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HOMOGENIZATION OF QUASILINEAR PARABOLIC PROBLEMS BY THE METHOD OF ROTHE AND TWO SCALE CONVERGENCE*

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Abstract. We consider a quasilinear parabolic problem with time dependent coefficients oscillating rapidly in the space variable. The existence and uniqueness results are proved by using Rothe's method combined with the technique of two-scale convergence.

Moreover, we derive a concrete homogenization algorithm for giving a unique and computable approximation of the solution.

Keywords: parabolic PDEs, Rothe's method, two-scale convergence, homogenization of periodic structures, homogenization algorithm

MSC 2010: 35K55, 74Q15

1. Introduction

Over the years PDEs with periodic rapidly oscillating coefficients have been studied by several authors, see e.g. [1], [2], [3], [4], [13], [14], [16], and [19]. These problems were mostly solved by using the method of multiple scale expansion or some mathematically based homogenization techniques, e.g. G-convergence, Γ-convergence or two scale convergence. However, recently J. Vala (see [18]) used Rothe's method (for more details on this method see e.g. [6], [7], [17]) and the technique of two scale convergence to solve a non-linear parabolic problem. In that paper the coefficient of the time derivative and that of the differential operator do not depend on time. In

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the present paper we continue this research for the corresponding quasilinear equation and solve the more general case when the coefficients also depend on time. In particular, this requires a partly new technique of proof. Moreover, we derive the corresponding homogenization result (see Theorem 3.1) and homogenization algorithm (see Corollary 3.3), which are useful for concrete numerical solutions of the actual problem.

The problem considered has the following form:

$$
\begin{cases} a(x,x/\varepsilon,t)\dfrac{\partial u^\varepsilon}{\partial t} - \nabla\cdot(b(x,x/\varepsilon,t)\nabla u^\varepsilon) = f(x,x/\varepsilon,t,u^\varepsilon) & \text{in } \Omega\times(0,T),\\ u^\varepsilon(x,0)=u_0 & \text{in } \Omega,\\ u^\varepsilon(x,t)=0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1}
$$

where $\varepsilon > 0$ is a small parameter, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $T < \infty$, a and b are functions defined in $\Omega\times\mathbb{R}^3\times(0,T)$ and the right-hand side function f is defined in $\Omega\times\mathbb{R}^3\times(0,T)\times\mathbb{R}$. The function u_0 is defined in Ω.

The paper is organized as follows: In Section 2 we present the necessary definitions and lemmas which are connected with two-scale convergence. In addition, we state some necessary assumptions and give a brief description of Rothe's method. Our main results are stated and discussed in Section 3 and the proofs are given in Section 4.

2. Preliminaries

In this section we first give some definitions and lemmas associated with two-scale convergence. Moreover, the space variable is represented by $x\in\Omega\subset\mathbb{R}^3$ while $t\in I=[0,T]\subset\mathbb{R}$ represents the time. The cell of periodicity is denoted by Y (i.e. the unit cube in $\mathbb{R}^3$). Moreover, we will use the space $C^\infty_{\mathrm{per}}(Y)$ and $W^{1,2}_{\mathrm{per}}(Y)$ of subspaces of $C^\infty(\mathbb{R}^3)$ and $W^{1,2}(\mathbb{R}^3)$, respectively, whose elements are periodic functions with periodicity Y.

Definition 2.1. Let u^0 be an element of $L_2(\Omega\times Y)$ and let $\varepsilon>0$. We say that a sequence u^ε from $L_2(\Omega)$ two-scale converges weakly to u^0 if

$$
\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon(x)\psi(x,x/\varepsilon)\,\mathrm{d}x = \int_\Omega\int_Y u^0(x,y)\psi(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad \forall\,\psi\in C_0^\infty(\Omega,C^\infty_{\mathrm{per}}(Y));
$$

briefly $u^\varepsilon \overset{2}{\rightharpoonup} u^0$.

Let us note that we can replace $C_0^\infty(\Omega,C^\infty_{\mathrm{per}}(Y))$ by $L_2(\Omega,C^\infty_{\mathrm{per}}(Y))$ in the definition, using the obvious density argument.

Definition 2.2. Let u^0 be an element of $L_2(\Omega \times Y)$. We say that a sequence u^ε from $L_2(\Omega)$ two-scale converges strongly to u^0 if $u^\varepsilon \overset{2}{\rightharpoonup} u^0$ and in addition

$$\lim_{\varepsilon \to 0} \int_\Omega |u^\varepsilon(x)|^2 \, \mathrm{d}x = \int_\Omega \int_Y |u^0(x,y)|^2 \, \mathrm{d}y \, \mathrm{d}x;$$

briefly $u^\varepsilon \overset{2}{\to} u^0$.

Lemma 2.3 ([2, Lemma 2.3])**.** *If $u^\varepsilon \overset{2}{\to} u^0$ and $v^\varepsilon \overset{2}{\rightharpoonup} v^0$, where $u^0, v^0 \in L_2(\Omega \times Y)$, then also*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) v^\varepsilon(x) \, \mathrm{d}x = \int_\Omega \int_Y u^0(x,y) v^0(x,y) \, \mathrm{d}y \, \mathrm{d}x.$$

In the sequel $H = L_2(\Omega)$ and $V = W_0^{1,2}(\Omega)$.

Lemma 2.4. *Let $\{u^\varepsilon\}$ be a bounded sequence in the space $C^{0,1}(I,H) \cap L_\infty(I,V)$. Then there exist functions $u \in C^{0,1}(I,H) \cap L_\infty(I,V)$ and $\tilde{u} \in L_\infty(I, L_2(\Omega, W^{1,2}_{\mathrm{per}}(Y)))$ such that up to a subsequence,*

a) $u^\varepsilon(t) \rightharpoonup u(t)$ *in* V *for every* $t \in I$,
b) $u^\varepsilon \to u$ *in* $C(I,H)$,
c) $u^\varepsilon(t) \overset{2}{\rightharpoonup} u(t)$ *for every* $t \in I$,
d) $\nabla u^\varepsilon(t) \overset{2}{\rightharpoonup} \nabla u(t) + \nabla_Y \tilde{u}(t)$ *for every* $t \in I$,
e) $(\partial u^\varepsilon / \partial t)(t) \overset{2}{\rightharpoonup} (\partial u / \partial t)(t)$ *for every* $t \in I$.

P r o o f. The lemma can be proved analogously to Lemma 5 in [18], and, thus, we leave out the details. □

To prove the uniqueness of the solution of the problem we will use the following Gronwall type lemma.

Lemma 2.5 ([15, Theorem 1.2.2])**.** *Let u and f be continuous and nonnegative functions defined on $J = [\alpha, \beta]$, and let C be a nonnegative constant. Then the inequality*

$$u(t) \leqslant C + \int_\alpha^t f(s) u(s) \, \mathrm{d}s, \quad t \in J,$$

implies that

$$u(t) \leqslant C \exp\biggl(\int_\alpha^t f(s) \, \mathrm{d}s\biggr), \quad t \in J.$$

Now we present a brief description of Rothe's method for the situation at hand. Using this method, we can solve the following parabolic problem, which is the weak form of problem (1.1):

$$
\begin{aligned}
(2.1)\ u^\varepsilon(t) \in V\colon \Big(a_\varepsilon(t)\frac{\partial u^\varepsilon(t)}{\partial t}, v\Big) + \langle b_\varepsilon(t)u^\varepsilon(t), v\rangle &= (f_\varepsilon(t, u^\varepsilon(t)), v) \quad \text{for all } v \in V,\\
u^\varepsilon(0) &= u_0,
\end{aligned}
$$

where

$$
\begin{aligned}
(2.2) \qquad u^\varepsilon(t) &:= u^\varepsilon(x,t), \quad a_\varepsilon(t) := a(x, x/\varepsilon, t), \quad b_\varepsilon(t) := b(x, x/\varepsilon, t),\\
f_\varepsilon(t, u^\varepsilon(t)) &:= f(x, x/\varepsilon, t, u^\varepsilon(x,t))
\end{aligned}
$$

for a fixed ε, and $(\cdot,\cdot)$ denotes the scalar product in H and

$$
(2.3) \qquad \langle b_\varepsilon(t)u^\varepsilon(t), v\rangle = \int_\Omega b_\varepsilon(t)\nabla u^\varepsilon(t)\cdot\nabla v\,\mathrm{d}x.
$$

We also need the following technical assumptions on the functions a, b, f, and u_0 in order to be able to solve problem (1.1).

