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Fengquan Li; Weiwei Sun

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HYPERBOLIC BOUNDARY VALUE PROBLEM WITH EQUIVALUED SURFACE ON A DOMAIN WITH THIN LAYER*

Fengquan Li, Weiwei Sun, Dalian

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Abstract. This paper deals with a kind of hyperbolic boundary value problems with equivalued surface on a domain with thin layer. Existence and uniqueness of solutions are given, and the limit behavior of solutions is studied in this paper.

Keywords: limit behavior of solutions, existence, uniqueness, equivalued surface, equivalued interface, hyperbolic equation

MSC 2010: 35A05, 35B40, 35L20

1. Introduction

In many practical applications, especially in resistivity well-logging in petroleum exploitation, boundary value problems with equivalued surface are formulated (see [8]–[10]). From the physical point of view, the equivalued surface boundary value condition corresponds to a source. When the equivalued surface boundary shrinks to an interior point or shrinks to a point on the boundary of the domain, the limit behaviour of solutions for hyperbolic equations has been discussed in [3] and [7].

In resistivity well-logging, one may encounter a formation with crack domain, the resistivity of which is often diffcult to be obtained. However, this crack domain is a thin layer compared with the whole formation (see [10]). In practical calculation, the variation of solutions near the thin layer should be quite large, and then in finite element procedure, it is necessary to have a refined partition of elements near the thin layer. This causes a complexity in computation. To get rid of this difficulty,

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when the thin layer is extremely thin (if measured by the mesh size parameter), the thin layer can be approximately regarded as an interface and corresponding the boundary value problem with equivalued surface on the thin layer can be approximately replaced by the boundary value problem with equivalued interface. To prove the above conclusion, we need to study existence, uniqueness and limit behavior of solutions for boundary value problems with equivalued surface on a domain with thin layer. For the case of elliptic equations this has been studied in [11]. In this paper, we will discuss the case of hyperbolic equations because this kind of boundary value problem can be used in acoustic well-logging (see [15]).

Here we consider the following hyperbolic boundary value problem with equivalued surface on the domain with thin layer:

$$
(\mathrm{P}_1)\qquad \begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) = F(x,t) & \text{in } Q_1 \cup Q_2,\\
u = 0 & \text{on } \Sigma,\\
u = C(t) \text{ (a function to be determined)} & \text{on } \tilde{\Sigma},\\
\displaystyle\int_{\tilde{\Gamma}_1} \frac{\partial u}{\partial n_L}\,\mathrm{d}s = \int_{\tilde{\Gamma}_2} \frac{\partial u}{\partial n_L}\,\mathrm{d}s + A(t) & \text{a.e. } t \in (0,T),\\
u(x,0) = \psi_0(x) & \text{in } \Omega_1 \cup \Omega_2,\\
\dfrac{\partial u}{\partial t}(x,0) = \psi_1(x) & \text{in } \Omega_1 \cup \Omega_2,
\end{cases}
$$

where $Q_1 = \Omega_1 \times (0,T)$, $Q_2 = \Omega_2 \times (0,T)$, $Q_T = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, $\tilde{\Sigma} = (\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2) \times (0,T)$, T is a fixed positive constant, and

$$
\frac{\partial u}{\partial n_L} = \sum_{i,j=1}^{N} a_{ij}(x,t) n_i \frac{\partial u}{\partial x_j}, \quad \mathbf{n} = (n_1, n_2, \ldots, n_N)
$$

denotes the conormal derivative.

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) be a bounded domain with smooth outside boundary Γ (see Fig. 1). Suppose that Ω is composed of three non-overlapping subdomains Ω_1, $\tilde{\Omega}$ and Ω_2, and $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are the interfaces of $\tilde{\Omega}$ with Ω_1 and Ω_2 respectively. The unit normal $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ takes the inward and outward directions (or vice versa) for the domain $\tilde{\Omega}$ on $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$. In this paper, we will deal with the existence, uniqueness and limit behavior of weak solutions to problem (P_1).

The paper is organized as follows: In Section 2 we will prove the existence and uniqueness of a weak solution to the problem (P_1). In Section 3 we will discuss a hyperbolic boundary value problem (P) with equivalued interface. In Section 4 the limit behavior of solutions to the problem (P_1) will be studied.

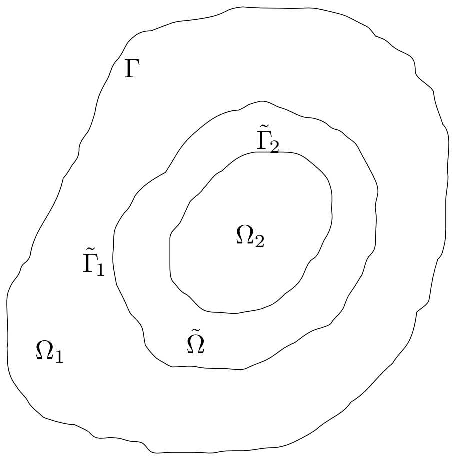

Figure 1.

2. Existence and uniqueness of weak solution to problem (P_1)

In this section, we will discuss the existence and uniqueness of a weak solution to the problem (P_1). We first give the following assumption:

($\tilde{\mathrm{H}}_1$) $a_{ij} \in W^{1,\infty}(Q_T)$, $a_{ij}(x,t) = a_{ji}(x,t)$, and there exist two positive constants α, β such that

$$\alpha|\xi|^2 \leqslant \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \leqslant \beta|\xi|^2, \quad \forall\, \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N, \quad \text{a.e. } (x,t) \in Q_T \tag{2.1}$$

Let

$$V_0 = \{v\colon\ v \in H_0^1(\Omega),\ v|_{\tilde{\Gamma}_1 \cup \tilde{\Omega} \cup \tilde{\Gamma}_2} = \text{constant}\}, \tag{2.2}$$

and

$$U_1 = \left\{ v \,\middle|\, \begin{array}{l} \varphi \in L^2(0,T; H_0^1(\Omega)),\ \varphi_t, \varphi_{tt} \in L^2(Q_T),\ \varphi(x,T) = 0, \\ \varphi_t(x,T) = 0,\ \varphi|_{\tilde{\Sigma}} = C(t) \end{array} \right\}, \tag{2.3}$$

where $C(t)$ is an arbitrary function of t.

Here we also assume $F \in L^2(Q_T)$, $A \in H^1(0,T)$, $\psi_0 \in V_0$, and $\psi_1 \in L^2(\Omega)$.

Definition 2.1. If there exists a measurable function $u \in L^2(0,T;V_0)$ such that $\forall\, \varphi \in U_1$,

$$\int_0^T\int_{\Omega_1\cup\Omega_2} u\varphi_{tt}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega}\sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t \tag{2.4}$$
$$= \int_0^T\int_{\Omega_1\cup\Omega_2} F(x,t)\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_1\cup\Omega_2}\psi_0(x)\varphi_t(x,0)\,\mathrm{d}x$$
$$+ \int_{\Omega_1\cup\Omega_2}\psi_1(x)\varphi(x,0)\,\mathrm{d}x + \int_0^T A(t)\varphi|_{\tilde{\Sigma}}\,\mathrm{d}t,$$

then we say that u is a weak solution to the problem (P_1).

Now we can state the existence and uniqueness of a weak solution to the problem (P_1) as follows.

Theorem 2.2. *Suppose that* $F \in L^2(Q_T)$, $\psi_0 \in V_0$, $\psi_1 \in L^2(\Omega)$, $A \in H^1(0,T)$ *and* $(\tilde{\mathrm{H}}_1)$ *hold, then there exists a unique weak solution* $u \in L^2(0,T;V_0)$ *to the problem* (P_1).

P r o o f. (1) Proof of existence: Let

$$V = \{v\colon\ v \in H^1(\Omega_1\cup\Omega_2),\ v|_{\Gamma} = 0,\ v|_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_2} = \text{constant}\} \tag{2.5}$$

and let V' be the dual space of V.

Here we will use the Galerkin method (see [14], [13], [6], [2], [5] and [12]). Take a basis $\{\omega_k\}_{k=1}^{\infty}$ of V such that it is a complete orthonormal basis of $L^2(\Omega_1\cup\Omega_2)$, too. For any fixed n, let $S_n = \operatorname{span}\{\omega_1,\omega_2,\ldots,\omega_n\}$. Let $\psi_{0n}(x) = \sum_{k=1}^n c_{0k}\omega_k$ and $\psi_{1n} = \sum_{k=1}^n c_{1k}\omega_k$ be the projections of $\psi_0(x)$ and $\psi_1(x)$ onto S_n, respectively.

