\ f(e) = -1\}$.

For any signed total graph (G, f), the two spanning subgraphs $G^+(f)$ and $G^-(f)$ of G are defined as $V(G^+(f)) = V(G^-(f)) = V(G)$ and $E(G^+(f)) = E^+(G, f)$ and $E(G^-(f)) = E^-(G, f)$. For every vertex $v \in V(G)$ we have $f(v) = \deg_{G^+(f)}(v) - \deg_{G^-(f)}(v)$.

2. A lower bound for SETDN of trees

In this section we study the signed edge total domination number of trees. We first prove that for every tree T of size $m \geqslant 2$, $\gamma'_{st}(T) \geqslant 2 - m/3$. Then we characterize all trees T for which $\gamma'_{st}(T) = 2 - m/3$.

Theorem 1. *For every tree T of size $m \geqslant 2$, $\gamma'_{st}(T) \geqslant 2 - m/3$.*

Proof. The proof is by induction on m. The statement holds for all trees of size $m = 2, 3, 4$. Assume T is an arbitrary tree of size $m \geqslant 5$ and that the statement holds for all trees with smaller sizes. Let f be a γ'_{st}-function of T. We consider two cases.

Case 1. There is a non-pendant edge $e = uv \in E$ for which $f(e) = -1$.

Let T_1 and T_2 be the connected components of $T - e$ with $u \in T_1$ and $v \in T_2$. Obviously, the sizes of T_1 and T_2 are greater than 1 and $\gamma'_{st}(T) = f(E(T_1)) - 1 + f(E(T_2))$. For $i = 1, 2$, the function f, restricted to T_i, is an SETDF of T_i, hence, $\gamma'_{st}(T_i) \leqslant f(E(T_i))$. By the inductive hypothesis, $\gamma'_{st}(T_i) \geqslant 2 - m_i/3$, where m_i is the size of T_i. Thus

$$(1) \qquad \gamma'_{ts}(T) \geqslant -1 + (2 - m_1/3) + (2 - m_2/3) = 3 - (m-1)/3 > 2 - m/3.$$

Case 2. The only edges e for which $f(e) = -1$ are pendant edges.

By assumption we have $f(v) \geqslant 0$ for each $v \in V(T)$ with $\deg(v) \geqslant 2$. Let $Z = \{v \in V(T);\ \deg(v) \geqslant 2 \text{ and } f(v) = 0\}$. First, let $Z = \emptyset$. Then f is an SEDF of T. Since $m \geqslant 5$, by Theorem A we have

$$(2) \qquad \gamma'_{st}(T) = f(E(T)) \geqslant \gamma'_s(T) \geqslant 1 > 2 - m/3.$$

Let $Z \neq \emptyset$. It is easy to see that Z is an independent set in T. Let $Z = \{u_i;\ 1 \leqslant i \leqslant k\}$. Obviously, there is no $+1$ pendant edge at u_i for each i. Let $N'(u_i) = \{u \in N(u_i);\ \deg(u) \geqslant 2\}$. Let first $|N'(u_i)| \geqslant 2$ for some i. Without loss of generality we may assume $|N'(u_1)| \geqslant 2$ and $v_1, v_2 \in N'(u_1)$. Let T_1 and T_2 be the connected components of $T - u_1v_1$ for which $v_1 \in V(T_1)$. Let T'_1 be obtained from T_1 by adding a new pendant edge v_1w_1 and let T'_2 be obtained from T_2 by deleting one of the -1 pendant edges at u_1. Now define $g_1\colon\ E(T'_1) \longrightarrow \{-1, +1\}$ by

$$g(v_1w_1) = +1 \text{ and } g(e) = f(e) \text{ if } e \in E(T_1).$$

Obviously, g is an SETDF of T'_1 and $f|_{T'_2}$ is an SETDF of T'_2. By the inductive hypothesis, $\gamma'_{st}(T'_i) \geqslant 2 - m_i/3$, where m_i is the size of T'_i and $m_1 + m_2 = m - 1$.

Thus

$$\gamma'_{st}(T) = f(E(T)) = g(E(T'_1)) + f|_{T'_2}(E(T'_2)) - 1 \geqslant -1 + (2 - m_1/3) + (2 - m_2/3) > 2 - m/3. \tag{3}$$

Now let $|N'(u_i)| = 1$ and $N'(u_i) = \{v_i\}$ for $1 \leqslant i \leqslant k$. It is clear that $f(v_i) \geqslant 3$ for each i. Let T' be obtained from T by deleting all leaves and the vertices of Z. Then since $|N'(u_i)| = 1$ for each i, T' is a tree. Let $w \in \{v_1, v_2, \ldots, v_k\}$. Hence, $f(w) \geqslant 3$ and $\deg(w) \geqslant 3$. We consider three subcases.