Assumption 2.6. Let C_1, C_2 be positive numbers and $y \in \mathbb{R}^3$. Then

(A1) the functions a, b satisfy the following conditions: for all $t \in (0,T)$, we have

$$
\begin{gathered}
C_1 \leqslant w(x,y,t) \leqslant C_2 \quad \text{for almost all } x \in \Omega,\\
\|w(\cdot,y,t) - w(\cdot,y,\tau)\|_{L_\infty(\Omega)} \leqslant C_2|t-\tau| \quad \text{for all } \tau \in (0,T),
\end{gathered}
$$

where $w = a$ (or $w = b$);

(A2) the function f satisfies the following condition:

$$
\|f(\cdot,y,t,u) - f(\cdot,y,\tau,v)\|_H \leqslant C_2(|t-\tau| + \|u-v\|_H)
$$

for all $t,\tau \in I$ and $u, v \in H$;

(A3) the function u_0 from V is such that:

$$
\nabla\cdot(b(x,y,0)\nabla u_0) \in H.
$$

(Here we write $b(x,y,0)$, which is the limit of $b(x,y,t)$ as $t \to 0$, since the existence of this limit is guaranteed by (A1).) Moreover, it is supposed that the functions a, b, f are Y-periodic in the second variable y.

Rothe's method. Let h be a positive number. We divide the interval $I=[0,T]$ into subintervals $I_1, I_2, \ldots, I_n$ ($I_j = [t_{j-1}, t_j)$, $t_j = jh$, $j = 1,2,\ldots,n-1$, and $I_n = [t_{n-1}, T]$, where $0 < T - t_{n-1} \leqslant h$) such that the interval I is covered by these intervals. Taking into account the initial condition of problem (2.1), we put

$$z_0 = u_0$$

for $t_0 = 0$ and successively for $j = 1,2,\ldots,n$ define vector functions z_j which are weak solutions of the elliptic problems

$$z_j \in V\colon\ \frac{1}{h}(a_j z_j, v) + \langle b_j z_j, v\rangle = \Big(f_j(z_{j-1}) + \frac{a_j}{h} z_{j-1}, v\Big) \quad \text{for all } v \in V, \tag{2.4}$$

where $a_j = a_\varepsilon(t_j)$, $b_j = b_\varepsilon(t_j)$ and $f_j(z_{j-1}) = f_\varepsilon(t_j, z_{j-1})$. We obtain these problems, if we replace the derivative $\partial u^\varepsilon(t)/\partial t$ by the differential quotient $(z_j - z_{j-1})/h$ at the points $t = t_j$, $j = 1,2,\ldots,n$, in (2.1).

Let $j = 1$. Then problem (2.4) takes the form

$$z_1 \in V\colon\ \frac{1}{h}(a_1 z_1, v) + \langle b_1 z_1, v\rangle = \Big(f_1(z_0) + \frac{a_0}{h} z_0, v\Big) \quad \text{for all } v \in V,$$

and it has exactly one solution (by virtue of Assumption 2.6 and as a consequence of the theory of elliptic boundary value problems; see e.g. [5]). Next we solve problem (2.4) for $j = 2$, i.e.

$$z_2 \in V\colon\ \frac{1}{h}(a_2 z_2, v) + \langle b_2 z_2, v\rangle = \Big(f_2(z_1) + \frac{1}{h} a_1 z_1, v\Big) \quad \text{for all } v \in V.$$

Repeating the above procedure for $j = 3,\ldots,n$, we get functions $z_1, z_2, \ldots, z_n \in V$ which are uniquely determined. It is thus possible to construct the *Rothe function* $u_n(t)$ as a function from I to V defined by

$$u_n(t) = z_{j-1} + \frac{t - t_{j-1}}{h}(z_j - z_{j-1}), \quad t \in I_j,\ j = 1,2,\ldots,n. \tag{2.5}$$

Hence, we obtain a sequence $\{u_n(t)\}_{n=1}^{\infty}$ which is called *Rothe's sequence* of approximative solutions of problem (2.1). Intuitively, we can expect that if $n \to \infty$, then this sequence will converge to some function $u^\varepsilon(t)$, which is a solution of problem (2.1).

In the next section we will in particular present and prove that this in fact holds in a special sense. Roughly speaking, first we use Rothe's method to prove the existence of $u^\varepsilon(t)$ as $n \to \infty$ (see Theorem 3.4). After that we use the technique of two-scale convergence to prove that $u^\varepsilon(t)$ actually converges to a unique function $u(t)$ as $\varepsilon \to 0$ and this is the approximative (homogenized) solution of (1.1) we are looking for (see Theorem 3.1 (a)). As expected this solution can be calculated by using a homogenization algorithm (see our Theorem 3.1 (b) and Corollary 3.3).

We are now ready to present and prove our main results.

3. Main results

In this section, the notation Ω, Y, V, H, and I have the same meaning as in our previous sections and, moreover, the functions $\tilde{a}(x,t)$ and $\tilde{f}(x,t,u)$ are defined by

$$\tilde{a}(x,t) := \int_Y a(x,y,t)\,\mathrm{d}y \quad \text{and} \quad \tilde{f}(x,t,u) := \int_Y f(x,y,t,u)\,\mathrm{d}y.$$

Our main result reads:

Theorem 3.1. *Let Assumption* 2.6 *be satisfied. Then*

(a) *problem* (1.1) *has a unique solution, and this solution can be approximated by a unique function* $u \in C^{0,1}(I,H) \cap L_\infty(I,V)$ *such that* $\tilde{u} \in L_\infty(I, L_2(\Omega, W^{1,2}_{\mathrm{per}}(Y)))$ *and*

$$\begin{aligned}\int_\Omega \tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t)v(x)\,\mathrm{d}x \qquad\qquad & \\ + \int_\Omega\int_Y b(x,y,t)(\nabla u(x,t) + \nabla_Y \tilde{u}(x,y,t)) \cdot \nabla v(x)\,\mathrm{d}y\,\mathrm{d}x & \\ = \int_\Omega \tilde{f}(x,t,u(x,t))v(x)\,\mathrm{d}x & \end{aligned} \tag{3.1}$$

for all $v \in V$ *and at almost every time* $t \in I$, *and* $u(x,0) = u_0(x)$ *for almost every* $x \in \Omega$;

(b) *the unique solution* $u(x,t)$ *in* (a) *can be obtained by solving the equation*

$$\tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t) - \nabla \cdot (B(x,t)\nabla u(x,t)) = \tilde{f}(x,t,u(x,t)), \tag{3.2}$$

where the matrix $B(x,t) = (b_{ij}(x,t))_{i,j=1,2,3}$ *is defined by*

$$\begin{pmatrix} b_{1j}(x,t) \\ b_{2j}(x,t) \\ b_{3j}(x,t)\end{pmatrix} = \int_Y b(e_j + \nabla_Y w_j)\,\mathrm{d}y \quad \begin{pmatrix} b_{12}(x,t) \\ b_{22}(x,t) \\ b_{32}(x,t)\end{pmatrix} = \int_Y b(e_2 + \nabla_Y w_2)\,\mathrm{d}y,$$
$$\begin{pmatrix} b_{13}(x,t) \\ b_{23}(x,t) \\ b_{33}(x,t)\end{pmatrix} = \int_Y b(e_3 + \nabla_Y w_3)\,\mathrm{d}y \tag{3.3}$$

and $w_i \in L_\infty(I, L_2(\Omega, W^{1,2}_{\mathrm{per}}(Y)))$, $i = 1,2,3$, *are the solutions of the local problems*

$$(3.4)\qquad \begin{cases} \displaystyle\int_Y b(x,y,t)(e_1 + \nabla_Y w_1)\cdot\nabla v(y)\,\mathrm{d}y = 0,\\ \displaystyle\int_Y b(x,y,t)(e_2 + \nabla_Y w_2)\cdot\nabla v(y)\,\mathrm{d}y = 0,\\ \displaystyle\int_Y b(x,y,t)(e_3 + \nabla_Y w_3)\cdot\nabla v(y)\,\mathrm{d}y = 0 \end{cases}$$

for all $v \in C^\infty_{\mathrm{per}}(Y)$, *where* $\{e_1, e_2, e_2\}$ *is the canonical basis in* $\mathbb{R}^3$.