Let $\tilde{u}_n = \sum_{k=1}^n c_{kn}\omega_k$; the Galerkin equations are as follows

$$\begin{cases} \displaystyle\int_{\Omega_1\cup\Omega_2}\frac{\partial^2\tilde{u}_n}{\partial t^2}\omega_k\,\mathrm{d}x + \int_{\Omega_1\cup\Omega_2}\sum_{i,j=1}^N a_{ij}\frac{\partial \tilde{u}_n}{\partial x_j}\frac{\partial \omega_k}{\partial x_i}\,\mathrm{d}x \\ \quad = \displaystyle\int_{\Omega_1\cup\Omega_2} F\omega_k\,\mathrm{d}x + A(t)\omega_k|_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_2}, \\ \tilde{u}_n(x,0) = \psi_{0n}(x), \\ \displaystyle\frac{\partial \tilde{u}_n}{\partial t}(x,0) = \psi_{1n}(x). \end{cases} \tag{2.6}$$

Namely, for almost all $t \in (0,T)$,

$$(2.7)\qquad \begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}\,t^2} c_{kn}(t) + \displaystyle\sum_{l=1}^{n} c_{ln}(t) \int_{\Omega_1\cup\Omega_2} \sum_{i,j=1}^{N} a_{ij} \frac{\partial \omega_l}{\partial x_j} \frac{\partial \omega_k}{\partial x_i}\,\mathrm{d}x \\ \quad = \displaystyle\int_{\Omega_1\cup\Omega_2} F\omega_k\,\mathrm{d}x + A(t)\omega_k|_{\tilde\Gamma_1\cup\tilde\Gamma_2}, \\ c_{kn}(0) = c_{0k}, \\ c'_{kn}(0) = c_{1k}. \end{cases}$$

By the theory of systems of ordinary differential equations, the problem (2.7) admits a unique solution $c_{kn} \in C^1$, $k = 1,\ldots,n$.

Multiplying (2.6) by $c'_{kn}(t)$ and summing over k, we obtain

$$(2.8)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\|\frac{\partial \tilde u_n}{\partial t}\Big\|^2_{L^2(\Omega_1\cup\Omega_2)} + \int_{\Omega_1\cup\Omega_2} \sum_{i,j=1}^{N} a_{ij} \frac{\partial \tilde u_n}{\partial x_j} \frac{\partial^2 \tilde u_n}{\partial x_i \partial t}\,\mathrm{d}x = \int_{\Omega_1\cup\Omega_2} F(x,t)\frac{\partial \tilde u_n}{\partial t}\,\mathrm{d}x + A(t)\frac{\partial \tilde u_n}{\partial t}\Big|_{\tilde\Gamma_1\cup\tilde\Gamma_2}.$$

Integrating (2.8) over $(0,\tau)$ with respect to t, we get

$$\begin{aligned}(2.9)\quad & \frac{1}{2}\Big\|\frac{\partial \tilde u_n}{\partial t}(\cdot,\tau)\Big\|^2_{L^2(\Omega_1\cup\Omega_2)} + \frac{1}{2}\int_{\Omega_1\cup\Omega_2} \sum_{i,j=1}^{N} a_{ij} \frac{\partial \tilde u_n}{\partial x_j}\frac{\partial \tilde u_n}{\partial x_i}\,\mathrm{d}x\Big|_{t=\tau} \\ &= \int_0^\tau\int_{\Omega_1\cup\Omega_2} F\frac{\partial \tilde u_n}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau A(t)\frac{\partial \tilde u_n}{\partial t}\Big|_{(\tilde\Gamma_1\cup\tilde\Gamma_2)\times(0,T)}\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^\tau\int_{\Omega_1\cup\Omega_2} \sum_{i,j=1}^{N} \frac{\partial a_{ij}(x,t)}{\partial t}\frac{\partial \tilde u_n}{\partial x_j}\frac{\partial \tilde u_n}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_{\Omega_1\cup\Omega_2} \sum_{i,j=1}^{N} a_{ij}\frac{\partial \tilde u_n}{\partial x_j}\frac{\partial \tilde u_n}{\partial x_i}\,\mathrm{d}x\Big|_{t=0} + \frac{1}{2}\Big\|\frac{\partial \tilde u_n}{\partial t}(\cdot,0)\Big\|^2_{L^2(\Omega_1\cup\Omega_2)}.\end{aligned}$$

By $(\tilde{\mathrm{H}}_1)$ and the Hölder inequality, we have

$$\begin{aligned}(2.10)\quad & \Big\|\frac{\partial \tilde u_n}{\partial t}(\cdot,\tau)\Big\|^2_{L^2(\Omega_1\cup\Omega_2)} + \alpha\|D\tilde u_n(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)} \\ &\leqslant \|F\|^2_{L^2(Q_T)} + \int_0^\tau\int_{\Omega_1\cup\Omega_2}\Big(\frac{\partial \tilde u_n}{\partial t}\Big)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 2\Big|\int_0^\tau A(t)\frac{\partial \tilde u_n}{\partial t}\Big|_{(\tilde\Gamma_1\cup\tilde\Gamma_2)\times(0,T)}\,\mathrm{d}t\Big| \\ &\quad + N^2\|a_{ij}\|_{W^{1,\infty}(Q_T)}\int_0^\tau \|D\tilde u_n\|^2_{L^2(\Omega_1\cup\Omega_2)}\,\mathrm{d}t \\ &\quad + \Big\|\frac{\partial \tilde u_n}{\partial t}(\cdot,0)\Big\|^2_{L^2(\Omega_1\cup\Omega_2)} + \beta\|D\tilde u_n(\cdot,0)\|^2_{L^2(\Omega_1\cup\Omega_2)}.\end{aligned}$$

Let

$$E(t)=\int_{\Omega_1\cup\Omega_2}\Big(\Big(\frac{\partial\tilde u_n(t)}{\partial t}\Big)^2+|D\tilde u_n(t)|^2+(\tilde u_n(t))^2\Big)\,\mathrm{d}x. \tag{2.11}$$

From (2.10)–(2.11), we obtain

$$\begin{aligned}\Big\|\frac{\partial\tilde u_n}{\partial t}(\cdot,\tau)\Big\|^2_{L^2(\Omega_1\cup\Omega_2)}&+\alpha\|D\tilde u_n(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)}\\ &\leqslant C_1\Big(\|F\|^2_{L^2(Q_T)}+E(0)+\int_0^\tau E(t)\,\mathrm{d}t\Big)\\ &\quad+2\Big|\int_0^\tau A(t)\frac{\partial\tilde u_n}{\partial t}\Big|_{(\tilde\Gamma_1\cup\tilde\Gamma_2)\times(0,T)}\,\mathrm{d}t\Big|,\end{aligned} \tag{2.12}$$

where $C_1=\max\{1,\beta,N^2\|a_{ij}\|_{W^{1,\infty}(Q_T)}\}$.

By integration by parts, the Sobolev imbedding theorem (see [12]), the Young inequality and the trace theorem, we obtain

$$\begin{aligned}2\Big|\int_0^\tau A(t)\frac{\partial\tilde u_n}{\partial t}\Big|_{(\tilde\Gamma_1\cup\tilde\Gamma_2)\times(0,T)}\,\mathrm{d}t\Big|&\\ \leqslant C_2\delta\Big(\int_{\Omega_1\cup\Omega_2}&[|D\tilde u_n(\cdot,\tau)|^2+(\tilde u_n(\cdot,\tau))^2]\,\mathrm{d}x\Big)\\ &+C_2\Big(\int_{\Omega_1\cup\Omega_2}[|D\tilde u_n(\cdot,0)|^2+(\tilde u_n(\cdot,0))^2]\,\mathrm{d}x\Big)\\ &+C_\delta\|A\|^2_{H^1(0,T)}\\ &+C_2\int_0^\tau(\|D\tilde u_n\|^2_{L^2(\Omega_1\cup\Omega_2)}+\|\tilde u_n\|^2_{L^2(\Omega_1\cup\Omega_2)})\,\mathrm{d}t,\end{aligned} \tag{2.13}$$

where C_2 is a positive constant depending on $|\tilde\Gamma_1|$, C_δ is a positive constant depending on δ, T, and δ is an arbitrary small postive constant.

According to the definition of $E(t)$, we have

$$\begin{aligned}2\Big|\int_0^\tau A(t)\frac{\partial\tilde u_n}{\partial t}\Big|_{(\tilde\Gamma_1\cup\tilde\Gamma_2)\times(0,T)}\,\mathrm{d}t\Big|&\\ \leqslant C_2\delta\Big(\int_{\Omega_1\cup\Omega_2}&[|D\tilde u_n(\cdot,\tau)|^2+(\tilde u_n(\cdot,\tau))^2]\,\mathrm{d}x\Big)\\ &+C_2E(0)+C_\delta\|A\|^2_{H^1(0,T)}+C_2\int_0^\tau E(t)\,\mathrm{d}t.\end{aligned} \tag{2.14}$$

Since $\tilde u_n(\cdot,\tau)=\tilde u_n(\cdot,0)+\int_0^\tau(\partial\tilde u_n/\partial t)\,\mathrm{d}t$,

$$\tilde u_n^2(\cdot,\tau)\leqslant 2(\tilde u_n(\cdot,0))^2+2T\int_0^\tau\Big(\frac{\partial\tilde u_n}{\partial t}\Big)^2\,\mathrm{d}t. \tag{2.15}$$

Integrating (2.15) over $\Omega_1 \cup \Omega_2$, we have

$$\|\tilde{u}_n(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)} \leqslant C_3\Big(E(0) + \int_0^\tau E(t)\,\mathrm{d}t\Big), \tag{2.16}$$

where $C_3 = \max\{2, 2T\}$.

