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A PARAMETRIZED NEWTON METHOD FOR NONSMOOTH EQUATIONS WITH FINITELY MANY MAXIMUM FUNCTIONS*

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Abstract. In this paper we propose a parametrized Newton method for nonsmooth equations with finitely many maximum functions. The convergence result of this method is proved and numerical experiments are listed.

Keywords: nonsmooth equations, Newton method, convergence

MSC 2010: 65H10, 90C30

1. Introduction

A lot of optimization problems can be transformed to a system of nonsmooth equations

$$F(x) = 0, \tag{1}$$

where $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, for instance, nonlinear complementarity problems, variational inequality problems and LC^1 optimization problems ([1], [3], [5], [7]–[10]).

In recent years, much attention has been devoted to various forms of methods for solving (1). Most popular methods are the Newton-type ones, which include Newton methods, inexact Newton methods and quasi-Newton methods. They are based on the Clarke-type subdifferentials in each iteration step. For instance, the Newton

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method is given as follows:

$$x_{k+1} = x_k - V_k^{-1} F(x_k), \tag{2}$$

where V_k is an element of Clarke generalized Jacobian [7], an element of B-differential [8], and an element of b-differential [11] of F an x_k. Under the assumption that all elements of $\partial_{\rm Cl} F(x^\star)$, of $\partial_B F(x^\star)$, or of $\partial_b F(x^\star)$, where $x^\star$ is the solution of (2), are nonsingular, the locally super-linear convergence properties were obtained. In [1], Chen and Qi presented a parametrized modification of the generalized Jacobian Newton-like method

$$x_{k+1} = x_k - \alpha_k (V_k + \lambda_k I)^{-1} F(x_k),$$

where $V_k \in \partial_B F(x_k)$, I is the $n \times n$ identity matrix, the parameters α_k and λ_k are chosen to ensure convergence and $V_k + \lambda_k I$ is invertible.

In this paper we study a new method for the equations with max-type functions proposed by Gao in [3]

$$\begin{aligned} \max_{j \in J_1} f_{1j}(x) &= 0, \\ &\vdots \\ \max_{j \in J_n} f_{nj}(x) &= 0, \end{aligned} \tag{3}$$

where $f_{ij} \colon \mathbb{R}^n \to \mathbb{R}$ for $j \in J_i$, $i = 1, \ldots, n$, are continuously differentiable, J_i for $i = 1, \ldots, n$ are finite index sets. Denote

$$\begin{gathered} f_i(x) = \max_{j \in J_i} f_{ij}(x), \quad x \in \mathbb{R}^n, \ i = 1, \ldots, n, \\ F(x) = (f_1(x), \ldots, f_n(x))^T, \quad x \in \mathbb{R}^n, \\ J_i(x) = \{j_i \in N \colon f_{ij}(x) = f_i(x)\}, \quad x \in \mathbb{R}^n, \ i = 1, \ldots, n; \end{gathered}$$

then the equation (3) can be rewritten as (1). Define a new kind of the differential for $F(x)$ by

$$\partial_\star F(x) = \{(\nabla f_{1j_1}, \ldots, \nabla f_{nj_n})^T \colon j_1 \in J_1(x), \ldots, j_n \in J_n(x)\}, \quad x \in \mathbb{R}^n. \tag{4}$$

Gao [3] gave the super convergence result. Based on [3], Śmietański constructed a new version of finite difference approximation of the generalized Jacobian for a finite maximum function in [9]. In paper [10], Śmietański also proposed a new class of the parametrized Newton-like method for semismooth equations.

Based on [10], we consider the max-type equations proposed in [3] and present a new parametrized Newton method for the system of nonsmooth equations with finite maximum functions. The well-known results on the generalized Jacobian and the semismoothness will be recalled in Section 2. In Section 3, we will give the new method for (3) and study the convergence of this method. Finally, we give a numerical example.