Remark 3.2. For the case when the coefficients a, b and the right-hand side f do not depend on t, Theorem 3.1 (a) coincides with that of Vala [18] in the quasilinear case. Moreover, Theorem 3.1 (b) is well suited to be directly applied for obtaining a good approximation of the solution of (1.1).

More exactly, we obtain the following homogenization algorithm for deriving an approximative solution of equation (1.1):

Corollary 3.3 (Homogenization algorithm). *An approximative solution of equation* (1.1) *can be obtained in the following way:*

Step 1: Solve the local problems (3.4).

Step 2: Insert the solutions of the local problems into (3.3) *and compute the homogenized coefficient* $B(x,t)$.

Step 3: Solve the homogenized equation (3.2), *which gives the approximative solution* $u(x,t)$ *we are looking for.*

In order to be able to prove Theorem 3.1 we need the following crucial result of independent interest:

Theorem 3.4. *Let Assumption* 2.6 *be satisfied. Then there exists a function* $u^\varepsilon \in C^{0,1}(I,H) \cap L_\infty(I,V)$ *which solves* (1.1) *and has the following properties (for each fixed* $\varepsilon > 0$*):*

1. $\|u^\varepsilon\|_{L_\infty(I,V)} \leqslant C,\quad \|\partial u^\varepsilon/\partial t\|_{L_\infty(I,H)} \leqslant C,$
2. $u^\varepsilon(0) = u_0,$
3.

$$(3.5)\qquad \int_\Omega a(x,x/\varepsilon,t)\frac{\partial u^\varepsilon}{\partial t}(x,t)v(x)\,\mathrm{d}x + \int_\Omega b(x,x/\varepsilon,t)\nabla u^\varepsilon(x,t)\cdot\nabla v(x)\,\mathrm{d}x = \int_\Omega f(x,x/\varepsilon,t,u^\varepsilon(x,t))v(x)\,\mathrm{d}x$$

for all $v \in V = W^{1,2}_0(\Omega)$ *and at almost every time* $t \in I$, *where* C *does not depend on* ε.

4. Proofs

P r o o f of Theorem 3.4. Let us consider the integral identity (2.4), i.e.

$$\frac{1}{h}(a_j(z_j - z_{j-1}), v) + \langle b_j z_j, v\rangle = (f_j(z_{j-1}), v) \quad \text{for all } v \in V. \tag{4.1}$$

We choose $v = z_j - z_{j-1}$ in (4.1). Then we get that

$$\frac{1}{h}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) + \langle b_j z_j, z_j - z_{j-1}\rangle = (f_j(z_{j-1}), z_j - z_{j-1})$$

for $j = 1, 2, \ldots, n$. By applying the Schwarz inequality to the right-hand side of the last equality we obtain that

$$\frac{1}{h}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) + \langle b_j z_j, z_j - z_{j-1}\rangle \leqslant \|f_j(z_{j-1})\|_H \|z_j - z_{j-1}\|_H.$$

Hence, according to (A1) and (A2) of Assumption 2.6, we find that

$$\frac{C_1}{h}\|z_j - z_{j-1}\|_H^2 + \langle b_j z_j, z_j - z_{j-1}\rangle \leqslant C_2(jh + \|z_{j-1}\|_H)\|z_j - z_{j-1}\|_H.$$

By applying first the trivial inequality $ab \leqslant a^2/2\theta + b^2\theta/2$ (for $\theta = 2C_1/hC_2 > 0$) followed by $(a+b)^2 \leqslant 2(a^2+b^2)$ to the right-hand side of the last estimate, we get that

$$\langle b_j z_j, z_j - z_{j-1}\rangle \leqslant \frac{hC_2^2}{4C_1}(T + \|z_{j-1}\|_H)^2,$$

i.e.

$$\langle b_j z_j, z_j - z_{j-1}\rangle \leqslant \frac{hC_2^2}{2C_1}(T^2 + \|z_{j-1}\|_H^2). \tag{4.2}$$

According to the Poincaré inequality, (2.3), and (A1) of Assumption 2.6 we have that

$$\|z_{j-1}\|_H \leqslant C_*\|z_{j-1}\|_V \leqslant C_*\Big[\frac{1}{C_1}\langle b_{j-1} z_{j-1}, z_{j-1}\rangle\Big]^{1/2} \tag{4.3}$$

and

$$|\langle (b_{j-1} - b_j) z_{j-1}, z_{j-1}\rangle| \leqslant \frac{C_2 h}{C_1}\langle b_{j-1} z_{j-1}, z_{j-1}\rangle,$$

so that

$$-\frac{C_2 h}{C_1}\langle b_{j-1} z_{j-1}, z_{j-1}\rangle \leqslant \langle (b_{j-1} - b_j) z_{j-1}, z_{j-1}\rangle. \tag{4.4}$$

Inserting (4.3) into (4.2), we see that

$$\langle b_j z_j, z_j - z_{j-1}\rangle \leqslant \frac{hC_2^2}{2C_1}\Big(T^2 + \Big[\frac{C_*^2}{C_1}\langle b_{j-1}z_{j-1}, z_{j-1}\rangle\Big]\Big). \tag{4.5}$$

Moreover, we estimate the left-hand side of (4.2) as follows:

$$\begin{aligned} &\langle b_j z_j, z_j - z_{j-1}\rangle \\ &\quad = \frac{1}{2}\Big[\langle b_j z_j, z_j\rangle + \langle b_j(z_j - z_{j-1}), z_j - z_{j-1}\rangle - \langle b_j z_{j-1}, z_{j-1}\rangle\Big] \\ &\quad = \frac{1}{2}[\langle b_j z_j, z_j\rangle + \langle b_j(z_j - z_{j-1}), z_j - z_{j-1}\rangle + \langle (b_{j-1} - b_j)z_{j-1}, z_{j-1}\rangle \\ &\qquad - \langle b_{j-1}z_{j-1}, z_{j-1}\rangle] \\ &\quad \geqslant \frac{1}{2}[\langle b_j z_j, z_j\rangle + \langle (b_{j-1} - b_j)z_{j-1}, z_{j-1}\rangle - \langle b_{j-1}z_{j-1}, z_{j-1}\rangle]. \end{aligned} \tag{4.6}$$

Inserting (4.4) into (4.6) and simplifying, we find that

$$\langle b_j z_j, z_j - z_{j-1}\rangle \geqslant \frac{1}{2}\Big[\langle b_j z_j, z_j\rangle - \Big(1 + \frac{C_2}{C_1}h\Big)\langle b_{j-1}z_{j-1}, z_{j-1}\rangle\Big]. \tag{4.7}$$

From (4.5) and (4.7) it follows that

$$\frac{1}{2}\Big[\langle b_j z_j, z_j\rangle - \Big(1 + \frac{C_2}{C_1}h\Big)\langle b_{j-1}z_{j-1}, z_{j-1}\rangle\Big] \leqslant \frac{hC_2^2}{2C_1}\Big(T^2 + \frac{C_*^2}{C_1}\langle b_{j-1}z_{j-1},\, z_{j-1}\rangle\Big).$$

By choosing now $C < \infty$ such that

$$\frac{C_2^2}{C_1}T^2 \leqslant C \quad \text{and} \quad \frac{C_*^2C_2^2 + C_2C_1}{C_1^2} \leqslant C,$$

we find that

$$\langle b_j z_j, z_j\rangle \leqslant Ch + (1 + Ch)\langle b_{j-1}z_{j-1},\, z_{j-1}\rangle.$$

Hence, using repeatedly this and the fact that $\ln(1+t) \leqslant t$, $t \geqslant 0$ yields that

$$\begin{aligned} \langle b_j z_j, z_j\rangle &\leqslant Ch + (1 + Ch)\langle b_{j-1}z_{j-1}, z_{j-1}\rangle \\ &\leqslant Ch + Ch(1 + Ch) + (1 + Ch)^2\langle b_{j-2}z_{j-2}, z_{j-2}\rangle \\ &\leqslant \ldots \leqslant (1 + Ch)^j + (1 + Ch)^j\langle b_0 z_0, z_0\rangle \\ &\qquad = \mathrm{e}^{j\ln(1+Ch)} + \mathrm{e}^{j\ln(1+Ch)}\langle b_0 z_0, z_0\rangle \\ &\leqslant \mathrm{e}^{CT} + \mathrm{e}^{CT}\langle b_0 z_0, z_0\rangle. \end{aligned}$$

By virtue of (A1) and (A3) of Assumption 2.6 we get that

$$\|z_j\|_V \leqslant C_3 \quad \text{for all } j = 1, 2, \ldots, n, \tag{4.8}$$

where C_3 does not depend on j and n.