Putting (2.16), (2.14), and (2.12) together, one can deduce that

$$\begin{aligned}\Big\|\frac{\partial\tilde{u}_n}{\partial t}(\cdot,\tau)\Big\|^2_{L^2(\Omega_1\cup\Omega_2)} &+ \alpha\|D\tilde{u}_n(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)} + \|\tilde{u}_n(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)} \\ &\leqslant C_4\Big(\|F\|^2_{L^2(Q_T)} + E(0) + \int_0^\tau E(t)\,\mathrm{d}t\Big) + C_\delta\|A\|^2_{H^1(0,T)} \\ &\quad + C_2\delta\Big(\int_{\Omega_1\cup\Omega_2}[|D\tilde{u}_n(\cdot,\tau)|^2 + (\tilde{u}_n(\cdot,\tau))^2]\,\mathrm{d}x\Big),\end{aligned} \tag{2.17}$$

where $C_4 = \max\{C_1, C_2, C_3\}$.

Let $\delta = \frac{1}{2}C_2^{-1}\min\{\alpha, 1\}$, then it is easy to get

$$\begin{aligned}\Big\|\frac{\partial\tilde{u}_n}{\partial t}(\cdot,\tau)\Big\|^2_{L^2(\Omega_1\cup\Omega_2)} &+ \|D\tilde{u}_n(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)} + \|\tilde{u}_n(\cdot,\tau)\|^2_{L^2(\Omega_1\cup\Omega_2)} \\ &\leqslant C_5\Big(\|F\|^2_{L^2(Q_T)} + E(0) + \int_0^\tau E(t)\,\mathrm{d}t + \|A\|^2_{H^1(0,T)}\Big).\end{aligned} \tag{2.18}$$

The above inequality may be written as

$$E(\tau) \leqslant C_5\Big(\|F\|^2_{L^2(Q_T)} + E(0) + \int_0^\tau E(t)\,\mathrm{d}t + \|A\|^2_{H^1(0,T)}\Big). \tag{2.19}$$

Using Gronwall's inequality, we get

$$\begin{aligned}E(\tau) &\leqslant C_6(\|F\|^2_{L^2(Q_T)} + \|\psi_{0n}\|^2_{V_0} + \|\psi_{1n}\|^2_{L^2(\Omega)} + \|A\|^2_{H^1(0,T)}) \\ &\leqslant C_6(\|F\|^2_{L^2(Q_T)} + \|\psi_0\|^2_{V_0} + \|\psi_1\|^2_{L^2(\Omega)} + \|A\|^2_{H^1(0,T)}).\end{aligned} \tag{2.20}$$

Hence for a.e. $t \in (0, T)$,

$$\begin{aligned}\|\tilde{u}_n(t)\|^2_{L^2(\Omega_1\cup\Omega_2)} &+ \|D\tilde{u}_n(t)\|^2_{L^2(\Omega_1\cup\Omega_2)} + \Big\|\frac{\partial\tilde{u}_n(t)}{\partial t}\Big\|^2_{L^2(\Omega_1\cup\Omega_2)} \\ &\leqslant C_6(\|F\|^2_{L^2(Q_T)} + \|\psi_0\|^2_{V_0} + \|\psi_1\|^2_{L^2(\Omega)} + \|A\|^2_{H^1(0,T)}).\end{aligned} \tag{2.21}$$

Thus we get

$$\|\tilde{u}_n\|_{L^2(0,T;V)} \leqslant (C_6T)^{1/2}(\|F\|_{L^2(Q_T)} + \|\psi_0\|_{V_0} + \|\psi_1\|_{L^2(\Omega)} + \|A\|_{H^1(0,T)}), \tag{2.22}$$

$$\begin{aligned}\Big\|\frac{\partial\tilde{u}_n}{\partial t}\Big\|_{L^\infty(0,T;L^2(\Omega_1\cup\Omega_2))} \\ \leqslant C_6^{1/2}(\|F\|_{L^2(Q_T)} + \|\psi_0\|_{V_0} + \|\psi_1\|_{L^2(\Omega)} + \|A\|_{H^1(0,T)}).\end{aligned} \tag{2.23}$$

Integrating (2.7) over $(t, t+\Delta t)$, we get

$$(2.24)\qquad c'_{kn}(t+\Delta t) - c'_{kn}(t) + \int_t^{t+\Delta t}\int_{\Omega_1\cup\Omega_2}\sum_{i,j=1}^{N} a_{ij}\frac{\partial \tilde{u}_n}{\partial x_j}\frac{\partial \omega_k}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}\tau = \int_t^{t+\Delta t}\int_{\Omega_1\cup\Omega_2} F\omega_k\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^{\tau} A(\tau)\omega_k|_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_2}\,\mathrm{d}\tau.$$

On the basis of $(\tilde{\mathrm{H}}_1)$, (2.22), and the trace theorem, we arrive at

$$(2.25)\qquad |c'_{kn}(t+\Delta t) - c'_{kn}(t)| \leqslant C_7\|\omega_k\|_V|\Delta t|^{1/2},$$

where C_7 is a positive constant independent of n, k.

From the above inequality, we can deduce that for any fixed positive integer k, c_{kn} is equicontinuous with respect to n in $[0,T]$. Thus by the Ascoli-Arzela theorem, we can extract a subsequence of $\{c'_{kn}\}$ (still denoted by $\{c'_{kn}\}$) such that as $n\to\infty$.

$$(2.26)\qquad c'_{kn} \to d_k \quad \text{uniformly in } [0,T].$$

Let $c_k(t) = \int_0^t d_k(\tau)\,\mathrm{d}\tau + c_{0k}$, where $c_{0k} = (\psi_0, \omega_k)$, then $c'_k(t) = d_k(t)$ and

$$(2.27)\qquad c_{kn} \to c_k \text{ uniformly in } [0,T].$$

For any postive integer $r \leqslant n$, (2.23) yields

$$(2.28)\qquad \sum_{k=1}^{r}(c'_{kn}(t))^2 \leqslant C_8, \quad \forall\, t\in(0,T),$$

where C_8 is a positive constant independent of n, k, t. Let $n\to\infty$ in (2.28), then for any postive integer r we have

$$(2.29)\qquad \sum_{k=1}^{r}(c'_k(t))^2 \leqslant C_8.$$

From (2.29), Hölder's inequality and Parseval's identity it follows that

$$(2.30)\qquad \sum_{k=1}^{r}(c_k(t))^2 \leqslant \sum_{k=1}^{r} 2\Big(\int_0^t d_k^2(\tau)\,\mathrm{d}\tau\Big)t + 2\sum_{k=1}^{r} c_{0k}^2 \leqslant 2T\int_0^t\sum_{k=1}^{r}(c'_k(\tau))^2\,\mathrm{d}\tau + 2\|\psi_0\|_{V_0}^2 \leqslant C_9,$$

where C_9 is a positive constant independent of n, k, r and t.

Let $\tilde{u}(x,t) = \sum_{k=1}^{\infty} c_k(t)\omega_k(x)$, then $\tilde{u}'(x,t) = \sum_{k=1}^{\infty} c_k'(t)\omega_k(x)$, and (2.30) and (2.29) imply that $\tilde{u}(x,t), \tilde{u}'(x,t) \in L^2(\Omega_1 \cup \Omega_2)$, $\forall\, t \in [0,T]$.

For any fixed postive integer k, it follows from (2.26) and (2.27) that

$$(2.31)\qquad (\tilde{u}_n'(\cdot,t) - \tilde{u}'(\cdot,t), \omega_k) = c_{kn}' - c_k' \to 0, \quad \text{uniformly in } [0,T]$$

and

$$(2.32)\qquad (\tilde{u}_n(\cdot,t) - \tilde{u}(\cdot,t), \omega_k) = c_{kn} - c_k \to 0, \quad \text{uniformly in } [0,T].$$

But $\{\omega_k\}_{k=1}^{\infty}$ is a complete orthonormal basis of $L^2(\Omega_1 \cup \Omega_2)$, thus

$$(2.33)\qquad \tilde{u}_n' \to \tilde{u}' \text{ weakly in } C([0,T]; L^2(\Omega_1 \cup \Omega_2))$$

and

$$(2.34)\qquad \tilde{u}_n \to \tilde{u} \text{ weakly in } C([0,T]; L^2(\Omega_1 \cup \Omega_2)).$$

Thus (2.22) and (2.34) imply that

$$(2.35)\qquad \tilde{u}_n \to \tilde{u} \text{ weakly in } L^2(0,T;V).$$

Next, (2.33)–(2.34) yield

$$(2.36)\qquad \tilde{u}_n'(\cdot,0) \to \tilde{u}'(\cdot,0) \text{ weakly in } L^2(\Omega_1 \cup \Omega_2)$$

and

$$(2.37)\qquad \tilde{u}_n(\cdot,0) \to \tilde{u}(\cdot,0) \text{ weakly in } L^2(\Omega_1 \cup \Omega_2).$$

Consequently $\tilde{u}(0) = \psi_0$, $\tilde{u}'(0) = \psi_1$.