2. Preliminaries

Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitzian function. According to Rademacher's Theorem, the local Lipschitz continuity of $F(x)$ implies that $F(x)$ is differentiable almost everywhere. Let D_F be the set where $F(x)$ is differentiable. Then

$$\partial_B F(x) = \{\lim_{x_i \to x} JF(x_i),\ x_i \in D_F\}$$

is called the B-differential of $F(x)$ at x ([8]), where $JF(x_k)$ denotes the usual Jacobian matrix of partial derivatives of $F(x)$ at x_k. The generalized Jacobian ([2]) of $F\colon \mathbb{R}^n \to \mathbb{R}^n$ at x in the sense of Clark is

$$\partial F(x) = \operatorname{conv} \partial_B F(x).$$

From [2] we know $\partial F(x)$ is nonempty, convex and compact and ∂F is upper semi-continuous at x. By the definition in [7], a locally Lipschitzian function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be semismooth at x provided that

$$\lim_{\substack{V \in \partial F(x+th') \\ h' \to h,\ t \downarrow 0}} Vh'$$

exists for any $h \in \mathbb{R}^n$.

The semismoothness was originally introduced for functions by Mifflin in [4]. Semismooth functions have many important properties, which are very important in convergence analysis of methods in nonsmooth optimization. We will give some properties for our discussion. If F is semismooth, let $F'(x;h)$ denote the classic directional derivative of F at x in the direction h, i.e.,

$$F'(x;h) = \lim_{t \downarrow 0} \frac{F(x+th) - F(x)}{t},$$

and

$$F'(x;h) = \lim_{\substack{V \in \partial F(x+th') \\ h' \to h,\ t \downarrow 0}} Vh', \quad h \in \mathbb{R}^n.$$

Lemma 1. *Suppose that $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitzian and semismooth at x. Then*
(1) $Vh - F'(x;h) = \mathcal{O}(\|h\|), \quad \forall V \in \partial F(x+h), \;\; h \in \mathbb{R}^n,$
(2) $F(x+h) - F(x) - F'(x;h) = \mathcal{O}(\|h\|), \quad h \in \mathbb{R}^n.$

If for any $V \in \partial F(x+h)$, $h \to 0$, we have $Vh - F'(x;h) = \mathcal{O}(\|h\|^{1+p})$, where $0 < p \leqslant 1$, then F is p-order semismooth at x. When $p = 1$, the function $F(x)$ is called strongly semismooth ([6]).

Lemma 2 ([3])**.** *Suppose that $F(x)$ and $\partial_\star F(x)$ are defined by* (3) *and by* (4)*, and all $V \in \partial_\star F(x)$ are nonsingular. Then there exits a scalar $\beta > 0$ such that*

$$\|V^{-1}\| \leqslant \beta, \quad \forall V \in \partial_\star F(x). \tag{5}$$

Furthermore, there exists a neighborhood $N(x)$ of x such that

$$\|V^{-1}\| \leqslant \frac{10}{9}\beta, \quad \forall V \in \partial_\star F(y), \;\; y \in N(x).$$

Scalar products and sums of semismooth functions are still semismooth functions (see [4]). Moreover, the equations in (3) are also semismooth.

Lemma 3. *Equations of max-type functions* (3) *form a system of semi-smooth equations.*

The terminology of the convergence rate, which is also used in this paper, refers to the following. Let $\{x_k\} \subset \mathbb{R}^n$ be a sequence of vectors tending to the limit $x^\star \neq x_k$ for all k. The convergence rate is said to be
(a) Q-linear if

$$\limsup_{k\to\infty} \frac{\|x_{k+1} - x^\star\|}{\|x_k - x^\star\|} < \infty;$$

(b) Q-superlinear if

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^\star\|}{\|x_k - x^\star\|} = 0;$$

(c) Q-quadratic if

$$\limsup_{k\to\infty} \frac{\|x_{k+1} - x^\star\|}{\|x_k - x^\star\|^2} < \infty;$$

(d) quadratic if

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^\star\|}{\|x_k - x^\star\|^2} = 0.$$

In these cases, we say that $\{x_k\}$ converges to $x^\star$ Q-linearly, Q-superlinearly, Q-quadratically, and quadratically, respectively.