From this estimate we obtain the uniform boundedness of Rothe's sequence, i.e. according to (2.5) and (4.8) it is obvious that

$$\|u_n(t)\|_V^2 \leqslant \max_j \|z_j\|_V^2 \leqslant C_3^2. \tag{4.9}$$

This estimate implies that the first estimate of the theorem holds.

Next we estimate the derivative of the Rothe's sequence, i.e. $\{\partial u_n(t)/\partial t\}$, which is also connected to the proof of the second estimate of the theorem. To this end we consider the identity (4.1), i.e.

$$\frac{1}{h}(a_j(z_j - z_{j-1}), v) + \langle b_j z_j, v\rangle = (f_j(z_{j-1}), v) \quad \text{for all } v \in V.$$

Subtracting from this identity the same identity written for $j-1$ and putting $v = z_j - z_{j-1}$, we obtain that

$$\begin{aligned} &\frac{1}{h}(a_j(z_j - z_{j-1}) - a_{j-1}(z_{j-1} - z_{j-2}), z_j - z_{j-1}) \\ &\qquad + \langle b_j z_j - b_{j-1} z_{j-1}, z_j - z_{j-1}\rangle \\ &\quad = (f_j(z_{j-1}) - f_{j-1}(z_{j-2}), z_j - z_{j-1}). \end{aligned} \tag{4.10}$$

We will separately estimate all terms of (4.10). Let us begin with the first term on the left-hand side. To estimate it we use the inequalities

$$(a_{j-1}u, u) - 2(a_{j-1}u, v) + (a_{j-1}v, v) \geqslant 0 \quad \text{for all } u, v \in V$$

and

$$\left|1 - \frac{a_{j-1}(x)}{a_j(x)}\right| \leqslant \frac{C_2}{C_1} h \quad \text{for a.e. } x \in \Omega$$

(for all $j = 1, 2, \ldots, n$), which immediately follows from (A1) of Assumption 2.6. By using these estimates we find that

$$\begin{aligned} &\frac{1}{h}(a_j(z_j - z_{j-1}) - a_{j-1}(z_{j-1} - z_{j-2}), z_j - z_{j-1}) \\ &= \frac{1}{2h}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) + \frac{1}{2h}[(a_{j-1}(z_j - z_{j-1}), z_j - z_{j-1}) \\ &\quad - 2(a_{j-1}(z_{j-1} - z_{j-2}), z_j - z_{j-1}) + (a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2})] \\ &\quad + \frac{1}{2h}((a_j - a_{j-1})(z_j - z_{j-1}), z_j - z_{j-1}) \\ &\quad - \frac{1}{2h}(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}) \end{aligned} \tag{4.11}$$

$$
\begin{aligned}
&\geqslant \frac{1}{2h}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) + \frac{1}{2h}\Big(a_j\Big(1 - \frac{a_{j-1}}{a_j}\Big)(z_j - z_{j-1}), z_j - z_{j-1}\Big) \\
&\quad - \frac{1}{2h}(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}) \\
&\geqslant \frac{1}{2h}\Big(1 - \frac{C_2}{C_1}h\Big)(a_j(z_j - z_{j-1}), z_j - z_{j-1}) \\
&\quad - \frac{1}{2h}(a_{j-1}(z_{j-1} - z_{j-2}), z_{j-1} - z_{j-2}).
\end{aligned}
$$

Next we use (A1) of Assumption 2.6 to obtain that

$$
\begin{aligned}
|\langle (b_{j-1} - b_j)z_{j-1}, z_j - z_{j-1}\rangle| &\leqslant C_2 h \int_\Omega |\nabla z_{j-1}||\nabla(z_j - z_{j-1})| \,\mathrm{d}x \\
&\leqslant C_2 h \|\nabla z_{j-1}\|_H \|\nabla(z_j - z_{j-1})\| \\
&\leqslant C_2 h \|z_{j-1}\|_V \|z_j - z_{j-1}\|_V,
\end{aligned} \tag{4.12}
$$

which implies that

$$
-\langle (b_{j-1} - b_j)z_{j-1}, z_j - z_{j-1}\rangle \geqslant -C_2 h\|z_{j-1}\|_V \|z_j - z_{j-1}\|_V. \tag{4.13}
$$

Thus, by using again (A1) of Assumption 2.6, (4.13), and (4.8), we can estimate the second term on the left-hand side of (4.10) as follows:

$$
\begin{aligned}
&\langle b_j z_j - b_{j-1}z_{j-1}, z_j - z_{j-1}\rangle \\
&\quad = \langle b_j(z_j - z_{j-1}), z_j - z_{j-1}\rangle - \langle (b_{j-1} - b_j)z_{j-1}, z_j - z_{j-1}\rangle \\
&\quad \geqslant C_1\|z_j - z_{j-1}\|_V^2 - C_2 h\|z_{j-1}\|_V \|z_j - z_{j-1}\|_V \\
&\quad = C_1\Big[\|z_j - z_{j-1}\|_V - \frac{C_2 h}{2C_1}\|z_{j-1}\|_V\Big]^2 - C_1\Big[\frac{C_2 h}{2C_1}\|z_{j-1}\|_V\Big]^2 \\
&\quad \geqslant -\frac{C_2^2 C_3^2}{4C_1}h^2.
\end{aligned} \tag{4.14}
$$

Moreover, for the right-hand side of (4.10) we use the Schwarz inequality, (A2) in Assumption 2.6, and elementary inequalities to find that

$$
\begin{aligned}
&(f_j(z_{j-1}) - f_{j-1}(z_{j-2}), z_j - z_{j-1}) \\
&\qquad \leqslant \|f_j(z_{j-1}) - f_{j-1}(z_{j-2})\|_H \|z_j - z_{j-1}\|_H \\
&\qquad \leqslant C_2(h + \|z_{j-1} - z_{j-2}\|_H)\|z_j - z_{j-1}\|_H \\
&\qquad \leqslant C_2 h^2 + C_2\|z_{j-1} - z_{j-2}\|_H^2 + \frac{C_2}{2}\|z_j - z_{j-1}\|_H^2.
\end{aligned} \tag{4.15}
$$

By using (A1) of Assumption 2.6 we see that

$$\begin{aligned}
&(a_{j-1}(z_{j-1}-z_{j-2}), z_{j-1}-z_{j-2}) \\
&\qquad = \int_\Omega a_{j-1}(z_{j-1}-z_{j-2})(z_{j-1}-z_{j-2})\,\mathrm{d}x \\
&\qquad \geqslant C_1 \int_\Omega |z_{j-1}-z_{j-2}|^2\,\mathrm{d}x = C_1\,\|z_{j-1}-z_{j-2}\|_H^2\,,
\end{aligned}$$

i.e.

$$\|z_{j-1}-z_{j-2}\|_H^2 \leqslant \frac{1}{C_1}(a_{j-1}(z_{j-1}-z_{j-2}), z_{j-1}-z_{j-2}). \tag{4.16}$$

Similarly

$$\|z_j-z_{j-1}\|_H^2 \leqslant \frac{1}{C_1}(a_j(z_j-z_{j-1}), z_j-z_{j-1}). \tag{4.17}$$