For any given sequence of smooth function $\{\upsilon_k(t)\}_{k=1}^{\infty}$ defined in $[0,T]$ with $\upsilon_k(T) = 0$ and $\upsilon_k'(T) = 0$, multiplying the Galerkin equation (2.6) by $\upsilon_k(t)$ and using integration by parts, we obtain

$$(2.38)\qquad \int_0^T\int_{\Omega_1\cup\Omega_2} \tilde{u}_n \upsilon_{ktt}\omega_k \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\Omega_1\cup\Omega_2} \sum_{i,j=1}^{N} a_{ij}\frac{\partial \omega_k}{\partial x_i}\frac{\partial \tilde{u}_n}{\partial x_j}\upsilon_k \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_{\Omega_1\cup\Omega_2} F\upsilon_k\omega_k \,\mathrm{d}x\,\mathrm{d}t + \int_0^T A\upsilon_k(t)\omega_k|_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_2}\,\mathrm{d}t$$
$$- \int_{\Omega_1\cup\Omega_2} \psi_{0n}(x)\upsilon_{kt}(0)\omega_k \,\mathrm{d}x + \int_{\Omega_1\cup\Omega_2} \psi_{1n}(x)\upsilon_k(0)\omega_k \,\mathrm{d}x.$$

According to (2.35)–(2.37), letting $n \to \infty$ in (2.38), it is easy to prove that

$$\begin{aligned}(2.39)\qquad &\int_0^T \int_{\Omega_1\cup\Omega_2} \tilde{u} \upsilon_{ktt}\omega_k \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_1\cup\Omega_2} \sum_{i,j=1}^N a_{ij} \frac{\partial \omega_k}{\partial x_i}\frac{\partial \tilde{u}}{\partial x_j}\upsilon_k \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\Omega_1\cup\Omega_2} F\upsilon_k\omega_k \,\mathrm{d}x\,\mathrm{d}t + \int_0^T A\upsilon_k(t)\omega_k|_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_2}\,\mathrm{d}t \\ &\quad - \int_{\Omega_1\cup\Omega_2} \psi_0(x)\upsilon_{kt}(0)\omega_k\,\mathrm{d}x + \int_{\Omega_1\cup\Omega_2} \psi_1(x)\upsilon_k(0)\omega_k\,\mathrm{d}x.\end{aligned}$$

For any postive integer r, let

$$(2.40)\qquad \varphi(x,t) = \sum_{k=1}^r \upsilon_k(t)\omega_k(x).$$

Replacing $\upsilon_k(t)\omega_k(x)$ by the above $\varphi(x,t)$ in (2.39), we have

$$\begin{aligned}(2.41)\qquad &\int_0^T \int_{\Omega_1\cup\Omega_2} \tilde{u}\varphi_{tt}(x,t) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_1\cup\Omega_2} \sum_{i,j=1}^N a_{ij} \frac{\partial \varphi(x,t)}{\partial x_i}\frac{\partial \tilde{u}}{\partial x_j} \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\Omega_1\cup\Omega_2} F\varphi(x,t) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T A\varphi(x,t)|_{(\tilde{\Gamma}_1\cup\tilde{\Gamma}_2)\times(0,T)}\,\mathrm{d}t \\ &\quad - \int_{\Omega_1\cup\Omega_2} \psi_0(x)\psi_t(x,0)\,\mathrm{d}x + \int_{\Omega_1\cup\Omega_2} \psi_1(x)\psi(x,0)\,\mathrm{d}x.\end{aligned}$$

Since the set composed of all functions like (2.40) is dense in the space U_1, (2.41) holds for any $\varphi \in U_1$ too.

Let

$$(2.42)\qquad u = \begin{cases} \tilde{u}, & x \in (Q_1 \cup\ Q_2), \\ C(t), & x \in \tilde{\Sigma}. \end{cases}$$

It is easy to verify that $u \in L^2(0,T;V_0)$ and satisfies (2.4). Thus we obtain that u is a weak solution to the problem (P_1).

(2) Proof of uniqueness: Assume that u_1 and u_2 are two weak solutions to (P_1) and let $u = u_1 - u_2$, then u satisfies

$$(2.43)\qquad \int_0^T \int_{\Omega_1\cup\Omega_2} u\varphi_{tt}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\, \varphi \in U_1.$$

For $0 < h < T$ and $0 < b < T - h$, let

(2.44)

$$u_h = \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} u(x,\tau)\,\mathrm{d}\tau, & 0 \leqslant t < T-h, \\ 0, & t \geqslant T-h; \end{cases} \qquad \hat{\varphi} = \begin{cases} 0, & t > b, \\ \displaystyle\int_b^t u_h(x,\tau)\,\mathrm{d}\tau, & 0 \leqslant t \leqslant b, \end{cases}$$

and

$$\text{(2.45)}\quad \eta = \begin{cases} 0, & t > b, \\ \displaystyle\int_b^t u(x,\tau)\,\mathrm{d}\tau, & 0 \leqslant t \leqslant b; \end{cases} \qquad \hat{\varphi}_{\bar{h}} = \begin{cases} \dfrac{1}{h}\displaystyle\int_{t-h}^t \hat{\varphi}(x,\tau)\,\mathrm{d}\tau, & h < t \leqslant T, \\ 0, & t \leqslant h. \end{cases}$$

It is easy to prove that $\hat{\varphi}_{\bar{h}} \in U_1$. Taking $\varphi = \hat{\varphi}_{\bar{h}}$ in (2.43), we have

$$\begin{aligned} \text{(2.46)}\quad I_1 &= \int_0^T \int_{\Omega_1 \cup \Omega_2} u(\hat{\varphi}_{\bar{h}})_{tt}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Omega_1 \cup \Omega_2} u \frac{(\hat{\varphi}_t(t) - \hat{\varphi}_t(t-h))}{h}\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^{T-h} \int_{\Omega_1 \cup \Omega_2} u\hat{\varphi}_t(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{h}\int_0^T \int_{\Omega_1 \cup \Omega_2} u\hat{\varphi}_t(x,t-h)\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^{T-h} \int_{\Omega_1 \cup \Omega_2} u\hat{\varphi}_t(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{h}\int_0^{T-h} \int_{\Omega_1 \cup \Omega_2} u(x,t+h)\hat{\varphi}_t(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^{T-h} \int_{\Omega_1 \cup \Omega_2} \frac{u(x,t+h) - u(x,t)}{h}\hat{\varphi}_t(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^{T-h} \int_{\Omega_1 \cup \Omega_2} (u_h)_t\hat{\varphi}_t\,\mathrm{d}x\,\mathrm{d}t = -\int_0^b \int_{\Omega_1 \cup \Omega_2} \hat{\varphi}_{tt}\hat{\varphi}_t\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} \text{(2.47)}\qquad I_2 &= \int_0^T \int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \hat{\varphi}_{\bar{h}}}{\partial x_i}\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^T \int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_j}\left(\frac{1}{h}\int_{t-h}^t \frac{\partial \hat{\varphi}(x,\tau)}{\partial x_i}\,\mathrm{d}\tau\right)\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{h}\int_\Omega \int_0^{T-h}\int_\tau^{\tau+h} \sum_{i,j=1}^N a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\frac{\partial \hat{\varphi}(x,\tau)}{\partial x_i}\,\mathrm{d}t\,\mathrm{d}\tau\,\mathrm{d}x \\ &= \int_0^b \int_\Omega \sum_{i,j=1}^N \left(a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\right)_h \frac{\partial \hat{\varphi}(x,\tau)}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Applying (2.46) and (2.47) to (2.43), we get

$$\text{(2.48)}\qquad \int_0^b \int_{\Omega_1 \cup \Omega_2} \hat{\varphi}_{tt}\hat{\varphi}_t\,\mathrm{d}x\,\mathrm{d}t = \int_0^b \int_\Omega \sum_{i,j=1}^N \left(a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\right)_h \frac{\partial \hat{\varphi}}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t.$$

Consequently,

$$\begin{aligned} \text{(2.49)}\qquad &\frac{1}{2}\|\hat{\varphi}_t(\cdot,b)\|^2_{L^2(\Omega_1 \cup \Omega_2)} - \frac{1}{2}\|\hat{\varphi}_t(\cdot,0)\|^2_{L^2(\Omega_1 \cup \Omega_2)} \\ &\qquad = \int_0^b \int_\Omega \sum_{i,j=1}^N \left(a_{ij}(x,t)\frac{\partial u(x,t)}{\partial x_j}\right)_h \frac{\partial \hat{\varphi}}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Taking $h \to 0$ in (2.49) and using Lemma 3.2 in [4], we have

$$\frac{1}{2}\|\hat{\varphi}_t(\cdot,b)\|^2_{L^2(\Omega_1\cup\Omega_2)} \leqslant \int_0^b\int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial\eta_t}{\partial x_j}\frac{\partial\eta}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t. \tag{2.50}$$