3. A parametrized Newton method

The classical generalized Jacobian method was introduced by Qi and Sun [7] in the form (2). The iteration is locally superlinearly convergent for the semismooth equations. Our parametrized Newton method would be considered as a new approach based on parametrization of the method (2) for solving (3). On the other hand, we extend the method of Gao [3] and Śmietański [10] to a system of nonsmooth equations with finite maximum functions. We give the new parametrization Newton method for (3) in the form

$$x_{k+1} = x_k - [\operatorname{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k]^{-1} F(x_k), \tag{6}$$

where $V_k \in \partial_\star F(x_k)$ and $\lambda_i^{(k)} \in \mathbb{R}$, $0 < |\lambda_i^{(k)}| < +\infty$ for $i = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$ is a parameter chosen such that the matrix $\operatorname{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k$ is nonsingular.

Lemma 4. *Suppose $x^\star$ is a solution of* (3). *Then*

$$\|\operatorname{diag}(\lambda_i^{(k)} f_i(x_k))\| \leqslant M \leqslant \frac{1}{\beta} \tag{7}$$

for all x in some neighborhood of $x^\star$ and $M + \varepsilon < 1/\beta$, where $\lambda_i^{(k)} \in \mathbb{R}$ and $0 < |\lambda_i^{(k)}| < +\infty$ for $i = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$

Since each f_{ij} of (3) is continuous, we get the lemma immediately.

Theorem. *Suppose that $x^\star$ is a solution of* (3), *and all $V \in \partial_\star F(x^\star)$ are nonsingular. Then the iteration method defined by* (6) *for solving* (3) *is well-defined and superlinearly convergent to $x^\star$ in a neighborhood of $x^\star$.*

Proof. Denote by $N_{x^\star}$ a neighborhood of $x^\star$ for any $x \in N_{x^\star}$. For $V_x \in \partial_\star F(x)$, Lemma 1 implies that

$$\|V_x(x - x^\star) - F'(x^\star; x - x^\star)\| = o(\|x - x^\star\|), \tag{8}$$

$$\|F(x) - F(x^\star) - F'(x^\star; x - x^\star)\| = o(\|x - x^\star\|). \tag{9}$$

By virtue of Lemma 2 and (7), we get

$$\|[\operatorname{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k]^{-1}\| \leqslant \frac{\beta}{1 - \beta(\varepsilon + M)}. \tag{10}$$

So (6) is well-defined for $x_k \in N_{x^\star}$.

By (8), (10) and the fact of Lipschitz continuity of $F(x)$, we have

$$\begin{aligned}\|x_{k+1}-x^\star\| &= \|x_k - x^\star - [\mathrm{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k]^{-1} F(x_k)\| \\ &\leqslant \|[\mathrm{diag}(\lambda_i^{(k)} f_i(x_k)) + V_k]^{-1}\| \\ &\quad \times (\|\mathrm{diag}(\lambda_i^{(k)} f_i(x_k))(x_k - x^\star)\| + \|F(x_k) - F(x^\star) - F'(x^\star; x_k - x^\star)\| \\ &\quad\quad + \|V_k(x_k - x^\star) - F'(x^\star; x_k - x^\star)\|) \\ &= o(\|x_k - x^\star\|).\end{aligned}$$

Therefore, the sequence $\{x_k\}$ converges to $x^\star$ superlinearly. Thus, we have completed the proof of the theorem. □

Corollary. *Under the condition of the theorem, if the iteration does not terminate after a finite number of steps, then*

$$\lim_{k\to\infty} \frac{\|F(x_{k+1})\|}{\|F(x_k)\|} = 0$$

holds in a neighborhood of $x^\star$.