Inserting (4.17) and (4.16) into (4.15) we see that

$$\begin{aligned}
&(f_j(z_{j-1})-f_{j-1}(z_{j-2}), z_j-z_{j-1}) \\
&\qquad \leqslant C_2h^2 + \frac{C_2}{C_1}(a_{j-1}(z_{j-1}-z_{j-2}), z_{j-1}-z_{j-2}) \\
&\qquad\quad + \frac{C_2}{2C_1}(a_j(z_j-z_{j-1}), z_j-z_{j-1}).
\end{aligned} \tag{4.18}$$

Now, according to (4.11), (4.14), (4.18), and (4.10) we get that

$$\begin{aligned}
&\frac{1}{2h}\Big(1-\frac{C_2}{C_1}h\Big)(a_j(z_j-z_{j-1}), z_j-z_{j-1}) \\
&\qquad - \frac{1}{2h}(a_{j-1}(z_{j-1}-z_{j-2}), z_{j-1}-z_{j-2}) - \frac{C_2^2C_3^2}{4C_1}h^2 \\
&\quad \leqslant C_2h^2 + \frac{C_2}{C_1}(a_{j-1}(z_{j-1}-z_{j-2}), z_{j-1}-z_{j-2}) \\
&\qquad + \frac{C_2}{2C_1}(a_j(z_j-z_{j-1}), z_j-z_{j-1}).
\end{aligned} \tag{4.19}$$

If we denote $\alpha_j = h^{-2}(a_j(z_j-z_{j-1}), z_j-z_{j-1})$ and insert it into (4.19) we find that

$$\Big(1-\frac{C_2}{C_1}h\Big)\alpha_j - \alpha_{j-1} - \frac{hC_2^2C_3^2}{2C_1} \leqslant 2C_2h + \frac{2C_2h}{C_1}\alpha_{j-1} + \frac{C_2h}{C_1}\alpha_j,$$

i.e.,

$$\begin{aligned}
\alpha_j - \alpha_{j-1} &\leqslant h\Big[\Big(\frac{C_2^2C_3^2}{2C_1} + 2C_2\Big) + \frac{2C_2}{C_1}(\alpha_j+\alpha_{j-1})\Big] \\
&\leqslant h[C + C(\alpha_j+\alpha_{j-1})] = Ch[1+\alpha_j+\alpha_{j-1}],
\end{aligned} \tag{4.20}$$

where $C < \infty$ is chosen such that

$$\frac{C_2^2 C_3^2}{2C_1} + 2C_2 \leqslant C \quad \text{and} \quad \frac{2C_2}{C_1} \leqslant C.$$

Simplifying further, we see that the last estimate takes the form

$$\alpha_j \leqslant \frac{1+Ch}{1-Ch}\alpha_{j-1} + \frac{Ch}{1-Ch}.$$

By using this estimate repeatedly together with another elementary estimate, we get that

$$\alpha_j \leqslant \Big(\frac{1+Ch}{1-Ch}\Big)^{j-1}\alpha_1 + \Big(\frac{1+Ch}{1-Ch}\Big)^{j-1}. \tag{4.21}$$

Without loss of generality it can be supposed that h is less than $1/2C$, which enables us to make the estimate

$$\begin{aligned}\Big(\frac{1+Ch}{1-Ch}\Big)^{j-1} &= \mathrm{e}^{(j-1)\ln(1+2Ch/(1-Ch))} \\ &\leqslant \mathrm{e}^{2C(j-1)h/(1-Ch)} \leqslant \mathrm{e}^{2CT/(1-Ch)} \leqslant \mathrm{e}^{4CT}.\end{aligned} \tag{4.22}$$

Our next goal is to estimate α_1 in a similar way. First we rewrite the identity (4.1) for $j = 1$ and put $v = z_1 - z_0$, i.e.

$$\frac{1}{h}(a_1(z_1 - z_0), z_1 - z_0) + \langle b_1 z_1, z_1 - z_0\rangle = (f_1(z_0), z_1 - z_0). \tag{4.23}$$

According to (4.12) for $j = 1$ we have that

$$\langle (b_1 - b_0)z_0, z_1 - z_0\rangle \geqslant -C_2 h\|z_0\|_V\|z_1 - z_0\|_V. \tag{4.24}$$

Moreover, by using Green's formula and the Schwarz inequality, we have that

$$\begin{aligned}|\langle b_0 z_0, z_1 - z_0\rangle| &= \Big|\int_\Omega b_0\nabla z_0 \cdot \nabla(z_1 - z_0)\,\mathrm{d}x\Big| \\ &= \Big|-\int_\Omega \nabla\cdot(b_0\nabla z_0)(z_1 - z_0)\,\mathrm{d}x\Big| \\ &\leqslant \|\nabla\cdot(b_0\nabla z_0)\|_H\|z_1 - z_0\|_H \\ &\leqslant h\|\nabla\cdot(b_0\nabla z_0)\|_H\Big\|\frac{z_1 - z_0}{h}\Big\|_H.\end{aligned} \tag{4.25}$$

From (4.17) we see that

$$\frac{C_1}{h^2}\,\|z_j - z_{j-1}\|_H^2 \leqslant \frac{1}{h^2}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) = \alpha_j,$$

so that, in particular for $j = 1$,

$$\left\| \frac{z_1 - z_0}{h} \right\|_H \leqslant \sqrt{\frac{\alpha_1}{C_1}}. \tag{4.26}$$

Also, according to (4.8) and Assumption 2.6 we have that

$$\|z_j\|_V \leqslant C_3 \quad \text{for all} \quad j = 0, 1, \dots, n, \tag{4.27}$$

so that, in particular,

$$\|z_1 - z_0\|_V \leqslant \|z_1\|_V + \|z_0\|_V \leqslant 2C_3. \tag{4.28}$$

Thus, by using (4.24)–(4.28), we can estimate the second term on the left-hand side of (4.23) in the following way:

$$\begin{aligned} \langle b_1 z_1, z_1 - z_0 \rangle &= \langle b_1(z_1 - z_0), z_1 - z_0 \rangle + \langle b_1 z_0, z_1 - z_0 \rangle \\ &\geqslant \langle b_1 z_0, z_1 - z_0 \rangle \\ &= \langle (b_1 - b_0) z_0, z_1 - z_0 \rangle + \langle b_0 z_0, z_1 - z_0 \rangle \\ &\geqslant -C_2 h \|z_0\|_V \|z_1 - z_0\|_V - h \|\nabla \cdot (b_0 \nabla z_0)\|_H \left\| \frac{z_1 - z_0}{h} \right\|_H \\ &\geqslant -2C_2 C_3^2 h - \frac{C_4}{C_1^{1/2}} \alpha_1^{1/2} h. \end{aligned} \tag{4.29}$$

By using the Schwarz inequality, (A2) of Assumption 2.6, and (4.26) the right-hand side of (4.23) can be estimated as follows:

$$(f_1(z_0), z_1 - z_0) \leqslant \|f_1(z_0)\|_H \|z_1 - z_0\|_H \leqslant \frac{C_5}{C_1^{1/2}} \alpha_1^{1/2} h. \tag{4.30}$$

Inserting (4.29) and (4.30) into (4.23), we see that

$$\alpha_1 - 2C_2 C_3^2 - \frac{C_4}{C_1^{1/2}} \alpha_1^{1/2} \leqslant \frac{C_5}{C_1^{1/2}} \alpha_1^{1/2},$$

i.e.

$$\alpha_1 \leqslant C_0 + C_0 \alpha_1^{1/2},$$

which implies that

$$\alpha_1 \leqslant 2C_0 + C_0^2, \tag{4.31}$$

where $C_0 < \infty$ is chosen such that

$$2C_2C_3^2 \leqslant C_0 \quad \text{and} \quad \frac{C_4 + C_5}{C_1^{1/2}} \leqslant C_0.$$

Hence, from (4.21), (4.22), and (4.31) we see that

$$\alpha_j \leqslant \mathrm{e}^{4CT}\,(1 + C_0)^2 := C_6 < \infty.$$

Therefore, according to (4.17),

$$C_1 \left\| \frac{z_j - z_{j-1}}{h} \right\|_H^2 \leqslant \frac{1}{h^2}(a_j(z_j - z_{j-1}), z_j - z_{j-1}) = \alpha_j \leqslant C_6 < \infty.$$