However,

$$\begin{aligned}\int_0^b\int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial\eta_t}{\partial x_j}\frac{\partial\eta}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t &\\ = \frac{1}{2}\int_0^b\int_\Omega \sum_{i,j=1}^N \frac{\partial}{\partial t}\Big(a_{ij}\frac{\partial\eta}{\partial x_j}\frac{\partial\eta}{\partial x_i}\Big)\,\mathrm{d}x\,\mathrm{d}t &\\ - \frac{1}{2}\int_0^b\int_\Omega \sum_{i,j=1}^N \frac{\partial a_{ij}(x,t)}{\partial t}\frac{\partial\eta}{\partial x_j}\frac{\partial\eta}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned} \tag{2.51}$$

Thus

$$\begin{aligned}\frac{1}{2}\|\hat{\varphi}_t(\cdot,b)\|^2_{L^2(\Omega_1\cup\Omega_2)} + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial\eta}{\partial x_j}\frac{\partial\eta}{\partial x_i}\,\mathrm{d}x\Big|_{t=0} &\\ \leqslant -\frac{1}{2}\int_0^b\int_\Omega \sum_{i,j=1}^N \frac{\partial a_{ij}(x,t)}{\partial t}\frac{\partial\eta}{\partial x_j}\frac{\partial\eta}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned} \tag{2.52}$$

By ($\tilde{\mathrm{H}}_1$), we get

$$\|\hat{\varphi}_t(\cdot,b)\|^2_{L^2(\Omega_1\cup\Omega_2)} + \int_\Omega \sum_{i=1}^N\Big(\frac{\partial\eta}{\partial x_i}\Big)^2\,\mathrm{d}x\Big|_{t=0} \leqslant C_{10}\int_0^b\int_\Omega \sum_{i=1}^N\Big(\frac{\partial\eta}{\partial x_i}\Big)^2\,\mathrm{d}x\,\mathrm{d}t, \tag{2.53}$$

where C_{10} is a positive constant only depending on $\|a_{ij}\|_{W^{1,\infty}(Q_T)}$, N and α.

According to the definition of η, we have

$$\begin{aligned}\|\hat{\varphi}_t(\cdot,b)\|^2_{L^2(\Omega_1\cup\Omega_2)} + \int_\Omega \sum_{i=1}^N\Big(\int_0^b \frac{\partial u}{\partial x_i}(x,\tau)\,\mathrm{d}\tau\Big)^2\,\mathrm{d}x &\\ \leqslant C_{10}\int_0^b\int_\Omega \sum_{i=1}^N\Big(\int_t^b \frac{\partial u}{\partial x_i}(x,\tau)\,\mathrm{d}\tau\Big)^2\,\mathrm{d}x\,\mathrm{d}t &\\ = C_{10}\int_0^b\int_\Omega \sum_{i=1}^N\Big(\int_0^b \frac{\partial u}{\partial x_i}(x,\tau)\,\mathrm{d}\tau - \int_0^t \frac{\partial u}{\partial x_i}(x,\tau)\,\mathrm{d}\tau\Big)^2\,\mathrm{d}x\,\mathrm{d}t &\\ \leqslant 2C_{10}b\int_\Omega \sum_{i=1}^N\Big(\int_0^b \frac{\partial u}{\partial x_i}(x,\tau)\Big)^2\,\mathrm{d}x &\\ + 2C_{10}\int_0^b\int_\Omega \sum_{i=1}^N\Big(\int_0^t \frac{\partial u}{\partial x_i}(x,\tau)\,\mathrm{d}\tau\Big)^2\,\mathrm{d}x\,\mathrm{d}t.&\end{aligned} \tag{2.54}$$

The chain of inequalities (2.54) yields

$$(2.55)\qquad (1-2C_{10}b)\int_\Omega \sum_{i=1}^N \Big(\int_0^b \frac{\partial u}{\partial x_i}(x,\tau)\,\mathrm{d}\tau\Big)^2\,\mathrm{d}x \\ \leqslant 2C_{10}\int_0^b\int_\Omega \sum_{i=1}^N \Big(\int_0^t \frac{\partial u}{\partial x_i}(x,\tau)\,\mathrm{d}\tau\Big)^2\,\mathrm{d}x\,\mathrm{d}t.$$

Set

$$(2.56)\qquad \tilde{E}(t) = \int_\Omega \sum_{i=1}^N \Big(\int_0^t \frac{\partial u}{\partial x_i}(x,\tau)\,\mathrm{d}\tau\Big)^2\,\mathrm{d}x.$$

Then (2.55) can be written as

$$(2.57)\qquad (1-2C_{10}b)\tilde{E}(b) \leqslant 2C_{10}\int_0^b \tilde{E}(t)\,\mathrm{d}t.$$

Choose a positive constant $b_0 = \frac{1}{4}C_{10}^{-1}$ such that

$$(2.58)\qquad \tilde{E}(b) \leqslant 4C_{10}\int_0^b \tilde{E}(t)\,\mathrm{d}t \quad \forall\, b\in[0,b_0].$$

By Gronwall's inequality, it is easy to get

$$(2.59)\qquad \tilde{E}(b) = 0 \quad \forall\, b\in[0,b_0].$$

Consequently, (2.54), (2.56), and (2.59) imply that

$$(2.60)\qquad u(x,b) = 0 \quad \text{a.e. } x\in(\Omega_1\cup\Omega_2),\ \forall\, b\in[0,b_0].$$

Applying the same argument on the intervals $[b_0, 2b_0]$, $[2b_0, 3b_0]\ldots$, we can thus prove that

$$(2.61)\qquad u(x,t) = 0 \quad \text{a.e. } (x,t)\in Q_1\cup Q_2.$$

Using (2.61) and the trace theorem, we can deduce that

$$(2.62)\qquad u(x,t) = 0 \quad \text{a.e. } (x,t)\in Q_T.$$

Thus the proof of uniqueness is completed. □

3. Hyperbolic boundary value problem with equivalued interface

In this section, we will study the existence and uniqueness of the weak solution to a hyperbolic boundary value problem with equivalued interface.

In order to study the limit behavior of solutions to the problem (P_1), we need to study the following equivalued interface problem (P). Here we give another division of Ω as shown in Fig. 2. Ω is composed of two non-overlapping subdomains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, and $\tilde{\Gamma}$ is the interface of $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$. Denote $\tilde{Q}_1 = \tilde{\Omega}_1 \times (0, T)$, $\tilde{Q}_2 = \tilde{\Omega}_2 \times (0, T)$, $\tilde{\Sigma}_0 = \tilde{\Gamma} \times (0, T)$.

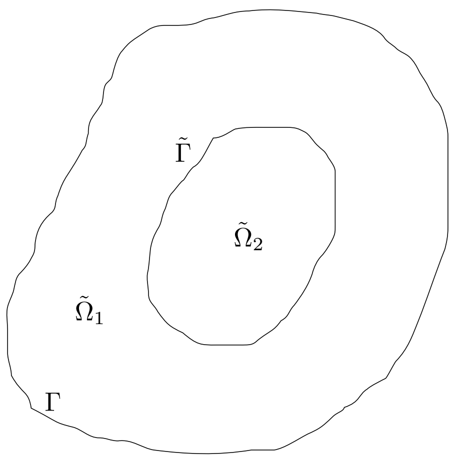

Figure 2.

In this section we will consider the following hyperbolic boundary value problem with equivalued interface:

$$
\text{(P)} \quad \begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \displaystyle\sum_{i,j=1}^{N} \dfrac{\partial}{\partial x_i}\Big(\tilde{a}_{ij}(x,t)\dfrac{\partial u}{\partial x_j}\Big) = F(x,t) & \text{in } \tilde{Q}_1 \cup \tilde{Q}_2, \\
u = 0 & \text{on } \Sigma, \\
u_+ = u_- = C(t) \ \ \text{(a function to be determined)} & \text{on } \tilde{\Sigma}_0, \\
\displaystyle\int_{\tilde{\Gamma}} \Big(\dfrac{\partial u}{\partial n_L}\Big)_+ \mathrm{d}s = \displaystyle\int_{\tilde{\Gamma}} \Big(\dfrac{\partial u}{\partial n_L}\Big)_- \mathrm{d}s + A(t) & \text{a.e. } t \in (0,T), \\
u(x,0) = \psi_0(x) & \text{in } \Omega, \\
\dfrac{\partial u}{\partial t} = \psi_1(x) & \text{in } \Omega,
\end{cases}
$$

where the subscripts + and − denote the values on both sides of $\tilde{\Gamma}$.

