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# LOCALLY LIPSCHITZ VECTOR OPTIMIZATION WITH INEQUALITY AND EQUALITY CONSTRAINTS

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*Abstract.* The present paper studies the following constrained vector optimization problem: $\min_C f(x)$, $g(x) \in -K$, $h(x) = 0$, where $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $g\colon \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz functions, $h\colon \mathbb{R}^n \to \mathbb{R}^q$ is $C^1$ function, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones. Two types of solutions are important for the consideration, namely $w$-minimizers (weakly efficient points) and $i$-minimizers (isolated minimizers of order 1). In terms of the Dini directional derivative first-order necessary conditions for a point $x^0$ to be a $w$-minimizer and first-order sufficient conditions for $x^0$ to be an $i$-minimizer are obtained. Their effectiveness is illustrated on an example. A comparison with some known results is done.

*Keywords*: vector optimization, locally Lipschitz optimization, Dini derivatives, optimality conditions

*MSC 2010*: 90C29, 90C30, 90C46, 49J52

## 1. Introduction

In this paper we deal with the local solutions of the constrained vector optimization problem

$$\min_C f(x), \quad g(x) \in -K, \quad h(x) = 0, \tag{1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $g\colon \mathbb{R}^n \to \mathbb{R}^p$ and $h\colon \mathbb{R}^n \to \mathbb{R}^q$ are given functions, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones. It is supposed that $f$ and $g$ are locally Lipschitz and $h$ is $C^1$ function. The inclusion $g(x) \in -K$ can be represented as a set of inequalities $\langle \eta, g(x)\rangle \leqslant 0$, $\eta \in K'$, where $K'$ is the positive polar cone of $K$. For this reason the problem is referred as one with inequality and equality constraints. Two types of solutions are important for the considerations, namely $w$-minimizers (weakly efficient points) and $i$-minimizers (isolated minimizers of order 1). In terms of the Dini directional derivative we obtain first-order necessary conditions for a point $x^0$ to be a $w$-minimizer and first-order sufficient conditions for $x^0$ to be an $i$-minimizer. The paper generalizes the results from [9], where problems with only inequality constraints are considered.

There is a growing interest toward optimality conditions for nonsmooth vector problems, though less papers study problems with equality constraints. In the smooth case the Fritz John optimality criterion is generalized in [16] and [13]. Unified first and second-order theory based on parabolic derivatives is proposed in [6]. Nonsmooth problems within Clarke subdifferentials are treated in [7] and [8]. Recently this problem is studied with the help of scalarization [2] or by second-order technique [15], [1]. Second-order technique based on Dini derivatives for problems without equality constraints and $C^{1,1}$ data (that is differentiable with locally Lipschitz derivatives) initiates in [14] (for problems with polyhedral cones) and goes on (for arbitrary cones) in [11] and [10]. A further generalization (toward relaxing the smoothness of the problem data) for (unconstrained) problems with $\ell$-stable data can be found in [5]. In [12] using suitable elimination procedure this technique is extended to problems with equality constraints (with $C^{1,1}$ objective function and inequality constraints and $C^2$ equality constraints). The present paper using similar elimination establishes first-order conditions for problems with locally Lipschitz objective function and inequality constraints and $C^1$ equality constraints. An example demonstrates the effectiveness of the obtained conditions and shows that they can work in some cases when the conditions from [7] and [8] fail.

## 2. Preliminaries

For the norm and the dual pairing in the considered finite-dimensional spaces we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. From the context it should be clear to what spaces exactly these notations are applied.

For a cone $M \subset \mathbb{R}^k$ its positive polar cone is $M' = \{\zeta \in \mathbb{R}^k \colon \langle\zeta,\varphi\rangle \geqslant 0$ for all $\varphi \in M\}$. If $\varphi \in \operatorname{cl}\operatorname{conv} M$ we set $M'[\varphi] = \{\zeta \in M' \colon \langle\zeta,\varphi\rangle = 0\}$. Then $M'[\varphi]$ is a closed convex cone and $M'[\varphi] \subset M'$. Consequently its positive polar cone $M[\varphi] := (M'[\varphi])'$ is a closed convex cone, $M \subset M[\varphi]$ and $(M[\varphi])' = M'[\varphi]$. In this paper we apply the notation $M[\varphi]$ for $M = K$ and $\varphi = -g(x^0)$.

The solutions of (1) (and similarly for the problem (2) considered further) are understood in a local sense. In any case a solution is a feasible point $x^0$, that is a point satisfying the constraints. The feasible point $x^0$ is said to be a $w$-minimizer (weakly efficient point) for the problem (1) if there exists a neighbourhood $U$ of $x^0$, such that $f(x) \notin f(x^0) - \operatorname{int} C$ for all feasible points $x \in U$.