Thus, it follows that

$$\left\| \frac{z_j - z_{j-1}}{h} \right\|_H \leqslant C^*.$$

The last estimate proves the uniform boundedness of the derivative of Rothe's functions, i.e. (see (2.5))

$$\max_{t \in [0,T]} \left\| \frac{\partial u_n}{\partial t}(t) \right\|_H = \max_{j=1,2,\ldots,n} \left\| \frac{z_j - z_{j-1}}{h} \right\|_H \leqslant C^*. \tag{4.32}$$

Next, let us introduce the sequence

$$v^{\varepsilon_n}(t) = u_n(t), \quad t \in I, \;\; n = 1, 2, \ldots,$$

where $\{\varepsilon_n\}_{n=1}^{\infty}$ is a parameter sequence such that $\varepsilon_n \to 0$ is equivalent to $n \to \infty$. According to (4.9) and (4.32) it follows that the sequence $v^{\varepsilon_n}(t)$ satisfies the conditions of Lemma 2.4. Therefore, in particular, we obtain that there exists a function $v \in C^{0,1}(I, H) \cap L_\infty(I, V)$ and, up to a subsequence,

a) $v^{\varepsilon_n}(t) \rightharpoonup v(t)$ in V for every $t \in I$,
b) $v^{\varepsilon_n} \to v$ in $C(I, H)$,
e) $(\partial v^{\varepsilon_n}/\partial t)(t) \overset{2}{\rightharpoonup} (\partial v/\partial t)(t)$ for every $t \in I$.

This together with the definition of v^{ε_n} yields that

a*) $u_n(t) \rightharpoonup u(t) = v(t)$ in V for every $t \in I$,
b*) $u_n \to u$ in $C(I, H)$,
e*) $(\partial u_n/\partial t)(t) \overset{2}{\rightharpoonup} (\partial u/\partial t)(t)$ for every $t \in I$.

The statements a*) and b*) are obvious. To obtain e*) we use the definitions of weak and two scale convergence.

According to (4.9), (4.32), and a*), b*), e*) there exist $u \in C^{0,1}(I,H) \cap L_\infty(I,V)$ with the time derivative $\partial u/\partial t \in L_\infty(I,H)$ which are also bounded by these constants. Moreover, since Rothe's sequence is uniformly convergent, we obtain that $u(0) = u_0$. This implies the correctness of the first two properties of the theorem.

Now we notice that all the above considerations have been done for a fixed ε, which implies that the obtained limit function $u(t)$ also depends on ε. Thus, we will in the sequel use the notation $u^\varepsilon(t)$ instead of $u(t)$.

Now we will prove that the function u^ε also has the third property from the theorem, i.e. the integral identity (3.5) holds. To this end we introduce step functions $\bar{u}_n$, $\bar{a}_n$, and $\bar{b}_n$ defined in I such that

$$z_j = \bar{u}_n(t), \quad a_j = \bar{a}_n(t), \quad b_j = \bar{b}_n(t),$$

and

$$\overline{f}_n(t,\cdot) = f_j(\cdot)$$

for $t \in I_j$, $j = 1,2,3,\ldots,n$, and we rewrite the integral identity (4.1) as

$$\Big(\bar{a}_n(t)\frac{\partial u_n(t)}{\partial t}, v(t)\Big) + \langle \bar{b}_n(t)\bar{u}_n(t), v(t)\rangle = (\overline{f}_n(t, \bar{u}_n(t-h)), v(t)), \tag{4.33}$$

where $v \in L_\infty(I,V)$. In view of a*), b*), e*) above, and Assumption 2.6 we get that

$$\begin{gathered}\Big(\bar{a}_n(t)\frac{\partial u_n}{\partial t}(t), v(t)\Big) \to \Big(a_\varepsilon(t)\frac{\partial u^\varepsilon}{\partial t}(t), v(t)\Big),\\ \langle \bar{b}_n(t)\bar{u}_n(t), v(t)\rangle \to \langle b_\varepsilon(t)u^\varepsilon(t), v(t)\rangle,\\ (\overline{f}_n(t,\bar{u}_n(t-h)), v(t)) \to (f_\varepsilon(t,u^\varepsilon(t)), v(t))\end{gathered} \tag{4.34}$$

as $n \to \infty$, for each fixed ε and almost all $t \in I$, since $\|\bar{a}_n(t) - a(t)\|_{L_\infty(\Omega)} \to 0$, $\|\bar{b}_n(t) - b(t)\|_{L_\infty(\Omega)} \to 0$, and $\|\overline{f}_n(t,\bar{u}_n(t-h)) - f(t,u(t))\|_H \to 0$ as $n \to \infty$. Moreover, to get the limits (4.34) we use also that $\bar{u}_n(t-h) \to u(t)$ in H, which follows from the estimate

$$\begin{aligned}\|\bar{u}_n(t-h) - u_n(t)\|_H &= \|u_n(t_{j-1}) - u_n(t)\|_H\\ &= \Big\|\int_{t_{j-1}}^t \frac{\partial u_n(\tau)}{\partial t}\,\mathrm{d}\tau\Big\|_H \leqslant \int_{t-h}^t \Big\|\frac{\partial u_n(\tau)}{\partial t}\Big\|_H \mathrm{d}\tau \leqslant Ch\end{aligned}$$

for $t \in \tilde{I}_j = (t_{j-1}, t_j]$, $j = 1,2,\ldots,n$. Taking the limit on both sides of equality (4.33), we obtain the identity

$$\Big(a_\varepsilon(t)\frac{\partial u^\varepsilon}{\partial t}(t), v(t)\Big) + \langle b_\varepsilon(t)u^\varepsilon(t), v(t)\rangle = (f_\varepsilon(t,u^\varepsilon(t)), v(t)),$$

which is the same as

$$\text{(4.35)}\quad \int_\Omega a(x,x/\varepsilon,t)\frac{\partial u^\varepsilon}{\partial t}(x,t)v(x)\,\mathrm{d}x+\int_\Omega b(x,x/\varepsilon,t)\nabla u^\varepsilon(x,t)\cdot\nabla v(x)\,\mathrm{d}x = \int_\Omega f(x,x/\varepsilon,t,u^\varepsilon(x,t))v(x)\,\mathrm{d}x$$

for all $v\in V=W_0^{1,2}(\Omega)$ and almost all $t\in I$. This shows that the function u^ε satisfies the integral identity (3.5), and Theorem 3.4 is proved. □

Proof of Theorem 3.1. *Existence.* (a) We will use Lemma 2.3 and Lemma 2.4, which involves the notion of two-scale convergence, to obtain the homogenized equation corresponding to problem (1.1). We note that by Theorem 3.4 and Lemma 2.4 there exists a certain $u\in C^{0,1}(I,H)\cap L_\infty(I,V)$ with the time derivative $\partial u/\partial t\in L_\infty(I,H)$, and a certain $\tilde u\in L_\infty(I,L_2(\Omega,W^{1,2}_{\mathrm{per}}(Y)))$ attained as limits of u^ε and $\partial u^\varepsilon/\partial t$ in the sense of Lemma 2.4. It remains to prove that these limits satisfy the weak formulation (3.1) of the theorem.