Remark. If f_{ij} in the equation system (3) are C^2 functions, then $F(x)$ is a strongly semismooth function. Since

$$\begin{aligned} Vh - F'(x;h) &= o(\|h\|^2), \\ F(x+h) - F(x) - F'(x;h) &= o(\|h\|^2), \end{aligned}$$

we obtain that

$$\begin{aligned} \|V_x(x - x^\star) - F'(x^\star; x - x^\star)\| &= o(\|x - x^\star\|^2), \\ \|F(x) - F(x^\star) - F'(x^\star; x - x^\star)\| &= o(\|x - x^\star\|^2), \\ \|\mathrm{diag}(\lambda_i^{(k)} f_i(x_k))(x - x^\star)\| &= o(\|x - x^\star\|^2). \end{aligned}$$

So the sequence $\{x_k\}$ converges to $x^\star$ quadratically.

4. Numerical test

The system of equations of max-type functions have concrete background, for instance, complementarity problems, variational inequality problems, Karush-Kuhn-Tucker systems of nonlinear programs and other mechanics and engineering problems. In order to show the performance of the parametrized Newton method (6), in this section, we present some numerical results of our method. All the experiments were run on a Pentium IV 1.8 GHz using Matlab 7.0.

Example 1.

$$\max\{f_{11}(x_1,x_2), f_{12}(x_1,x_2)\} = 0,$$
$$\max\{f_{21}(x_1,x_2), f_{22}(x_1,x_2)\} = 0,$$

where

$$f_{11} = \tfrac{1}{3}x_1^2, \quad f_{12} = x_1^2, \quad f_{21} = \tfrac{1}{2}x_1^2, \quad f_{22} = x_1^2.$$

From (3) we know

$$F(x) = (f_1(x), f_2(x))^T,$$

where $f_1(x) = x_1^2$, $f_2(x) = x_1^2$, $x \in \mathbb{R}^2$.

$x_0 = (1,10)^T$	$\lambda_1 = 0.01$	$\lambda_2 = 10$
Step	x	$F(x)$
1	$(1.0000, 10.0000)^T$	$(1.0000, 1.0000)^T$
2	$(0.5025, 9.9995)^T$	$(0.2525, 0.2525)^T$
3	$(0.2519, 9.9993)^T$	$(0.0634, 0.0634)^T$
4	$(0.1261, 9.9991)^T$	$(0.0159, 0.0159)^T$
5	$(0.0631, 9.99906)^T$	$(0.00398, 0.00398)^T$
6	$(0.031554, 9.99903)^T$	$1.0e-003*(0.99563, 0.99563)^T$
7	$(0.01578, 9.99902)^T$	$1.0e-003*(0.248986, 0.248986)^T$
8	$(0.00789, 9.99901)^T$	$1.0e-004*(0.62256, 0.62256)^T$
9	$(0.003945, 9.99900)^T$	$1.0e-004*(0.155653, 0.155653)^T$
10	$(0.001973, 9.99900)^T$	$1.0e-005*(0.38915, 0.38915)^T$
11	$(0.000986, 9.999001)^T$	$1.0e-006*(0.972888, 0.972888)^T$
12	$(0.000493, 9.999000)^T$	$1.0e-006*(0.243225, 0.243225)^T$
13	$(0.000247, 9.999000)^T$	$1.0e-007*(0.608064, 0.608064)^T$
14	$(0.000123, 9.999000)^T$	$1.0e-007*(0.152016, 0.152016)^T$
15	$(0.0000616, 9.9990001)^T$	$1.0e-008*(0.3800416, 0.3800416)^T$
16	$(0.0000308, 9.9990000)^T$	$1.0e-009*(0.9501046, 0.9501046)^T$
17	$(0.0000154, 9.9990000)^T$	$1.0e-009*(0.2375262, 0.2375262)^T$
18	$(0.0000077, 9.9990000)^T$	$1.0e-010*(0.5938156, 0.5938156)^T$
19	$(0.0000039, 9.9990000)^T$	$1.0e-010*(0.1484539, 0.1484539)^T$
20	$(0.0000019, 9.9990000)^T$	$1.0e-011*(0.3711348, 0.3711348)^T$
21	$(0.000000963, 9.999000001)^T$	$1.0e-012*(0.9278370, 0.9278370)^T$

Table 1. Results for Example 1 with initial point $x_0 = (1,10)^T$ and $\lambda_1 = 0.01$, $\lambda_2 = 10$.