To define an $i$-minimizer we need the concept of an oriented distance. Given a set $A \subset \mathbb{R}^k$, then the distance from $y \in \mathbb{R}^k$ to $A$ is $d(y,A) = \inf\{\|a-y\| \colon a \in A\}$. The oriented distance from $y$ to $A$ is defined by $D(y,A) = d(y,A) - d(y, \mathbb{R}^k \setminus A)$. When $A = -C$, where $C$ is a convex cone, then $D(y,-C) = \sup\{\langle \xi, y\rangle \colon \xi \in C',\ \|\xi\| = 1\}$ (here $\|\xi\|$ means the dual norm to the one given in $\mathbb{R}^k$).

We say that the feasible point $x^0$ is an $i$-minimizer (isolated minimizer of order 1) for the problem (1) (and similarly for (2)) if there exists a neighbourhood $U$ of $x^0$ and a constant $A > 0$ such that

$$D\big(f(x) - f(x^0), -C\big) \geqslant A\|x - x^0\| \quad \text{for all feasible } x \in U.$$

The above definition generalizes to vector optimization problems the definition of an isolated minimizer for scalar problems from [4]. Some authors (e. g. [3]) use to say strict minimizers instead of isolated minimizers. The definition of an $i$-minimizer involves the norm. However, since any two norms in a finite dimensional real space are equivalent, the concept of an $i$-minimizer is actually norm-independent. Obviously, each $i$-minimizer is a $w$-minimizer.

For a given locally Lipschitz function $\Phi\colon \mathbb{R}^n \to \mathbb{R}^k$ the Dini derivative $\Phi'_u(x^0)$ of $\Phi$ at $x^0$ in direction $u \in \mathbb{R}^n$ is defined as the set-valued Kuratowski limit

$$\Phi'_u(x^0) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t}\big(\Phi(x^0 + tu) - \Phi(x^0)\big).$$

If $\Phi$ is Fréchet differentiable at $x^0$ then the Dini derivative is a singleton and can be expressed in terms of the Jacobian $\Phi'_u(x^0) = \Phi'(x^0)u$. We will deal with the Dini derivative of the function $\Phi\colon \mathbb{R}^n \to \mathbb{R}^{m+p}$, $\Phi(x) = (f(x), g(x))$. Then we use the notation $\Phi'_u(x^0) = \big(f(x^0), g(x^0)\big)'_u$. Let us note that always $\big(f(x^0), g(x^0)\big)'_u \subset f'_u(x^0) \times g'_u(x^0)$, but in general these two sets do not coincide.

## 3. Problems with only inequality constraints

In this section following [9] we recall some necessary and sufficient optimality conditions for the problem with only inequality constraints

$$\min_C f(x), \quad g(x) \in -K. \tag{2}$$

The following constraint qualification of Kuhn-Tucker type appears in the Sufficient Conditions part of Theorem 1:

$$\mathbb{Q}_{0,1}(x^0) \qquad \begin{cases} \text{if } g(x^0) \in -K \text{ and } \dfrac{1}{t_k}\big(g(x^0 + t_k u^0) - g(x^0)\big) \to z^0 \in -K[-g(x^0)], \\ \text{then } \exists\, u^k \to u^0\ \exists\, k_0 \in \mathbb{N}\ \forall\, k > k_0 \colon\ g(x^0 + t_k u^k) \in -K. \end{cases}$$

**Theorem 1** ([9]). *Let $f$, $g$ be locally Lipschitz functions and consider the problem* (2).
(Necessary Conditions) *Let $x^0$ be a $w$-minimizer of the problem* (2). *Then for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied:*

$$\mathbb{N}'_{0,1} \qquad \begin{cases} \forall\,(y^0,z^0) \in \big(f(x^0),g(x^0)\big)'_u \ \exists\,(\xi^0,\eta^0) \in C' \times K'[-g(x^0)]: \\ (\xi^0,\eta^0) \neq (0,0) \text{ and } \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle \geqslant 0. \end{cases}$$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied:*

$$\mathbb{S}'_{0,1} \qquad \begin{cases} \forall\,(y^0,z^0) \in \big(f(x^0),g(x^0)\big)'_u \ \exists\,(\xi^0,\eta^0) \in C' \times K'[-g(x^0)]: \\ (\xi^0,\eta^0) \neq (0,0) \text{ and } \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle > 0. \end{cases}$$

*Then $x^0$ is an $i$-minimizer of order one for the problem* (2).