We give the following assumption:

(H_1) $\tilde{a}_{ij} \in W^{1,\infty}(Q_T)$, $\tilde{a}_{ij}(x,t) = \tilde{a}_{ji}(x,t)$, and there exist two positive constants α, β such that

$$\alpha|\xi|^2 \leqslant \sum_{i,j=1}^{N} \tilde{a}_{ij}(x,t)\xi_i\xi_j \leqslant \beta|\xi|^2, \quad \forall\, \xi = (\xi_1,\xi_2,\ldots,\xi_N) \in \mathbb{R}^N, \quad \text{a.e. } (x,t) \in Q_T. \tag{3.1}$$

Let

$$V_1 = \{v\colon\ v \in H_0^1(\Omega);\ v|_{\tilde{\Gamma}} = \text{constant}\}, \tag{3.2}$$

$$U = \left\{ \varphi \left| \begin{array}{l} \varphi \in L^2(0,T;H_0^1(\Omega)),\ \varphi_t,\varphi_{tt} \in L^2(Q_T),\ \varphi(x,T) = 0, \\ \varphi_t(x,T) = 0,\ \varphi|_{\tilde{\Sigma}_0} = C(t). \end{array} \right. \right\} \tag{3.3}$$

Definition 3.1. *If there exists a measurable function $u \in L^2(0,T;V_1)$ such that $\forall\, \varphi \in U_2$*

$$\begin{aligned} \int_0^T\!\!\int_\Omega u\varphi_{tt}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \sum_{i,j=1}^{N} \tilde{a}_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\!\!\int_\Omega F(x,t)\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \psi_0(x)\varphi_t(x,0)\,\mathrm{d}x \\ &\quad + \int_\Omega \psi_1(x)\varphi(x,0)\,\mathrm{d}x + \int_0^T A(t)\varphi|_{\tilde{\Sigma}_0}\,\mathrm{d}t, \end{aligned} \tag{3.4}$$

then we say that u is a weak solution to the problem (P).

Theorem 3.2. *Suppose that $F \in L^2(Q_T)$, $\psi_0 \in V_1$, $\psi_1 \in L^2(\Omega)$, $A \in H^1(0,T)$ and* (H_1) *hold, then there exists a unique weak solution $u \in L^2(0,T;V_1)$ to the problem* (P).

P r o o f. The proof of this theorem is similar to Theorem 2.2, we omit the details. □

4. Limit behavior of solutions to the problem (P_1)

In this section, we will study the limit behavior of solutions to the boundary value problem (P). More precisely, let $\varepsilon > 0$ be a small parameter and replace Ω_1, Ω_2, $\tilde{\Omega}$ by Ω_1^ε, Ω_2^ε, $\tilde{\Omega}^\varepsilon$, and the interfaces $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ by the interface $\tilde{\Gamma}_1^\varepsilon$ and $\tilde{\Gamma}_2^\varepsilon$, respectively, as shown in Fig. 3. Let $Q_1^\varepsilon = \Omega_1^\varepsilon \times (0,T)$, $Q_2^\varepsilon = \Omega_2^\varepsilon \times (0,T)$, $\tilde{\Sigma}_\varepsilon = (\tilde{\Gamma}_1^\varepsilon \cup \tilde{\Omega}^\varepsilon \cup \tilde{\Gamma}_2^\varepsilon) \times (0,T)$.

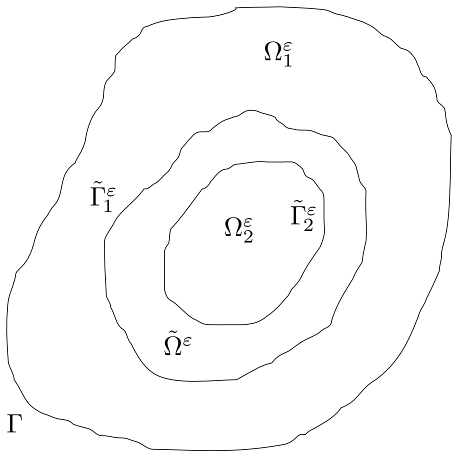

Figure 3.

Here we will discuss the following problem:

$$
(\mathrm{P}_\varepsilon)\qquad \begin{cases}
\dfrac{\partial^2 u_\varepsilon}{\partial t^2} - \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}^\varepsilon(x,t)\frac{\partial u_\varepsilon}{\partial x_j}\Big) = F(x,t) & \text{in } Q_1^\varepsilon \cup Q_2^\varepsilon,\\
u_\varepsilon = 0 & \text{on } \Sigma,\\
u_\varepsilon = C_\varepsilon(t) \text{ (a function to be determined)} & \text{on } \tilde{\Sigma}_\varepsilon,\\
\displaystyle\int_{\tilde{\Gamma}_1^\varepsilon} \frac{\partial u_\varepsilon}{\partial n_{L^\varepsilon}}\,\mathrm{d}s = \int_{\tilde{\Gamma}_2^\varepsilon} \frac{\partial u}{\partial n_{L^\varepsilon}}\,\mathrm{d}s + A(t) & \text{a.e. } t \in (0,T),\\
u_\varepsilon(x,0) = \psi_0^\varepsilon(x) & \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon,\\
\dfrac{\partial u_\varepsilon}{\partial t}(x,0) = \psi_1^\varepsilon(x) & \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon.
\end{cases}
$$

We give the following assumptions:

- (H_2) $\tilde{\Gamma} \subset \tilde{\Omega}^\varepsilon$, $\forall\,\varepsilon > 0$; $\tilde{\Omega}^\varepsilon$ shrinks to $\tilde{\Gamma}$, as $\varepsilon \to 0$.
- (H_3) Given any domain $\tilde{\Omega}'$ such that $\tilde{\Gamma} \subset \tilde{\Omega}' \subset \Omega$, then for any $\varepsilon > 0$ small enough, we have $\tilde{\Omega}^\varepsilon \subset \tilde{\Omega}'$.
- (H_4) $a_{ij}^\varepsilon \in W^{1,\infty}(Q_T)$, $a_{ij}^\varepsilon(x,t) = a_{ji}^\varepsilon(x,t)$, and there exist three positive constants K_1, α and β independent of ε such that

$$
\|a_{ij}^\varepsilon\|_{W^{1,\infty}(Q_T)} \leqslant K_1 \tag{4.1}
$$

and

$$\alpha|\xi|^2 \leqslant \sum_{i,j=1}^{N} a_{ij}^{\varepsilon}(x,t)\xi_i\xi_j \leqslant \beta|\xi|^2,\ \forall\,\xi=(\xi_1,\xi_2,\ldots,\xi_N)\in\mathbb{R}^N, \quad \text{a.e. } (x,t)\in Q_T. \tag{4.2}$$

(H_5) Given any domain $\tilde{\Omega}'$ such that $\tilde{\Gamma}\subset\tilde{\Omega}'\subset\Omega$, then as $\varepsilon\to 0$,

$$a_{ij}^{\varepsilon}(x,t)\to\tilde{a}_{ij}(x,t) \ \text{ strongly in } L^{\infty}((\Omega\setminus\tilde{\Omega}')\times(0,T)). \tag{4.3}$$

Set

$$V_0^{\varepsilon}=\{v\colon\ v\in H_0^1(\Omega),\ v|_{\tilde{\Omega}^{\varepsilon}\cup\tilde{\Gamma}_1^{\varepsilon}\cup\tilde{\Gamma}_2^{\varepsilon}}=\text{constant}\}, \tag{4.4}$$

$$U_{\varepsilon}=\left\{\varphi_{\varepsilon}\,\middle|\,\begin{matrix}\varphi_{\varepsilon}\in L^2(0,T;H_0^1(\Omega)),\ \varphi_{\varepsilon t},\varphi_{\varepsilon tt}\in L^2(Q_T),\ \varphi_{\varepsilon}(x,T)=0,\\ \varphi_{\varepsilon t}(x,T)=0,\ \varphi_{\varepsilon}|_{\tilde{\Sigma}_{\varepsilon}}=C_{\varepsilon}(t).\end{matrix}\right\} \tag{4.5}$$

Definition 4.1. *If there exists a measurable function $u_{\varepsilon}\in L^2(0,T;V_0^{\varepsilon})$ such that $\forall\,\varphi\in U_{\varepsilon}$,*

$$\begin{aligned}\int_0^T\int_{\Omega_1^{\varepsilon}\cup\Omega_2^{\varepsilon}} u_{\varepsilon}\varphi_{tt}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\Omega}\sum_{i,j=1}^{N}a_{ij}^{\varepsilon}\frac{\partial u_{\varepsilon}}{\partial x_j}\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\Omega_1^{\varepsilon}\cup\Omega_2^{\varepsilon}}F(x,t)\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_{\Omega_1^{\varepsilon}\cup\Omega_2^{\varepsilon}}\psi_0^{\varepsilon}(x)\varphi_t(x,0)\,\mathrm{d}x\\ &\quad+\int_{\Omega_1\cup\Omega_2}\psi_1^{\varepsilon}(x)\varphi(x,0)\,\mathrm{d}x+\int_0^T A(t)\varphi|_{\tilde{\Sigma}_{\varepsilon}}\,\mathrm{d}t,\end{aligned} \tag{4.6}$$

then we say that u_{ε} is a weak solution to the problem (P_{ε}).

Remark 4.2. For every fixed $\varepsilon>0$, if (4.2) and $\psi_0^{\varepsilon}\in V_0^{\varepsilon}$, $\psi_1^{\varepsilon}\in L^2(\Omega)$, $F\in L^2(Q_T)$, and $A\in H^1(0,T)$ hold, we can similarly prove that the problem (P_{ε}) admits a unique weak solution $u_{\varepsilon}\in L^2(0,T;V_0^{\varepsilon})$ in the sense of Definition 4.1.

Now we give the limit behavior of solutions to the problem (P_{ε}) as follows.