Subcase 2.1. $\deg_{T'}(w) \geqslant 1$, $e = ww_1 \in E(T')$ and $f(w_1) = 1$ in T.

By the construction of T' we have $deg_T(w_1) \geqslant 2$. Since $f(w_1) = 1$ and each edge at w_1 in T' is a $+1$ edge, there exists a pendant edge e' in T at w_1. Let T_1 and T_2 be the connected components of $T - e$ containing w, w_1, respectively. Let T'_1 be obtained from T_1 by adding a new pendant edge ww' at w and $T'_2 = T_2 - e'$. It is easy to see that the sizes of T'_1 and T'_2 are greater than 1. Define $g_1\colon E(T'_1) \longrightarrow \{-1, +1\}$ by

$$g(ww') = 1 \text{ and } g(e) = f(e) \text{ if } e \in E(T_1).$$

Obviously, g and $f|_{T'_2}$ are SETDFs of T'_1 and T'_2, respectively. By the inductive hypothesis, $\gamma'_{st}(T'_i) \geqslant 2 - m_i/3$ where m_i is the size of T'_i and $m_1 + m_2 = m - 1$. Thus

$$\gamma'_{st}(T) = f(E(T)) = g(E(T'_1)) + f|_{T'_2}(E(T'_2)) - 1 > 2 - m/3. \tag{4}$$

Subcase 2.2. $\deg_{T'}(w) \geqslant 1$, $e = ww_1 \in E(T')$ and $f(w_1) \geqslant 2$ in T.

Let T_1 and T_2 be the connected components of $T - e$. Let T'_1 and T'_2 be obtained from T_1 and T_2 by adding new pendant edges ww' and $w_1w'_1$, respectively. Define $g_1\colon E(T'_1) \longrightarrow \{-1, +1\}$ by

$$g(ww') = 1 \text{ and } g(e) = f(e) \text{ if } e \in E(T_1),$$

and $g_2\colon E(T'_2) \longrightarrow \{-1, +1\}$ by

$$g(w_1w'_1) = 1 \text{ and } g(e) = f(e) \text{ if } e \in E(T_2).$$

Obviously, g_i is an SETDF of T'_i for $i = 1, 2$. Let $m_i = |E(T'_i)|$. Then we have $m_1 + m_2 = m + 1$. By the inductive hypothesis,

$$\gamma'_{st}(T) = f(E(T)) = g_1(E(T'_1)) + g_2(E(T'_2)) - 1 > 2 - m/3. \tag{5}$$

S u b c a s e 2.3. $\deg_{T'}(w) = 0$.

This implies that $wu_i \in E(T)$ for each $1 \leqslant i \leqslant k$. If there exist two pendant edges at w, say e', e'', such that $f(e') = -1$ and $f(e'') = 1$, then using the inductive hypothesis on $T - \{e', e''\}$ we have

$$\gamma'_{st}(T) \geqslant 2 - (m-2)/3 > 2 - m/3. \tag{6}$$

Finally, let f assign -1 to all pendant edges at w and let r be the number of pendant edges at w. By assumption $k - r = f(w) \geqslant 3$. Furthermore, since $f(u_i) = 0$, there exists a pendant edge $u_i v_i$ for each i. Therefore, $m \geqslant 2k + r$ and hence, $r \leqslant m/3 - 2$. On the other hand, we have $\gamma'_{st}(T) = -r$. Therefore, $\gamma'_{st}(T) \geqslant 2 - m/3$. This completes the proof. □

Now we characterize all trees that attain this bound. We use the notation of Theorem 1.

Theorem 2. *Let $T = (V, E)$ be a tree of size $m \geqslant 2$. Then $\gamma'_{st}(T) = 2 - m/3$ if and only if $V = \{w, u_i, v_i, w_j;\ 1 \leqslant i \leqslant k,\ k \geqslant 3$ and $1 \leqslant j \leqslant k-3\}$, and $E(T) = \{ww_j, wu_i, u_i v_i;\ 1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k-3\}$.*

P r o o f. Let $\gamma'_{st}(T) = 2 - m/3$. Obviously, $m \equiv 0 \pmod 3$. By Theorems C, D and E we must have $m \geqslant 6$. Let f be a γ'_{st}-function of T. By (1), f must assign 1 to all non-pendant edges of T. Obviously, $f(v) \geqslant 0$ for each $v \in V(T)$ with $\deg(v) \geqslant 2$. By (2), we have $Z \neq \emptyset$. Let $Z = \{u_i;\ 1 \leqslant i \leqslant k\}$. Obviously, there is no $+1$ pendant edge at u_i for each i and Z is an independent set of T. By (3), $|N'(u_i)| = 1$ for each i. Since $f(u_i) = 0$, there exists precisely one pendant edge at u_i, hence $\deg(u_i) = 2$ for each i. By (4) and (5), the subtree T' of T is of order one. Let $w \in T'$. Then $w \in \cap_{i=1}^{k} N'(u_i)$. By (6), f assigns -1 to all pendant edges at w. Let r be the number of pendant edges at w. Then we have $2 - (2k + r)/3 = f(E(T)) = -r$, which implies $r = k - 3$ and $k \geqslant 3$.