*Conversely, if $x^0$ is an $i$-minimizer of order one for the problem* (2) *and the constraint qualification $\mathbb{Q}_{0,1}(x^0)$ holds, then the condition $\mathbb{S}'_{0,1}$ is satisfied.*

## 4. Problems with inequality and equality constraints

In this section we generalize Theorem 1 to problems with both inequality and equality constraints. We prove our result under the assumption that at the feasible point $x^0$ the vectors $h'_1(x^0),\dots,h'_q(x^0)$, which are the components of $h'(x^0)$, are linearly independent. Under this assumption the considered problem (1) can be reduced to an equivalent problem with only inequality constraints to which Theorem 1 can be applied. Here we explain this reduction.

Let the vectors $\bar{u}^j \in \mathbb{R}^n$, $j = 1,\dots,q$, be determined by

$$(3) \qquad h'_k(x^0)\bar{u}^j = 0 \text{ for } k \neq j, \quad \text{and} \quad h'_j(x^0)\bar{u}^j = 1.$$

For each $j = 1,\dots,q$, the equalities (3) constitute a system of linear equations with respect to the components of $\bar{u}^j$, which due to the linear independence of $h'_1(x^0),\dots,h'_q(x^0)$ has a unique solution. Moreover, the vectors $\bar{u}^1,\dots,\bar{u}^q$ solving this system are linearly independent and $\mathbb{R}^n$ is decomposed into a direct sum $\mathbb{R}^n = L \oplus L'$, where $L = \ker h'(x^0)$ and $L' = \operatorname{lin}\{\bar{u}^1,\dots,\bar{u}^q\}$. Let $u^1,\dots,u^{n-q}$ be $n-q$ linearly independent vectors in $L = \ker h'(x^0)$. We consider the system of equations

$$(4) \qquad h_k\Bigg(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j\Bigg) = 0, \quad k = 1,\dots,q.$$

Taking $\tau_1, \dots, \tau_{n-q}$ as independent variables and $\sigma_1, \dots, \sigma_q$ as dependent variables, we see that this system satisfies the requirements of the implicit function theorem at the point $\tau_1 = \dots = \tau_{n-q} = 0$, $\sigma_1 = \dots = \sigma_q = 0$ (at this point $h_k$ take the values $h_k(x^0) = 0$ because $x^0$ is feasible, and the Jacobian $\partial h/\partial \sigma$ is the unit matrix and hence is non degenerate). The implicit function theorem gives that in a neighbourhood of $x^0$ given by $|\tau_i| < \overline{\tau}$, $i = 1, \dots, n-q$, $|\sigma_j| < \overline{\sigma}$, $j = 1, \dots, q$, this system possesses a unique solution $\sigma_j = \sigma_j(\tau_1, \dots, \tau_{n-q})$, $j = 1, \dots, q$. The functions $\sigma_j = \sigma_j(\tau_1, \dots, \tau_{n-q})$ are $C^1$, and

$$\sigma_j\Big|_{\tau^0} = \sigma_j(0, \dots, 0) = 0, \quad j = 1 \dots, q, \tag{5}$$

$$\frac{\partial \sigma_j}{\partial \tau_i}\Big|_{\tau^0} = 0, \quad j = 1, \dots, q, \quad i = 1, \dots, n-q, \tag{6}$$

where $\tau^0 = (0, \dots, 0)$. To show the latter we differentiate (4) with respect to $\tau_i$ obtaining

$$h_k'\left(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j \bar{u}^j\right)\left(u^i + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_i} \bar{u}^j\right) = 0.$$

For $\tau = \tau^0 = 0$ we get

$$h_k'(x^0)\left(u^i + \sum_{j=1}^{q} \frac{\partial \sigma_j}{\partial \tau_i}\Big|_{\tau^0} \bar{u}^j\right) = 0,$$

whence on account of $u^i \in \ker h'(x^0)$ and (3) we obtain (6).

The equivalence of the problem (1) with a problem with only inequality constraints is given in the next lemma.