Let us choose an arbitrary $v\in V$ and introduce

$$\omega^\varepsilon(t):=\omega^\varepsilon(x,t)=a(x,x/\varepsilon,t)v(x)$$

and

$$\omega(t):=\omega(x,y,t)=a(x,y,t)v(x).$$

Evidently $\omega^\varepsilon(t)\xrightarrow{2}\omega(t)$. By using assertion e) of Lemma 2.4 and Lemma 2.3, we find that

$$\begin{aligned}\text{(4.36)}\quad &\lim_{\varepsilon\to0}\int_\Omega a(x,x/\varepsilon,t)\frac{\partial u^\varepsilon}{\partial t}(x,t)v(x)\,\mathrm{d}x\\ &=\lim_{\varepsilon\to0}\int_\Omega \omega^\varepsilon(x,t)\frac{\partial u^\varepsilon}{\partial t}(x,t)\,\mathrm{d}x=\int_\Omega\int_Y\omega(x,y,t)\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}y\,\mathrm{d}x\\ &=\int_\Omega\int_Y a(x,y,t)\frac{\partial u}{\partial t}(x,t)v(x)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

This shows that the first integral in (4.35) tends to the corresponding one in (3.1) as $\varepsilon\to0$. Next we evaluate the limit of the right-hand side of (4.35) when $\varepsilon\to0$ as follows:

$$\begin{aligned}\text{(4.37)}\quad &\lim_{\varepsilon\to0}\int_\Omega f(x,x/\varepsilon,t,u^\varepsilon(x,t))\,v(x)\,\mathrm{d}x\\ &=\int_\Omega\int_Y f(x,y,t,u(x,t))v(x)\,\mathrm{d}y\,\mathrm{d}x\\ &\quad-\lim_{\varepsilon\to0}\int_\Omega(f(x,x/\varepsilon,t,u(x,t))-f(x,x/\varepsilon,t,u^\varepsilon(x,t)))v(x)\,\mathrm{d}x\\ &\quad-\lim_{\varepsilon\to0}\int_\Omega\Big(\int_Y f(x,y,t,u(x,t))\,\mathrm{d}y-f(x,x/\varepsilon,t,u(x,t))\Big)v(x)\,\mathrm{d}x;\end{aligned}$$

here the last two integrals converge to zero as $\varepsilon \to 0$. The second integral on the right-hand side converges to zero, since f satisfies (A2) of Assumption 2.6 and the sequence $u^\varepsilon(t)$ converges strongly to $u(t)$ in H. The convergence to zero of the third integral on the right-hand side follows from the definition of two-scale convergence. It also holds that

$$\begin{aligned}(4.38)\qquad \lim_{\varepsilon\to 0}\int_\Omega b(x,x/\varepsilon,t)\nabla u^\varepsilon(x,t)\cdot\nabla v(x)\,\mathrm{d}x \\ = \int_\Omega\int_Y b(x,y,t)(\nabla u(x,t)+\nabla_Y\tilde{u}(x,y,t))\cdot\nabla v(x)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

This statement holds according to Lemma 2.3, since $b_\varepsilon(t) \xrightarrow{2} b(t)$, which follows from Definitions 2.1 and 2.2, and $\nabla u^\varepsilon(t) \stackrel{2}{\rightharpoonup} \nabla u(t)+\nabla_Y\tilde{u}(t)$, which follows from the assertion d) of Lemma 2.4.

By combining (4.35)–(4.38), we find that the function u satisfies the equality

$$\begin{aligned}(4.39)\qquad \int_\Omega \tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t)v(x)\,\mathrm{d}x \\ +\int_\Omega\int_Y b(x,y,t)(\nabla u(x,t)+\nabla_Y\tilde{u}(x,y,t))\cdot\nabla v(x)\,\mathrm{d}y\,\mathrm{d}x \\ = \int_\Omega \tilde{f}(x,t,u(x,t))v(x)\,\mathrm{d}x,\end{aligned}$$

and (4.39) coincides with (3.1), so we are done.

(b) Let us now choose for the test function v in (3.5) (i.e. (4.35)) a function $\varepsilon\psi(x)v(x/\varepsilon)$, where $\psi\in C_0^\infty(\Omega)$ and $v\in C^\infty_{\mathrm{per}}(Y)$. Then we get that

$$\begin{aligned}\varepsilon\int_\Omega a(x,x/\varepsilon,t)\frac{\partial u^\varepsilon}{\partial t}(x,t)\psi(x)v(x/\varepsilon)\,\mathrm{d}x \\ +\int_\Omega (b(x,x/\varepsilon,t)\nabla u^\varepsilon(x,t))\cdot\nabla[\varepsilon\psi(x)v(x/\varepsilon)]\,\mathrm{d}x \\ =\varepsilon\int_\Omega f(x,x/\varepsilon,t,u^\varepsilon(x,t))\psi(x)v(x/\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Simplifying we see that

$$\begin{aligned}\varepsilon\int_\Omega a(x,x/\varepsilon,t)\frac{\partial u^\varepsilon}{\partial t}(x,t)\psi(x)v(x/\varepsilon)\,\mathrm{d}x \\ +\int_\Omega b(x,x/\varepsilon,t)\nabla u^\varepsilon(x,t)\cdot[\psi(x)\nabla v(x/\varepsilon)]\,\mathrm{d}x \\ +\varepsilon\int_\Omega (b(x,x/\varepsilon,t)\nabla u^\varepsilon(x,t))\cdot[\nabla\psi(x)v(x/\varepsilon)]\,\mathrm{d}x \\ =\varepsilon\int_\Omega f(x,x/\varepsilon,t,u^\varepsilon(x,t))\psi(x)v(x/\varepsilon)\,\mathrm{d}x.\end{aligned}$$

As $\varepsilon \to 0$ we find that

$$\int_\Omega \int_Y [b(x,y,t)(\nabla u(x,t) + \nabla_Y \tilde{u}(x,y,t))] \cdot [\psi(x) \nabla v(y)] \,\mathrm{d}y \,\mathrm{d}x = 0.$$

Since $\psi \in C_0^\infty(\Omega)$ is arbitrary, we have that (for a.e. x) $\tilde{u}(x,y,t)$ is the unique solution of the following periodic problem: Find $\tilde{u} \in L_\infty(I, L_2(\Omega, W^{1,2}_{\mathrm{per}}(Y)))$ such that

$$\int_Y [b(x,y,t)(\nabla u(x,t) + \nabla_Y \tilde{u}(x,y,t))] \cdot \nabla v(y) \,\mathrm{d}y = 0$$

for almost all $x \in \Omega$. Rearranging we find that

$$\begin{aligned}
(4.40) \quad & \int_Y [b(x,y,t) \nabla_Y \tilde{u}(x,y,t)] \cdot \nabla v(y) \,\mathrm{d}y \\
&= -\int_Y [b(x,y,t) \nabla u(x,t)] \cdot \nabla v(y) \,\mathrm{d}y \\
&= -\int_Y b(x,y,t) \frac{\partial u}{\partial x_1} \frac{\partial v(y)}{\partial y_1} \,\mathrm{d}y - \int_Y b(x,y,t) \frac{\partial u}{\partial x_2} \frac{\partial v(y)}{\partial y_2} \,\mathrm{d}y \\
&\quad - \int_Y b(x,y,t) \frac{\partial u}{\partial x_3} \frac{\partial v(y)}{\partial y_3} \,\mathrm{d}y.
\end{aligned}$$

By linearity

$$\tilde{u}(x,y,t) = w_1(x,y,t) \frac{\partial u}{\partial x_1} + w_2(x,y,t) \frac{\partial u}{\partial x_2} + w_3(x,y,t) \frac{\partial u}{\partial x_3}, \tag{4.41}$$

where $w_i \in L_\infty(I, L_2(\Omega, W^{1,2}_{\mathrm{per}}(Y)))$ $(i = 1,2,3)$ are the solutions of the following *local problems*:

$$\begin{cases}
\int_Y b(x,y,t)(\nabla_Y w_1 + e_1) \cdot \nabla v(y) \,\mathrm{d}y = 0, \\
\int_Y b(x,y,t)(\nabla_Y w_2 + e_2) \cdot \nabla v(y) \,\mathrm{d}y = 0, \\
\int_Y b(x,y,t)(\nabla_Y w_3 + e_3) \cdot \nabla v(y) \,\mathrm{d}y = 0.
\end{cases} \tag{4.42}$$

Finally, to obtain the homogenized equation, we insert (4.41) into (4.39) to get

$$\begin{aligned}
(4.43) \quad & \int_\Omega \int_Y a(x,y,t) \frac{\partial u}{\partial t}(x,t)\, v(x) \,\mathrm{d}y \,\mathrm{d}x \\
&\quad + \int_\Omega \int_Y b(x,y,t) \Big(\nabla u(x,t) + \nabla_Y w_1 \frac{\partial u}{\partial x_1} + \nabla_Y w_2 \frac{\partial u}{\partial x_2} + \nabla_Y w_3 \frac{\partial u}{\partial x_3} \Big) \\
&\qquad\qquad \times \nabla v(x) \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_\Omega \int_Y f(x,y,t,u(x,t)) v(x) \,\mathrm{d}y \,\mathrm{d}x.
\end{aligned}$$