Theorem 4.3. *Suppose that* (H_1)–(H_5) *and $F\in L^2(Q_T)$, $A\in H^1(0,T)$ hold. If as $\varepsilon\to 0$,*

$$\psi_0^{\varepsilon}(x)\to\psi_0(x)\quad\textit{weakly in }\ V_1 \tag{4.7}$$

and

$$\psi_1^\varepsilon(x) \to \psi_1(x) \quad \textit{weakly in } L^2(\Omega), \tag{4.8}$$

then for every weak solution u_ε *to* (P_ε) *we have*

$$u_\varepsilon \to u \quad \textit{weakly in } L^2(0,T;V_1), \tag{4.9}$$

where u *is the weak solution to the problem* (P) *and the definition of* V_1 *can be seen in* (3.2).

Before we give the proof of Theorem 4.3, we need the following Lemma.

Lemma 4.4. *Under the hypotheses* (H_2) *and* (H_3), *for any given* $\varphi \in U$, *there exist* $\varphi_\varepsilon \in U_\varepsilon$ *such that as* $\varepsilon \to 0$,

$$\varphi_\varepsilon \to \varphi \quad \textit{strongly in } U, \tag{4.10}$$

where U *is as in* (3.3).

P r o o f. For convenience, we may assume that the origin is an interior point of $\tilde{\Omega}_2$ (see Fig. 2).

For fixed $\varepsilon > 0$ small enough, let $\Omega_2^\varepsilon = \{x(1-\varepsilon)\colon\ x \in \tilde{\Omega}_2\}$, $\tilde{\Omega}_1' = \{x/(1-\varepsilon)\colon\ x \in \tilde{\Omega}_2\}$, $\Omega_1^\varepsilon = \Omega \setminus \tilde{\Omega}_1'$, $\tilde{\Omega}^\varepsilon = \tilde{\Omega}_1' \setminus \Omega_2^\varepsilon$.

Defining $\Gamma^\varepsilon = \{x(1-\varepsilon)\colon\ x \in \Gamma\}$ and assuming $\tilde{\Gamma}_1^\varepsilon$, $\tilde{\Gamma}_2^\varepsilon$ are the interfaces of $\tilde{\Omega}^\varepsilon$ with Ω_1^ε and Ω_2^ε, we can write $\Gamma^\varepsilon \times (0,T) = \Sigma_\varepsilon$ (see Fig. 4).

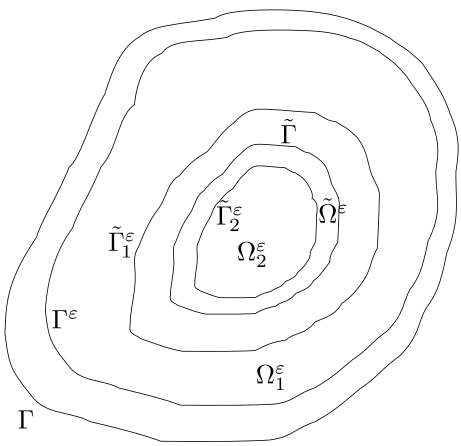

Figure 4.

Let

$$\varphi_\varepsilon^+ = \begin{cases} \Big(\varphi((1-\varepsilon)x,t) - \sup\limits_{\Gamma^\varepsilon\times(0,T)} \varphi^+(x,t)\Big)^+, & (x,t)\in\Omega_1^\varepsilon\times(0,T), \\ \Big(\varphi(x,t)|_{\tilde\Gamma\times(0,T)} - \sup\limits_{\Gamma^\varepsilon\times(0,T)} \varphi^+(x,t)\Big)^+, & (x,t)\in(\tilde\Gamma_1^\varepsilon\cup\tilde\Omega^\varepsilon\cup\tilde\Gamma_2^\varepsilon)\times(0,T), \\ \Big(\varphi\Big(\dfrac{x}{1-\varepsilon},t\Big) - \sup\limits_{\Gamma^\varepsilon\times(0,T)} \varphi^+(x,t)\Big)^+, & (x,t)\in\Omega_2^\varepsilon\times(0,T), \end{cases}$$

and

$$\varphi_\varepsilon^- = \begin{cases} \Big(\varphi((1-\varepsilon)x,t) - \inf\limits_{\Gamma^\varepsilon\times(0,T)} \varphi^-(x,t)\Big)^-, & (x,t)\in\Omega_1^\varepsilon\times(0,T), \\ \Big(\varphi(x,t)|_{\tilde\Gamma\times(0,T)} - \inf\limits_{\Gamma^\varepsilon\times(0,T)} \varphi^-(x,t)\Big)^-, & (x,t)\in(\tilde\Gamma_1^\varepsilon\cup\tilde\Omega^\varepsilon\cup\tilde\Gamma_2^\varepsilon)\times(0,T), \\ \Big(\varphi\Big(\dfrac{x}{1-\varepsilon},t\Big) - \inf\limits_{\Gamma^\varepsilon\times(0,T)} \varphi^-(x,t)\Big)^-, & (x,t)\in\Omega_2^\varepsilon\times(0,T). \end{cases}$$

Obviously $\varphi_\varepsilon^+\in U_\varepsilon$, $\varphi_\varepsilon^-\in U_\varepsilon$, so we have $\varphi_\varepsilon\in U_\varepsilon$. It is easy to prove that φ_ε^+ and φ_ε^- strongly converge to φ^+ and φ^- in U respectively. We omit the details. □

P r o o f of Theorem 4.3. For any given $\varepsilon>0$, the problem (P_ε) admits a unique weak solution $u_\varepsilon\in L^2(0,T;V_0^\varepsilon)$ by Theorem 2.2 and (4.6) holds. Furthermore, checking the proof of Theorem 2.2, we can deduce that $u_\varepsilon'\in C([0,T];L^2(\Omega_1^\varepsilon\cup\Omega_2^\varepsilon))$. Thus (4.6) can be written as

$$\tag{4.11} \begin{aligned} -\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u_{\varepsilon t}\varphi_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} F(x,t)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_1\cup\Omega_2}\psi_1^\varepsilon(x)\varphi(x,0)\,\mathrm{d}x \\ + \int_0^T A(t)\varphi|_{\tilde\Sigma_\varepsilon}\,\mathrm{d}t, \quad \forall\,\varphi\in U_\varepsilon. \end{aligned}$$

For $0<\tau\leqslant T$ and $0<h<\tau$, let

$$\tag{4.12} \bar\varphi_{\varepsilon\bar h} = \begin{cases} \dfrac{1}{h}\displaystyle\int_{t-h}^t u_{\varepsilon h}'(x,\sigma)\,\mathrm{d}\sigma, & h<t\leqslant T, \\ 0, & t\leqslant h, \end{cases}$$

where

$$\tag{4.13} u_{\varepsilon h} = \begin{cases} \dfrac{1}{h}\displaystyle\int_\sigma^{\sigma+h} u_\varepsilon(x,\varsigma)\,\mathrm{d}\varsigma, & 0\leqslant\sigma<\tau-h, \\ 0, & \sigma\geqslant\tau-h, \end{cases}$$

and $u_{\varepsilon h}'=\partial u_{\varepsilon h}/\partial t$. □

Taking $\varphi = \bar{\varphi}_{\varepsilon\bar{h}}$ in (4.11), we have

$$(4.14)\quad -\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u_{\varepsilon t}\bar{\varphi}_{\varepsilon\bar{h}t}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\bar{\varphi}_{\varepsilon\bar{h}}}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} F(x,t)\bar{\varphi}_{\varepsilon\bar{h}}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_1\cup\Omega_2}\psi_1^\varepsilon(x)\bar{\varphi}_{\varepsilon\bar{h}}(x,0)\,\mathrm{d}x$$
$$+\int_0^T A(t)\bar{\varphi}_{\varepsilon\bar{h}}|_{\tilde{\Sigma}_\varepsilon}\,\mathrm{d}t.$$

Similarly to (2.46) and (2.47), it is easy to prove that

$$(4.15)\quad -\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u_{\varepsilon t}\bar{\varphi}_{\varepsilon\bar{h}t}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{1}{2}\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}\|u'_{\varepsilon h}\|^2_{L^2(\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)}\,\mathrm{d}t + \frac{1}{h}\int_0^h\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u'_\varepsilon(t)u'_{\varepsilon h}\,\mathrm{d}x\,\mathrm{d}t,$$

$$(4.16)\quad \int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\bar{\varphi}_{\varepsilon\bar{h}}}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^{\tau-h}\int_\Omega\sum_{i,j=1}^N \frac{\partial u'_{\varepsilon h}}{\partial x_i}\Big(a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\Big)_h\,\mathrm{d}x\,\mathrm{d}t$$
$$-\frac{1}{h}\int_\Omega\mathrm{d}x\int_0^h\frac{\partial u'_{\varepsilon h}}{\partial x_i}\,\mathrm{d}\sigma\int_\sigma^h\sum_{i,j=1}^N a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\,\mathrm{d}t$$
$$= \int_\Omega\sum_{i,j=1}^N\Big(a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\Big)_h\frac{\partial u_{\varepsilon h}}{\partial x_i}\,\mathrm{d}x\Big|_0^{\tau-h}$$
$$-\int_0^{\tau-h}\int_\Omega\sum_{i,j=1}^N\frac{\partial u_{\varepsilon h}(t)}{\partial x_i}(a_{ij}^\varepsilon)'_h\frac{\partial u_\varepsilon(t+h)}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^{\tau-h}\int_\Omega\sum_{i,j=1}^N\frac{\partial u'_{\varepsilon h}}{\partial x_i}a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t$$
$$-\frac{1}{h}\int_\Omega\mathrm{d}x\int_0^h\frac{\partial u'_{\varepsilon h}}{\partial x_i}\,\mathrm{d}\sigma\int_\sigma^h\sum_{i,j=1}^N a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\,\mathrm{d}t,$$