**Lemma 1** ([12]). *Consider the problem* (1) *with* $h \in C^1$, *for which* $h_1'(x^0), \dots, h_q'(x^0)$, *are linearly independent, and* $C$ *and* $K$ *are closed convex cones. Then* $x^0$ *is a* $w$*-minimizer or* $i$*-minimizer for* (1) *if and only if* $\tau^0 = 0$ *is respectively a* $w$*-minimizer or* $i$*-minimizer for the problem*

$$\min_C \overline{f}(\tau_1, \dots, \tau_{n-q}), \quad \overline{g}(\tau_1, \dots, \tau_{n-q}) \in -K, \tag{7}$$

*where*

$$\overline{f}(\tau_1, \dots, \tau_{n-q}) = f\left(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j(\tau_1, \dots, \tau_{n-q}) \bar{u}^j\right),$$

$$\overline{g}(\tau_1, \dots, \tau_{n-q}) = g\left(x^0 + \sum_{i=1}^{n-q} \tau_i u^i + \sum_{j=1}^{q} \sigma_j(\tau_1, \dots, \tau_{n-q}) \bar{u}^j\right).$$

The next theorem is our main result.

**Theorem 2.** *Consider the problem* (1) *with $f$, $g$ being locally Lipschitz functions, $h \in C^1$, and $C$ and $K$ closed convex cones. Let $x^0$ be a feasible point and let the vectors $h_1'(x^0), \ldots, h_q'(x^0)$, the components of $h'(x^0)$, be linearly independent.*

(Necessary Conditions). *Let $x^0$ be a $w$-minimizer of the problem* (1). *Then for each $u \in \ker h'(x^0) \setminus \{0\}$ the following condition is satisfied:*

$$\mathbb{N}' \qquad \begin{cases} \forall\,(y^0, z^0) \in \big(f(x^0), g(x^0)\big)'_u \ \exists\,(\xi^0, \eta^0)\colon \\ (\xi^0, \eta^0) \in C' \times K'[-g(x^0)],\ (\xi^0, \eta^0) \neq (0,0) \\ \textit{and } \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle \geqslant 0. \end{cases}$$

(Sufficient Conditions). *Suppose that for each $u \in \ker h'(x^0) \setminus \{0\}$ the following condition is satisfied:*

$$\mathbb{S}' \qquad \begin{cases} \forall\,(y^0, z^0) \in \big(f(x^0), g(x^0)\big)'_u \ \exists\,(\xi^0, \eta^0)\colon \\ (\xi^0, \eta^0) \in C' \times K'[-g(x^0)],\ (\xi^0, \eta^0) \neq (0,0) \\ \textit{and } \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle > 0. \end{cases}$$

*Then $x^0$ is an $i$-minimizer of the problem* (1).

Proof. According to Lemma 1 the feasible point $x^0$ is a $w$-minimizer or $i$-minimizer of the problem (1) if and only if $\tau^0 = (0, \ldots, 0)$ is respectively a $w$-minimizer or $i$-minimizer of the problem with only inequality constraints (7). It remains to apply Theorem 1 to (7) and to express the necessary and sufficient conditions through the data of the problem (1).

We deal first with the necessary conditions. Lemma 1 gives that if $\tau^0$ is a $w$-minimizer of (7), then for each $\tau = (\tau_1, \ldots, \tau_{n-q}) \in \mathbb{R}^{n-q} \setminus \{0\}$ it holds

$$\begin{gathered} \forall\,(y^0, z^0) \in \big(\overline{f}(\tau^0), \overline{g}(\tau^0)\big)'_\tau \ \exists\,(\xi^0, \eta^0) \in C' \times K'[-\overline{g}(\tau^0)]\colon \\ (\xi^0, \eta^0) \neq (0,0) \quad \text{and} \quad \langle \xi^0, y^0\rangle + \langle \eta^0, z^0\rangle \geqslant 0. \end{gathered} \tag{8}$$

To the fixed vector $\tau = (\tau_1, \ldots, \tau_{n-q})$ we juxtapose the vector

$$u = \sum_{i=1}^{n-q} \tau_i u^i. \tag{9}$$

Since the vectors $u^1, \ldots, u^{n-q}$ form a base in $\ker h'(x^0)$, obviously (9) gives a one-to-one correspondence between the vectors $\tau$ in $\mathbb{R}^{n-q} \setminus \{0\}$ and the vectors $u$ in $\ker h'(x^0) \setminus \{0\}$. Now we express the condition (8) using the vector $u$ instead of $\tau$ and $x^0$, $f$, $g$ instead of $\tau^0$, $\overline{f}$, $\overline{g}$.