According to (6), we have

$$\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{pmatrix} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} - \begin{pmatrix} \lambda_1 x_1^{(k)} * x_1^{(k)} + 2x_1^{(k)} & 0 \\ 2x_1^{(k)} & \lambda_2 x_1^{(k)} * x_1^{(k)} \end{pmatrix}^{-1} \begin{pmatrix} x_1^{(k)} * x_1^{(k)} \\ x_1^{(k)} * x_1^{(k)} \end{pmatrix}.$$

To test the points, we use

$$|x^{(k+1)} - x^k| < \varepsilon.$$

In this example, we let $\varepsilon = 10^{-6}$.

$x_0 = (1000, 1000)^T$	$\lambda_1 = 0.0001$	$\lambda_2 = 2$
Step	x	$F(x)$
1	$(1000, 1000)^T$	$(1000000, 1000000)^T$
2	$1.0e+002*(5.2381, 9.9998)^T$	$1.0e+005*(2.74376, 2.74376)^T$
3	$1.0e+002*(2.68589, 9.99963)^T$	$1.0e+004*(7.21401, 7.21401)^T$
4	$1.0e+002*(1.36074, 9.99957)^T$	$1.0e+004*(1.85162, 1.85162)^T$
5	$1.0e+002*(0.68497, 9.99953)^T$	$1.0e+003*(4.69182, 4.69182)^T$
6	$1.0e+002*(0.34365, 9.99952)^T$	$1.0e+003*(1.18098, 1.18098)^T$
7	$1.0e+002*(0.17212, 9.99951)^T$	$1.0e+002*(2.96258, 2.96258)^T$
8	$1.0e+002*(0.08613, 9.99950)^T$	$(74.19182, 74.19182)^T$
9	$1.0e+002*(0.04309, 9.99950)^T$	$(18.56393, 18.56393)^T$
10	$1.0e+002*(0.02155, 9.99950)^T$	$(4.64298, 4.64298)^T$
11	$1.0e+002*(0.01077, 9.99950)^T$	$(1.16010, 1.16010)^T$
12	$1.0e+002*(0.00539, 9.99950)^T$	$(0.29028, 0.29028)^T$
13	$1.0e+002*(0.00269, 9.99950)^T$	$(0.07257, 0.07257)^T$
14	$1.0e+002*(0.00135, 9.99950)^T$	$(0.01814, 0.01814)^T$
15	$1.0e+002*(0.00067, 9.99950)^T$	$(0.00454, 0.00454)^T$
16	$1.0e+002*(0.00034, 9.99950)^T$	$(0.00113, 0.00113)^T$
17	$1.0e+002*(0.00017, 9.99950)^T$	$1.0e-003*(0.28351, 0.28351)^T$
18	$1.0e+002*(0.000084, 9.999500)^T$	$1.0e-004*(0.70877, 0.70877)^T$
19	$1.0e+002*(0.000042, 9.999500)^T$	$1.0e-004*(0.17719, 0.17719)^T$
20	$1.0e+002*(0.000021, 9.999500)^T$	$1.0e-005*(0.44298, 0.44298)^T$
21	$1.0e+002*(0.000011, 9.999500)^T$	$1.0e-005*(0.11074, 0.11074)^T$
22	$1.0e+002*(0.000005, 9.999500)^T$	$1.0e-006*(0.27686, 0.27686)^T$
23	$1.0e+002*(0.000003, 9.999500)^T$	$1.0e-007*(0.69216, 0.69216)^T$
24	$1.0e+002*(0.000001, 9.999500)^T$	$1.0e-007*(0.17304, 0.17304)^T$
25	$1.0e+002*(0.00000066, 9.99950000)^T$	$1.0e-008*(0.432598, 0.432598)^T$
26	$1.0e+002*(0.00000033, 9.99950000)^T$	$1.0e-008*(0.108149, 0.108149)^T$
27	$1.0e+002*(0.00000016, 9.99950000)^T$	$1.0e-009*(0.270374, 0.270374)^T$
28	$1.0e+002*(0.00000008, 9.99950000)^T$	$1.0e-010*(0.675934, 0.675934)^T$
29	$1.0e+002*(0.00000004, 9.99950000)^T$	$1.0e-010*(0.168983, 0.168983)^T$
30	$1.0e+002*(0.00000002, 9.99950000)^T$	$1.0e-011*(0.422459, 0.422459)^T$
31	$1.0e+002*(0.00000001, 9.99950000)^T$	$1.0e-011*(0.105615, 0.105615)^T$
32	$1.0e+002*(0.000000005, 9.999500000)^T$	$1.0e-012*(0.264037, 0.264037)^T$