Conversely, let G be a graph with the structure described in the theorem. By Theorem 1 we have $\gamma'_{st}(G) \geqslant 2 - (3k-3)/3$. Define $g\colon E(T) \longrightarrow \{-1, +1\}$ by

$$g(wu_i) = 1,\ g(u_i v_i) = -1\ (1 \leqslant i \leqslant k) \text{ and } g(ww_j) = -1\ (1 \leqslant j \leqslant k-3).$$

Obviously, g is an SETDF of T and $g(E(T)) = 2 - (3k-3)/3$. This completes the proof. □

3. Lower bounds

In this section we find some lower bounds for signed edge total domination numbers of simple connected graphs. Let G be a simple connected graph of order n and size $m \geqslant 2$. For every edge $e = uv \in V(G)$, the degree of e, $d(e)$, is defined by $d(e) = \deg(u) + \deg(v) - 2$. First we present a lower bound in terms of n, m, δ and Δ.

Theorem 3. *For every simple connected graph of order $n \geqslant 3$, size m and $\delta \geqslant 2$,*

$$\gamma'_{st}(G) \geqslant \left\lceil \frac{m - (\Delta - \delta)(\Delta - 1)(n - \delta)}{2(\Delta - 1)} \right\rceil.$$

Proof. Let f be a γ'_{st}-function of G. We have

$$\begin{aligned} (7) \qquad 2\gamma'_{st}(G) &= 2f(E(T) = 2(|E^+(G,f)| - |E^-(G,f)|) \\ &= \sum_{u \in V(G^+(f))} \deg_{G^+(f)}(u) - \sum_{u \in V(G^-(f))} \deg_{G^-(f)}(u) \\ &= \sum_{u \in V(G)} f(u). \end{aligned}$$

For $uv \in E(G)$ we have $f(u) + f(v) - 2f(uv) \geqslant 1$. Therefore

$$\begin{aligned} (8) \qquad m + 2\gamma'_{st}(G) &\leqslant \sum_{uv \in E(G)} (f(u) + f(v) - 2f(uv)) + 2 \sum_{uv \in E(G)} f(uv) \\ &= \sum_{uv \in E(G)} (f(u) + f(v)) \\ &= \sum_{u \in V(G)} f(u) \deg_G(u). \end{aligned}$$

Let $B_1 = \{u \in V(G);\ f(u) \geqslant 1\}$, $B_2 = \{u \in V(G);\ f(u) \leqslant -1\}$ and $B_3 = \{u \in V(G);\ f(u) = 0\}$. Obviously, for each $u \in B_2$ we have $N_G(u) \subseteq B_1 \cup B_3$. Hence,

$$(9) \qquad \delta \leqslant |N_G(u)| \leqslant |B_1| + |B_3| = n - |B_2|.$$

Thus by (7) and (8) we have

$$\begin{aligned} m + 2\gamma'_{st}(G) &\leqslant \sum_{u\in V(G)} f(u)\deg_G(u) \\ &= \sum_{u\in B_1} f(u)\deg_G(u) + \sum_{u\in B_2} f(u)\deg_G(u) \\ &\leqslant \Delta \sum_{u\in B_1} f(u) + \delta \sum_{u\in B_2} f(u) \\ &= \Delta \sum_{u\in V(G)} f(u) + (\delta-\Delta)\sum_{u\in B_2} f(u) \\ &= 2\Delta\gamma'_{st}(G) + (\delta-\Delta)\sum_{u\in B_2} f(u). \end{aligned}$$

Hence,

$$2(\Delta-1)\gamma'_{st}(G) \geqslant m + (\Delta-\delta)\sum_{u\in B_2} f(u). \tag{10}$$

Now for each $u \in B_2$ there exists $v \in N_G(u)$ such that $f(uv) = -1$. So we have $f(u)+f(v) \geqslant 1+2f(uv) = -1$. Since $f(v) \leqslant \Delta-2$, it follows that $f(u) \geqslant -(\Delta-1)$. Using (9) and (10) we have $2(\Delta-1)\gamma'_{st}(G) \geqslant m-(\Delta-\delta)(n-\delta)(\Delta-1)$. Now the result follows. □

The following result is an immediate consequence of Theorem 3.