We will show that (8) transforms into $\mathbb{N}'$. Observe that $K'[-\overline{g}(\tau^0)] = K'[-g(x^0)]$ due to $\overline{g}(\tau^0) = g(x^0)$. Therefore, $(\xi^0, \eta^0) \in C' \times K'[-\overline{g}(\tau^0)]$ can be written as

$(\xi^0, \eta^0) \in C' \times K'[-g(x^0)]$. It remains to show that $(y^0, z^0) \in \left(\overline{f}(\tau^0), \overline{g}(\tau^0)\right)'_\tau$ is equivalent to $(y^0, z^0) \in \left(f(x^0), g(x^0)\right)'_u$, where $u$ and $\tau$ are related by (9). Indeed, let

$$y^0 = \lim_k \frac{1}{t_k}\left(\overline{f}(\tau^0 + t_k\tau) - \overline{f}(\tau^0)\right), \quad z^0 = \lim_k \frac{1}{t_k}\left(\overline{g}(\tau^0 + t_k\tau) - \overline{g}(\tau^0)\right),$$

with some sequence $t_k \to 0^+$. In order to prove that $(y^0, z^0) \in \left(f(x^0), g(x^0)\right)'_u$ it is enough to show that

$$y^0 = \lim_k \frac{1}{t_k}\left(f(x^0 + t_k u) - f(x^0)\right), \quad z^0 = \lim_k \frac{1}{t_k}\left(g(x^0 + t_k u) - g(x^0)\right).$$

We show only the first equality. The second one is derived similarly. Assume that $f$ is Lipschitz with constant $\lambda$ in a neighbourhood of $x^0$. Then

$$\begin{aligned}
&\frac{1}{t_k}\left(f(x^0 + t_k u) - f(x^0)\right) \\
&\quad = \frac{1}{t_k}\left(\overline{f}(\tau^0 + t_k\tau) - \overline{f}(\tau^0)\right) \\
&\qquad + \frac{1}{t_k}\left(f(x^0 + t_k u) - f\left(x^0 + t_k u + \sum_{j=1}^{q} \sigma_j(t_k\tau_1, \dots, t_k\tau_{n-q})\bar{u}^j\right)\right) \to y^0.
\end{aligned}$$

In the above limit the first term tends toward $y^0$ and the second toward 0. The latter follows by the following chain of inequalities, true for sufficiently large $k$:

$$\begin{aligned}
&\left|\frac{1}{t_k}\left(f(x^0 + t_k u) - f\left(x^0 + t_k u + \sum_{j=1}^{q} \sigma_j(t_k\tau_1, \dots, t_k\tau_{n-q})\bar{u}^j\right)\right)\right| \\
&\qquad \leqslant \frac{\lambda}{t_k} \sum_{j=1}^{q} \left|\sigma_j(t_k\tau_1, \dots, t_k\tau_{n-q}) - \sigma_j(\tau^0)\right| \cdot \|\bar{u}^j\| \\
&\qquad \leqslant \lambda \sum_{j=1}^{q} \sum_{i=1}^{n-q} \left|\frac{\partial \sigma_j}{\partial \tau_i}(\theta_k t_k\tau_1, \dots, \theta_k t_k\tau_{n-q})\right| \cdot \|\bar{u}^j\| \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Here $0 < \theta_k < 1$ is given by the mean-value theorem. We have also used the fact that $\sigma_j \in C^1$ and the equalities (5) and (6).

The above reasonings prove the Necessary Conditions of the theorem. The Sufficient Conditions are proved in a similar way. □

Let us make the following remark. Theorem 1 gives also the converse of the sufficient conditions. To obtain a similar converse for the problem (1) with both equalities and inequalities constraints we can write the constraint qualification $\mathbb{Q}_{0,1}(\tau^0)$ for the problem (7) and reformulate it in terms of the problem (1). What we get is the following constraint qualification:

$$\mathbb{Q}(x^0) \qquad \begin{cases} \text{if } g(x^0) \in -K,\ h(x^0)=0,\ \bar{u} = \sum_{i=1}^{n-q} \overline{\tau}_i u^i \in \ker h'(x^0) \\ \text{and } \dfrac{1}{t_k}\big(g(x^0+t_k\bar{u}) - g(x^0)\big) \to z^0 \in -K[-g(x^0)], \\ \text{then } \exists\, \bar{u}^k = \sum_{i=1}^{n-q} \overline{\tau}_i^k u^i \to \bar{u}\ \exists\, k_0 \in \mathbb{N} \\ \forall\, k > k_0\colon\ g\Big(x^0 + t_k\bar{u}^k + \sum_{j=1}^{q} \sigma_j(t_k\overline{\tau}_1^k, \ldots, t_k\overline{\tau}_{n-q}^k)\Big) \in -K. \end{cases}$$

It should be noted here that if at some feasible point $x^0$ the constrained qualification $\mathbb{Q}(x^0)$ holds, then the condition $\mathbb{S}'$ is implied by the fact that $x^0$ is an $i$-minimizer of the problem (1).