Table 2. Results for Example 1 with initial point $x_0 = (1000, 1000)^T$ and $\lambda_1 = 0.0001$, $\lambda_2 = 2$.

$x_0 = (10, 10)^T$	$\lambda_1 = 0.003$	$\lambda_2 = 0.002$
Step	x	$F(x)$
1	$(10.0000, 10.0000)^T$	$(150.0000, 100.0000)^T$
2	$(5.0491, 5.0868)^T$	$(38.6227, 25.4937)^T$
3	$(2.5373, 2.5657)^T$	$(9.8018, 6.4379)^T$
4	$(1.2719, 1.2885)^T$	$(2.4691, 1.6177)^T$
5	$(0.6368, 0.6457)^T$	$(0.6196, 0.4055)^T$
6	$(0.3186, 0.3232)^T$	$(0.1552, 0.1015)^T$
7	$(0.1593, 0.1617)^T$	$(0.0388, 0.0254)^T$
8	$(0.0797, 0.0809)^T$	$(0.0097, 0.0063)^T$
9	$(0.0398, 0.0404)^T$	$(0.0024, 0.0016)^T$
10	$(0.0199, 0.0202)^T$	$1.0\text{e} - 003 * (0.6073, 0.3970)^T$
11	$(0.00996, 0.01011)^T$	$1.0\text{e} - 003 * (0.1518, 0.0992)^T$
12	$(0.00498, 0.00506)^T$	$1.0\text{e} - 004 * (0.3796, 0.2481)^T$
13	$(0.00249, 0.00253)^T$	$1.0\text{e} - 005 * (0.9491, 0.6203)^T$
14	$(0.00125, 0.00126)^T$	$1.0\text{e} - 005 * (0.2373, 0.1551)^T$
15	$1.0\text{e} - 003 * (0.6226, 0.6319)^T$	$1.0\text{e} - 006 * (0.5932, 0.3877)^T$
16	$1.0\text{e} - 003 * (0.31132, 0.31597)^T$	$1.0\text{e} - 006 * (0.1483, 0.0969)^T$
17	$1.0\text{e} - 003 * (0.15566, 0.15798)^T$	$1.0\text{e} - 007 * (0.3707, 0.2423)^T$
18	$1.0\text{e} - 004 * (0.77830, 0.78992)^T$	$1.0\text{e} - 008 * (0.9268, 0.6058)^T$
19	$1.0\text{e} - 004 * (0.38915, 0.39496)^T$	$1.0\text{e} - 008 * (0.2317, 0.1514)^T$
20	$1.0\text{e} - 004 * (0.19458, 0.19748)^T$	$1.0\text{e} - 009 * (0.5793, 0.3786)^T$
21	$1.0\text{e} - 005 * (0.97288, 0.98740)^T$	$1.0\text{e} - 009 * (0.1448, 0.0946)^T$
22	$1.0\text{e} - 005 * (0.48644, 0.49370)^T$	$1.0\text{e} - 010 * (0.3620, 0.2366)^T$
23	$1.0\text{e} - 005 * (0.24322, 0.24685)^T$	$1.0\text{e} - 011 * (0.9051, 0.5916)^T$
24	$1.0\text{e} - 005 * (0.12161, 0.12342)^T$	$1.0\text{e} - 011 * (0.2263, 0.1479)^T$
25	$1.0\text{e} - 006 * (0.60805, 0.61712)^T$	$1.0\text{e} - 012 * (0.5657, 0.3697)^T$