Moreover, we note that the second term on the left-hand side can be written as

$$
\begin{aligned}
(4.44)\qquad & \int_\Omega\int_Y b(x,y,t)\Big(\frac{\partial u}{\partial x_1}e_1+\frac{\partial u}{\partial x_2}e_2+\frac{\partial u}{\partial x_3}e_3 \\
&\quad +\nabla_Y w_1\frac{\partial u}{\partial x_1}+\nabla_Y w_2\frac{\partial u}{\partial x_2}+\nabla_Y w_3\frac{\partial u}{\partial x_3}\Big)\cdot\nabla v(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_\Omega\Big\{\frac{\partial u}{\partial x_1}\Big(\int_Y b(x,y,t)(e_1+\nabla_Y w_1)\,\mathrm{d}y\Big) \\
&\quad +\frac{\partial u}{\partial x_2}\Big(\int_Y b(x,y,t)(e_2+\nabla_Y w_2)\,\mathrm{d}y\Big) \\
&\quad +\frac{\partial u}{\partial x_3}\Big(\int_Y b(x,y,t)(e_3+\nabla_Y w_3)\,\mathrm{d}y\Big)\Big\}\cdot\nabla v(x)\,\mathrm{d}x \\
&= \int_\Omega\left\{\frac{\partial u}{\partial x_1}\begin{pmatrix} b_{11}(x,t)\\ b_{21}(x,t)\\ b_{31}(x,t)\end{pmatrix}+\frac{\partial u}{\partial x_2}\begin{pmatrix} b_{12}(x,t)\\ b_{22}(x,t)\\ b_{32}(x,t)\end{pmatrix}\right. \\
&\quad \left.+\frac{\partial u}{\partial x_3}\begin{pmatrix} b_{13}(x,t)\\ b_{23}(x,t)\\ b_{33}(x,t)\end{pmatrix}\right\}\cdot\nabla v(x)\,\mathrm{d}x \\
&= \int_\Omega (B(x,t)\nabla u(x,t))\cdot\nabla v(x)\,\mathrm{d}x,
\end{aligned}
$$

where the matrix $B(x,t)=(b_{ij}(x,t))_{i,j=1,2,3}$ is defined by

$$
\begin{pmatrix} b_{1j}(x,t)\\ b_{2j}(x,t)\\ b_{3j}(x,t)\end{pmatrix}=\int_Y b(x,y,t)(e_j+\nabla_Y w_j)\,\mathrm{d}y\quad\text{for } j=1,2,3
$$

and (3.3) is proved. By inserting (4.44) into (4.43), we see that

$$
\begin{aligned}
(4.45)\quad \int_\Omega\int_Y a(x,y,t)\,\frac{\partial u}{\partial t}(x,t)v(x)\,\mathrm{d}y\,\mathrm{d}x &+ \int_\Omega (B(x,t)\nabla u(x,t))\cdot\nabla v(x)\,\mathrm{d}x \\
&= \int_\Omega\int_Y f(x,y,t,u(x,t))v(x)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Introducing the notation

$$
\begin{aligned}
\tilde f(x,t,u(x,t)) &= \int_Y f(x,y,t,u(x,t))\,\mathrm{d}y \\
\text{and }\ \tilde a(x,t) &= \int_Y a(x,y,t)\,\mathrm{d}y,
\end{aligned}
$$

the equality (4.45) takes the form

$$
\int_\Omega \tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t)v(x)\,\mathrm{d}x + \int_\Omega (B(x,t)\nabla u(x,t))\cdot\nabla v(x)\,\mathrm{d}x = \int_\Omega \tilde{f}(x,t,u(x,t))v(x)\,\mathrm{d}x, \tag{4.46}
$$

which is the weak form of

$$
\tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t) - \nabla\cdot(B(x,t)\nabla u(x,t)) = \tilde{f}(x,t,u(x,t)).
$$

The proof of existence of the solution is complete.

Uniqueness. Assume that u^1 and u^2 are solutions of problem (3.2), i.e. $u^i \in C^{0,1}(I,H)\cap L_\infty(I,V)$ is such that $u^i(0)=u_0$ and

$$
\int_\Omega \tilde{a}(x,t)\frac{\partial u^i}{\partial t}(x,t)v(x)\,\mathrm{d}x + \int_\Omega (B(x,t)\nabla u^i(x,t))\cdot\nabla v(x)\,\mathrm{d}x = \int_\Omega \tilde{f}(x,t,u^i(x,t))v(x)\,\mathrm{d}x, \tag{4.47}
$$

$(i=1,2)$. If we denote $u(t)=u^1(t)-u^2(t)$, then $u(0)=0$. Subtracting the identity (4.47) written for $i=2$ from the same identity written for $i=1$ and choosing $v=u$, we get that

$$
\int_\Omega \tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t)u(x,t)\,\mathrm{d}x + \int_\Omega (B(x,t)\nabla u(x,t))\cdot\nabla u(x,t)\,\mathrm{d}x = \int_\Omega [\tilde{f}(x,t,u^1(x,t)) - \tilde{f}(x,t,u^2(x,t))]u(x,t)\,\mathrm{d}x. \tag{4.48}
$$

By using the assumption on f (see (A2) of Assumption 2.6) we estimate the right-hand side as

$$
\int_\Omega [\tilde{f}(x,t,u^1(x,t)) - \tilde{f}(x,t,u^2(x,t))]u(x,t)\,\mathrm{d}x \leqslant C_2\int_\Omega u(x,t)^2\,\mathrm{d}x.
$$

From this and from (4.48) we get that

$$
\int_\Omega \tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t)u(x,t)\,\mathrm{d}x + \int_\Omega (B(x,t)\nabla u(x,t))\cdot\nabla u(x,t)\,\mathrm{d}x \leqslant C_2\int_\Omega (u(x,t))^2\,\mathrm{d}x.
$$

From the nonnegativity of the second term (which is guaranteed by (A1) of Assumption 2.6) on the left-hand side of the last estimate we obtain that

$$
\int_\Omega \tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t)u(x,t)\,\mathrm{d}x \leqslant C_2\int_\Omega (u(x,t))^2\,\mathrm{d}x.
$$

Now we integrate both sides with respect to t from 0 to τ, i.e.

$$\int_0^\tau \int_\Omega \tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t)u(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C_2 \int_0^\tau \int_\Omega (u(x,t))^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.49}$$

According to (A1) of Assumption 2.6 we can estimate the left-hand side of (4.49) as

$$\begin{aligned}
&\int_0^\tau \int_\Omega \tilde{a}(x,t)\frac{\partial u}{\partial t}(x,t)u(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \frac{1}{2}\int_\Omega \int_0^\tau \tilde{a}(x,t)\frac{\partial(u(x,t)^2)}{\partial t}\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad = \frac{1}{2}\int_\Omega \tilde{a}(x,\tau)u(x,\tau)^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega \int_0^\tau \frac{\partial \tilde{a}(x,t)}{\partial t}u(x,t)^2\,\mathrm{d}t\,\mathrm{d}x \\
&\qquad \geqslant \frac{C_1}{2}\int_\Omega u(x,\tau)^2\,\mathrm{d}x - \frac{C_2}{2}\int_0^\tau \int_\Omega u(x,t)^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

This together with (4.49) yields that

$$\int_\Omega u(x,\tau)^2\,\mathrm{d}x \leqslant \frac{3C_2}{C_1}\int_0^\tau \int_\Omega u(x,t)^2\,\mathrm{d}x\,\mathrm{d}t,$$

i.e.

$$\|u(\tau)\|_H^2 \leqslant \frac{3C_2}{C_1}\int_0^\tau \|u(t)\|_H^2\,\mathrm{d}t.$$

Hence, by applying Lemma 2.5 we get that

$$u(t) = 0, \quad \text{i.e. } u^1(t) = u^2(t) \quad \text{for a.e. } t \in I.$$

This proves the uniqueness of the solution of the homogenized equation (3.2).

The uniqueness of the solution implies that not only some subsequence of $\{u_\varepsilon\}$ converges to the solution, but also the whole sequence converges. The proof is complete. $\square$

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