The next example shows the effectiveness of the conditions from Theorem 2 for particular problems. This example is used in the next section to compare Theorem 2 with some results of [8] and [7]. For brevity we omit some of the calculations. Applying Theorem 2 we follow the usual procedure. First we find the set $N_w$ of the critical points, that is, the points satisfying the Necessary Conditions, which contains all the $w$-minimizers. Among the critical points we distinguish the set of the $i$-minimizers satisfying the Sufficient Conditions. The problem considered in this example is with locally Lipschitz data, but not with $C^1$ data (the function $g$ is not $C^1$).

Example 1. Consider the problem (1), for which $n=2$, $m=2$, $p=1$, $q=1$, the cones are $C=\mathbb{R}^2_+$ and $K=\mathbb{R}_+$, and the functions $f$, $g$, $h$, are given by

$$f(x_1,x_2) = (x_1,-x_2), \quad g(x_1,x_2) = \min(x_1,x_2),$$
$$h(x_1,x_2) = x_1^2 - 2x_1x_2 + x_2^2 - x_1 - x_2.$$

Then the sets $N_w$ and $S_i$ of the feasible points satisfying respectively the Necessary Conditions $\mathbb{N}'$ and the Sufficient Conditions $\mathbb{S}'$ are given by $N_w = N_w^1 \cup N_w^2$ and $S_i = S_i^1 \cup S_i^2$, where

$$N_w^1 = \Big\{\Big(x_1, \frac{1}{2}(2x_1+1-\sqrt{8x_1+1})\Big)\colon\ \frac{3}{8} \leqslant x_1 \leqslant 1\Big\},$$
$$N_w^2 = \Big\{\Big(\frac{1}{2}(2x_2+1-\sqrt{8x_2+1}), x_2\Big)\colon\ \frac{3}{8} \leqslant x_2 \leqslant 1\Big\},$$
$$S_i^1 = \Big\{\Big(x_1, \frac{1}{2}(2x_1+1-\sqrt{8x_1+1})\Big)\colon\ \frac{3}{8} < x_1 \leqslant 1\Big\},$$
$$S_i^2 = \Big\{\Big(\frac{1}{2}(2x_2+1-\sqrt{8x_2+1}), x_2\Big)\colon\ \frac{3}{8} < x_2 \leqslant 1\Big\}.$$

Indeed, the set of the feasible points in this example is $F = F^1 \cup F^2$, where

$$\begin{aligned} F^1 &= \Big\{\Big(x_1, \frac{1}{2}(2x_1 + 1 - \sqrt{8x_1+1})\Big) \colon\ 0 \leqslant x_1 \leqslant 1\Big\}, \\ F^2 &= \Big\{\Big(\frac{1}{2}(2x_2 + 1 - \sqrt{8x_2+1}), x_2\Big) \colon\ 0 \leqslant x_2 \leqslant 1\Big\}. \end{aligned}$$

We have $h_1'(x) = h'(x) = (2x_1 - 2x_2 - 1, -2x_1 + 2x_2 - 1)$. Obviously, the two components of $h_1'(x)$ cannot vanish simultaneously, which guarantees the linear independence of the single-valued set $\{h_1'(x)\}$ at any feasible point $x$. Clearly, if $u \in \mathbb{R}^2$, then

$$\begin{aligned} h'(x)u &= (2x_1 - 2x_2 - 1)u_1 + (-2x_1 + 2x_2 - 1)u_2, \\ \ker h'(x) &= \{(2x_1 - 2x_2 + 1, 2x_1 - 2x_2 - 1)t \colon\ t \in \mathbb{R}\}. \end{aligned}$$

The Dini derivatives are given by

$$\begin{aligned} f_u'(x) &= f'(x)u = (u_1, -u_2), \\ g_u'(x) &= \begin{cases} u_1, & x_1 < x_2, \\ u_1, & x_1 = x_2,\ u_1 \leqslant u_2, \\ u_2, & x_1 = x_2,\ u_2 \leqslant u_1, \\ u_2, & x_2 < x_1. \end{cases} \end{aligned}$$

Obviously $C' = C = \mathbb{R}^2_+$ and $K' = K = \mathbb{R}_+$. For $z^0 \in K'$ we have also $K'[z^0] = \{0\}$ when $z^0 < 0$, and $K'[z^0] = \mathbb{R}_+$ when $z^0 = 0$.