Corollary 4. *For every simple k-regular graph G with $k \geqslant 2$, $\gamma'_{st}(G) \geqslant \lceil \frac{1}{2}m \times (k-1)\rceil$.*

Theorem 5. *For every simple connected graph G with $2 \leqslant \delta \leqslant \Delta \leqslant 4$, $\gamma'_{st}(G) \geqslant 0$.*

Proof. Let f be a γ'_{st}-function of G. Since $2 \leqslant \delta \leqslant \Delta \leqslant 4$, we have $|N_G(e) \cap E^+(G,f)| \geqslant 2$ and $|N_G(e)\cap E^-(G,f)| \leqslant 2$. Now it is clear that

$$\begin{aligned} 2|E^-(G,f)| &\leqslant \sum_{e\in E^-(G,f)} |N_G(e)\cap E^+(G,f)| \\ &= \sum_{e\in E^+(G,f)} |N_G(e)\cap E^-(G,f)| \\ &\leqslant 2|E^+(G,f)|. \end{aligned}$$

Thus $|E^-(G,f)| \leqslant |E^+(G,f)|$ and hence, $\gamma'_{st}(G) = |E^+(G,f)| - |E^-(G,f)| \geqslant 0$. □

Theorem 6. *For every simple connected graph G of order $n \geqslant 3$ and size m,*

$$\gamma'_{st}(G) \geqslant m\Big(\frac{2m}{n(\Delta-1)} - \frac{\varepsilon_{\mathrm{o}}}{2m(\Delta-1)} - 1\Big)$$

where ε_{o} is the number of edges of odd degree. Furthermore, this bound is sharp.

P r o o f. Let A be the set of edges of even degree. It is easy to see that if $uv \in A$, then $|N_G(e) \cap E^+(G,f)| \geqslant \frac{1}{2}(\deg(u)+\deg(v))$ and if $e \in E(G) \setminus A$, then $|N_G(e) \cap E^+(G,f)| \geqslant \frac{1}{2}(\deg(u)+\deg(v)-1)$. Thus

$$\begin{aligned}\sum_{uv\in E(G)} |N(e)\cap E^+(G,f)| &\geqslant \frac{1}{2}\sum_{uv\in E(G)}(\deg(u)+\deg(v)) - \frac{1}{2}\varepsilon_{\mathrm{o}}\\ &= \frac{1}{2}\sum_{u\in V(G)}\deg(u)^2 - \frac{1}{2}\varepsilon_{\mathrm{o}}\\ &\geqslant \frac{1}{2n}\Big(\sum_{u\in V(G)}\deg(u)\Big)^2 - \frac{1}{2}\varepsilon_{\mathrm{o}}\\ &= \frac{2m^2}{n} - \frac{1}{2}\varepsilon_{\mathrm{o}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}2(\Delta-1)|E^+(G,f)| &\geqslant \sum_{e\in E^+(G,f)} |N_G(e)|\\ &= \sum_{e\in E^+(G,f)} (|N_G(e)\cap E^+(G,f)| + |N_G(e)\cap E^-(G,f)|)\\ &= \sum_{e\in E^+(G,f)} |N_G(e)\cap E^+(G,f)| + \sum_{e\in E^+(G,f)} |N_G(e)\cap E^-(G,f)|\\ &= \sum_{e\in E^+(G,f)} |N_G(e)\cap E^+(G,f)| + \sum_{e\in E^-(G,f)} |N_G(e)\cap E^+(G,f)|\\ &= \sum_{e\in E(G)} |N_G(e)\cap E^+(G,f)|.\end{aligned}$$

Therefore $|E^+(G,f)| \geqslant \dfrac{m^2}{n(\Delta-1)} - \dfrac{\varepsilon_{\mathrm{o}}}{4(\Delta-1)}$. This implies that

$$\gamma'_{st}(G) = 2|E^+(G,f)| - m \geqslant m\Big(\frac{2m}{n(\Delta-1)} - \frac{\varepsilon_{\mathrm{o}}}{2m(\Delta-1)} - 1\Big).$$

Theorem D shows that this bound is sharp and the proof is complete. □

References

[1] *H. Karami, A. Khodkar and S. M. Sheikholeslami*: Signed edge domination numbers in trees. Ars Combinatoria. To appear.
[2] *D. B. West*: Introduction to Graph Theory. Prentice-Hall, Inc, 2000.
[3] *B. Xu*: On signed edge domination numbers of graphs. Discrete Mathematics *239* (2001), 179–189.
[4] *B. Xu*: On lower bounds of signed edge domination numbers in graphs. J. East China Jiaotong Univ. *1* (2004), 110–114. (In Chinese.)
[5] *B. Zelinka*: On signed edge domination numbers of trees. Math. Bohem. *127* (2002), 49–55.

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