Table 3. Results for Example 2 with initial point $x_0 = (10, 10)^T$ and $\lambda_1 = 0.003$, $\lambda_2 = 0.002$.

Example 2.

$$\max\{f_{11}(x_1, x_2), f_{12}(x_1, x_2)\} = 0,$$
$$\max\{f_{21}(x_1, x_2), f_{22}(x_1, x_2)\} = 0,$$

where

$$f_{11} = \tfrac{1}{2}x_1^2 + \tfrac{1}{3}x_2^2, \quad f_{12} = \tfrac{1}{2}x_1^2 + x_2^2, \quad f_{21} = \tfrac{1}{4}x_1^2, \quad f_{22} = x_1^2.$$

From (3) we know

$$F(x) = (f_1(x), f_2(x))^T.$$

According to (6), we also have

$$\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{pmatrix} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} - \begin{pmatrix} \lambda_1 * (f_1(x^{(k)})) + x_1^{(k)} & 2 * x_2^{(k)} \\ 2 * x_1^{(k)} & \lambda_2 * (f_2(x^{(k)})) \end{pmatrix}^{-1} \begin{pmatrix} f_1(x^{(k)}) \\ f_2(x^{(k)}) \end{pmatrix}$$

where

$$f_1(x) = \tfrac{1}{2}x_1^2 + x_2^2, \quad f_2(x) = x_1^2, \quad x \in \mathbb{R}^2.$$

To test the points, we use

$$|x^{(k+1)} - x^k| < \varepsilon.$$

In this example, we also let $\varepsilon = 10^{-6}$.

We also can test the example with other initial points and λ.

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References

[1] *X. Chen, L. Qi*: A parameterized Newton method and a quasi-Newton method for nonsmooth equations. Comput. Optim. Appl. *3* (1994), 157–179.

[2] *F. H. Clarke*: Optimization and Nonsmooth Analysis. John Wiley & Sons, New York, 1983.

[3] *Y. Gao*: Newton methods for solving two classes of nonsmooth equations. Appl. Math. *46* (2001), 215–229.

[4] *R. Mifflin*: Semismooth and semiconvex functions in constrained optimization. SIAM J. Control. Optim. *15* (1997), 959–972.

[5] *J. S. Pang, L. Qi*: Nonsmooth equations: Motivation and algorithms. SIAM J. Optim. *3* (1993), 443–465.

[6] *F. A. Potra, L. Qi, D. Sun*: Secant methods for semismooth equations. Numer. Math. *80* (1998), 305–324.

[7] *L. Qi, J. Sun*: A nonsmooth version of Newton's method. Math. Program. Ser. A *58* (1993), 353–367.

[8] *L. Qi*: Convergence analysis of some algorithms for solving nonsmooth equations. Math. Oper. Res. *18* (1993), 227–244.

[9] *M. J. Śmietański*: An approximate Newton method for non-smooth equations with finite max functions. Numer. Algorithms *41* (2006), 219–238.

[10] *M. J. Śmietański*: On a new class parametrized Newton-like methods for semismooth equations. Appl. Math. Comput. *193* (2007), 430–437.

[11] *D. Sun, J. Han*: Newton and quasi-Newton methods for a class of nonsmooth equations and related problems. SIAM J. Optim. *7* (1997), 463–480.

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