Further we denote for brevity

$$\mathcal{L} = \mathcal{L}(\xi^0, \eta^0; y^0, z^0) = \langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle = \xi_1^0 y_1^0 + \xi_2^0 y_2^0 + \eta^0 z^0.$$

Let $x$ be a feasible point and $u \in \ker h'(x) \setminus \{(0,0)\}$. We can distinguish the following cases:

1. $x_1 = \frac{1}{2}(2x_2 + 1 - \sqrt{8x_2+1})$, $\frac{3}{8} \leqslant x_2 \leqslant 1$.
   Now $y^0 = (u_1, -u_2)$, $z^0 = u_1$, $\mathcal{L} = \xi_1^0 u_1 - \xi_2^0 u_2 + \eta^0 u_1$, where

$$\begin{aligned} u_1 &= (2x_1 - 2x_2 + 1)t = (2 - \sqrt{8x_2+1})t, \\ u_2 &= (2x_1 - 2x_2 - 1)t = -\sqrt{8x_2+1}\,t,\ \ t \neq 0. \end{aligned}$$

   We have the possibilities:
   - 1a. $t > 0$. Taking $\xi^0 = (0,1)$, $\eta^0 = 0$, we get $\mathcal{L} = \sqrt{8x_2+1}\,t > 0$.
   - 1b. $t < 0$. Taking $\xi^0 = (1, 0)$, $\eta^0 = 0$, we get $\mathcal{L} = (2 - \sqrt{8x_2+1})t \geqslant 0$ with strict inequality for $x_2 > \frac{3}{8}$ and equality for $x_2 = \frac{3}{8}$.
2. $x_1 = \frac{1}{2}(2x_2 + 1 - \sqrt{8x_2+1})$, $0 < x_2 < \frac{3}{8}$.

Now $y^0$, $z^0$, $u$ and $\mathcal{L}$ are expressed as in the case **1**. In particular

$$\mathcal{L} = (\xi_1^0 + \eta^0)(2 - \sqrt{8x_2+1})t + \xi_2^0\sqrt{8x_2+1}\,t < 0$$

for all $t < 0$ and $(\xi^0, \eta^0) \in C' \times K'[-g(x)] = \mathbb{R}_+^2 \times \{0\}$, $(\xi^0, \eta^0) \neq (0,0,0)$, since

$$(2 - \sqrt{8x_2+1})t < 0 \quad \text{and} \quad \sqrt{8x_2+1}\,t < 0.$$

3. $x_2 = \frac{1}{2}(2x_1 + 1 - \sqrt{8x_1+1})$, $\frac{3}{8} \leqslant x_1 \leqslant 1$.
Now $y^0 = (u_1, -u_2)$, $z^0 = u_2$, $\mathcal{L} = \xi_1^0 u_1 - \xi_2^0 u_2 + \eta^0 u_2$, where

$$u_1 = (2x_1 - 2x_2 + 1)t = \sqrt{8x_1+1}\,t,$$
$$u_2 = (2x_1 - 2x_2 - 1)t = (-2 + \sqrt{8x_1+1})t, \quad t \neq 0.$$

We have the possibilities:
3a. $t > 0$. Taking $\xi^0 = (1,0)$, $\eta^0 = 0$, we get $\mathcal{L} = \sqrt{8x_1+1}\,t > 0$.
3b. $t < 0$. Taking $\xi^0 = (0,1)$, $\eta^0 = 0$, we get $\mathcal{L} = (2 - \sqrt{8x_1+1})t \geqslant 0$ with strict inequality for $x_1 > \frac{3}{8}$ and equality for $x_1 = \frac{3}{8}$.
4. $x_2 = \frac{1}{2}(2x_1 + 1 - \sqrt{8x_1+1})$, $0 < x_1 < \frac{3}{8}$.
Now $y^0$, $z^0$, $u$ and $\mathcal{L}$ are expressed as in the case 3. In particular

$$\mathcal{L} = \xi_2^0\sqrt{8x_1+1}\,t + (\xi_2^0 - \eta^0)(2 - \sqrt{8x_1+1})t < 0$$

for all $t < 0$ and $(\xi^0, \eta^0) \in C' \times K'[-g(x)] = \mathbb{R}_+^2 \times \{0\} \setminus \{(0,0,0)\}$, since

$$\sqrt{8x_1+1}\,t < 0 \quad \text{and} \quad (2 - \sqrt{8x_1+1})t < 0.$$

5. $x_1 = 0$, $x_2 = 0$.
Now $y^0 = (u_1, -u_2)$, $z^0 = u_1$ when $u_1 \leqslant u_2$ and $z^0 = u_2$ when $u_2 \leqslant u_1$,

$$u_1 = (2x_1 - 2x_2 + 1)t = t,$$
$$u_2 = (2x_1 - 2x_2 - 1)t = -t, \quad t \neq 0,$$
$$\mathcal{L} = \xi_1^0 u_1 - \xi_2^0 u_2 + \eta^0 z^0 = \begin{cases} (\xi_1^0 + \xi_2^0 - \eta^0)t, & t > 0, \\ (\xi_1^0 + \xi_2^0 + \eta^0)t, & t < 0. \end{cases}$$

Obviously, when $t < 0$ we have $\mathcal{L} < 0$.

Thus, on the basis of Theorem 2 we see that the points which do not belong to the set $N_w$ determined above are not $w$-minimizers, and the points from the set $S_i$ are $i$-minimizers. The efficiency for points in the set $N_w \backslash S_i = \{(-1/8, 3/8), (3/8, -1/8)\}$ needs a separate investigation. It can be shown directly from the definition that the point $(-1/8, 3/8)$ is a $w$-minimizer but not an $i$-minimizer (actually it is an isolated minimizer of order 2, a concept defined in [10]), while the point $(3/8, -1/8)$ is not a $w$-minimizer.

## 5. Some comparison

First-order optimality conditions for the problem (1) with locally Lipschitz functions are well-known from the classical monograph of Clarke [7] (see Theorem 6.3.1 therein), where the particular case $C = K = \mathbb{R}^n_+$ is treated. A generalization to problems with arbitrary cones $C$ and $K$ is presented in [8] and involves Clarke's generalized Jacobians. Recall that Clarke's generalized Jacobian for the vector function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ at a point $x^0$, denoted by $\partial f(x^0)$, is defined as the convex hull of all limits of sequences $f'(x^k)$, where $x^k \to x^0$ and the gradient $f'(x^k)$ exists. The following result is a particular case of Theorem 2 in [8].

**Theorem 3.** *Consider the problem* (1) *with $f$, $g$ being locally Lipschitz functions, $h \in C^1$, and $C$ and $K$ closed convex cones. Let $x^0$ be a feasible point and assume it is a $w$-minimizer of the problem* (1). *Then there exist vectors $\tau \in C'$, $\lambda \in K'[-g(x^0)]$, $\mu \in \mathbb{R}^q$, not all zero, such that*

$$0 \in \partial(\tau f + \lambda g + \mu h)(x^0). \tag{10}$$

The following observation gives some comparison between Theorems 3 and 2.

Observation. Consider the problem (1) with $f$, $g$ and $h$ as defined in Example 1 and let $N_w$ be the set described there. Then the set of points satisfying the condition (10) is $N_w^C = N_w \cup \{(0,0)\}$. Therefore, Theorem 3 does not reject the point $(0,0)$ as a $w$-minimizer, while Theorem 2 does (because $(0,\,0) \notin N_w$).

Indeed, it is easy to check that all the points in the set $N_w$ satisfy the necessary conditions of Theorem 3. This is easily seen, since the functions $f$, $g$, and $h$ are continuously differentiable at the points $x \in N_w$. Let $\operatorname{conv} A$ denote the convex hull of the set $A$. At the point $(0,0)$, which clearly is not a $w$-minimizer, we have $\partial g(0,0) = \operatorname{conv}\{(1,0),(0,1)\}$, while $g_1(x) = x_1$ and $g_2(x) = x_2$ are continuously differentiable at $(0,0)$ and their generalized Jacobian coincides with their gradient. Straightforward calculations show that the condition (10) is satisfied choosing $\tau = (0,1)$, $\lambda = 1$, and $\mu = 0$. Hence, the necessary conditions of Theorem 3 are satisfied at $(0,0)$, although $(0,0)$ is not a $w$-minimizer.

Similarly, one can show that also the necessary optimality conditions given in Clarke [7, Theorem 6.3.1] hold at the point $(0,0)$. On the contrary, the necessary conditions of Theorem 2 do not hold at $(0,0)$ and on this basis it follows that this point is not a $w$-minimizer.

This observation is significant, since in fact $(0,0)$ is the only point requiring special attention. Indeed, Clarke's generalized Jacobian is introduced to treat nonsmooth problems. But $(0,0)$ is the only point among those satisfying the equality constraints at which the problem fails to be $C^1$.

It is also worth recalling that neither Theorem 3 nor Theorem 6.3.1 in [7] give sufficient optimality conditions, while Theorem 2 does. Moreover, Theorem 2 allows to distinguish the $i$-minimizers, which as Example 1 shows are rather typical type of solutions for vector optimization problems.

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