$$(4.17)\quad \int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} F(x,t)\bar{\varphi}_{\varepsilon\bar{h}}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^{\tau-h}\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u'_{\varepsilon h}F_h\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{h}\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon}\mathrm{d}x\int_0^h u'_{\varepsilon h}\,\mathrm{d}\sigma\int_\sigma^h F\,\mathrm{d}t,$$

$$\int_{\Omega_1\cup\Omega_2}\psi_1^\varepsilon(x)\bar{\varphi}_{\varepsilon\bar{h}}(x,0)\,\mathrm{d}x=0, \tag{4.18}$$

$$\begin{aligned}\int_0^T A(t)\hat{\varphi}_h|_{\tilde{\Sigma}_\varepsilon}\,\mathrm{d}t &= \frac{1}{|\tilde{\Gamma}|}\int_0^{\tau-h}\int_{\tilde{\Gamma}}u'_{\varepsilon h}A_h\,\mathrm{d}s\,\mathrm{d}t-\frac{1}{h|\tilde{\Gamma}|}\int_{\tilde{\Gamma}}\mathrm{d}s\int_0^h u'_{\varepsilon h}\,\mathrm{d}\sigma\int_\sigma^h A\,\mathrm{d}t\\ &= \frac{1}{|\tilde{\Gamma}|}\int_{\tilde{\Gamma}}u_{\varepsilon h}A_h\,\mathrm{d}s|_0^{\tau-h}-\frac{1}{|\tilde{\Gamma}|}\int_0^{\tau-h}\int_{\tilde{\Gamma}}u_{\varepsilon h}A'_h\,\mathrm{d}s\,\mathrm{d}t\\ &\quad+\frac{1}{h|\tilde{\Gamma}|}\int_{\tilde{\Gamma}}u_{\varepsilon h}(0)\,\mathrm{d}s\int_0^h A(t)\,\mathrm{d}t-\frac{1}{h|\tilde{\Gamma}|}\int_{\tilde{\Gamma}}\int_0^h u_{\varepsilon h}A\,\mathrm{d}t.\end{aligned} \tag{4.19}$$

Taking $h\to 0$ in (4.14)–(4.19), we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^\tau\frac{\mathrm{d}}{\mathrm{d}t}\|u'_\varepsilon\|^2_{L^2(\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)}\,\mathrm{d}t+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N a_{ij}^\varepsilon\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_i}\,\mathrm{d}x\Big|_0^\tau\\ &\quad-\frac{1}{2}\int_0^\tau\int_\Omega\sum_{i,j=1}^N(a_{ij}^\varepsilon)'\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^\tau\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon}u'_\varepsilon F\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{|\tilde{\Gamma}|}\int_{\tilde{\Gamma}}u_\varepsilon(\tau)A(\tau)\,\mathrm{d}s-\frac{1}{|\tilde{\Gamma}|}\int_0^\tau\int_{\tilde{\Gamma}}u_\varepsilon A'\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{4.20}$$

Thus, by (H_4), (4.7)–(4.8), the Sobolev imbedding theorem (see [12]), the Young inequality, and the trace theorem, we obtain

$$\begin{aligned}&\|u'_\varepsilon(\tau)\|^2_{L^2(\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)}+\alpha\|Du_\varepsilon(\tau)\|^2_{L^2(\Omega)}\\ &\qquad\leqslant\|F\|^2_{L^2(Q_T)}+\beta\|\psi_0^\varepsilon\|^2_{V_1}+\|\psi_1^\varepsilon\|^2_{L^2(\Omega)}+C(\delta)\|A\|^2_{H^1(0,T)}\\ &\qquad\quad+(N^2K_1+C_{11})\int_0^\tau\|Du_\varepsilon(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\ &\qquad\quad+\int_0^\tau\|u'_\varepsilon(t)\|^2_{L^2(\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)}\,\mathrm{d}t+\delta\|Du_\varepsilon(\tau)\|^2_{L^2(\Omega)},\end{aligned} \tag{4.21}$$

where C_{11} is a positive constant independent of ε.

Taking $\delta=\alpha/2$ in (4.21) and using Gronwall's inequality, we get

$$\begin{aligned}&\|u'_\varepsilon(\tau)\|^2_{L^2(\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)}+\|Du_\varepsilon(\tau)\|^2_{L^2(\Omega)}\\ &\quad\leqslant C_{12}(\|F\|^2_{L^2(Q_T)}+\|\psi_0^\varepsilon\|^2_{V_1}+\|\psi_1^\varepsilon\|^2_{L^2(\Omega)}+\|A\|^2_{H^1(0,T)}),\quad\forall\tau\in(0,T),\end{aligned} \tag{4.22}$$

where C_{12} is a positive constant independent of ε.

Thus, from (4.7)–(4.8) and (4.22) it follows that

$$\|u_\varepsilon\|_{L^2(0,T;V_1)} \leqslant C_{13}, \tag{4.23}$$

where C_{13} is a positive constant independent of ε.

Hence, there exists a subsequence of $\{u_\varepsilon\}$ (still denoted by $\{u_\varepsilon\}$) and a measurable function u such that as $\varepsilon \to 0$,

$$u_\varepsilon \to u \quad \text{weakly in } \ L^2(0,T;V_1). \tag{4.24}$$

By Lemma 4.4, for any given $\varphi \in U$, there exists $\varphi_\varepsilon \in U_\varepsilon$ such that

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } \ U. \tag{4.25}$$

For a fixed $\varepsilon_0 > 0$ and for any $0 < \varepsilon < \varepsilon_0$, we have $\tilde{\Omega}^\varepsilon \subset \tilde{\Omega}^{\varepsilon_0}$ and $\varphi_{\varepsilon_0} \in U_\varepsilon$, so taking $\varphi = \varphi_{\varepsilon_0}$ in (4.6), we have

$$\begin{aligned}\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u_\varepsilon \varphi_{\varepsilon_0 tt}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial \varphi_{\varepsilon_0}}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} F(x,t)\varphi_{\varepsilon_0}\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} \psi_0^\varepsilon(x)\varphi_{\varepsilon_0 t}(x,0)\,\mathrm{d}x \\ + \int_{\Omega_1\cup\Omega_2} \psi_1^\varepsilon(x)\varphi_{\varepsilon_0}(x,0)\,\mathrm{d}x + \int_0^T A(t)\varphi_{\varepsilon_0}|_{\tilde{\Sigma}_{\varepsilon_0}}\,\mathrm{d}t.\end{aligned} \tag{4.26}$$

By (4.24), (4.7)–(4.8), and the absolute continuity of the Lebesgue integral, as $\varepsilon \to 0$, it is easy to prove that

$$\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} u_\varepsilon\varphi_{\varepsilon_0 tt}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega u\varphi_{\varepsilon_0 tt}\,\mathrm{d}x\,\mathrm{d}t, \tag{4.27}$$

$$\int_0^T\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} F(x,t)\varphi_{\varepsilon_0}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega F\varphi_{\varepsilon_0}\,\mathrm{d}x\,\mathrm{d}t, \tag{4.28}$$

$$\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} \psi_0^\varepsilon(x)\varphi_{\varepsilon_0 t}(x,0)\,\mathrm{d}x \to \int_\Omega \psi_0(x)\varphi_{\varepsilon_0 t}(x,0)\,\mathrm{d}x, \tag{4.29}$$

$$\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} \psi_1^\varepsilon(x)\varphi_{\varepsilon_0}(x,0)\,\mathrm{d}x \to \int_\Omega \psi_1(x)\varphi_{\varepsilon_0}(x,0)\,\mathrm{d}x. \tag{4.30}$$

We now prove that

$$\int_0^T\int_\Omega \sum_{i,j=1}^N a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi_{\varepsilon_0}}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \sum_{i,j=1}^N \tilde{a}_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_{\varepsilon_0}}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.31}$$

For any given $\tilde{\Omega}'$ such that $\tilde{\Gamma} \subset \tilde{\Omega}' \subset \Omega$, we have

$$
\begin{aligned}
(4.32)\qquad & \int_0^T \int_\Omega a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \varphi_{\varepsilon_0}}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \tilde{a}_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi_{\varepsilon_0}}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\tilde{\Omega}'} a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \varphi_{\varepsilon_0}}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\tilde{\Omega}} \tilde{a}_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi_{\varepsilon_0}}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega \setminus \tilde{\Omega}'} (a_{ij}^\varepsilon - \tilde{a}_{ij}) \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \varphi_{\varepsilon_0}}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega \setminus \tilde{\Omega}'} \tilde{a}_{ij} \frac{\partial (u_\varepsilon - u)}{\partial x_j} \frac{\partial \varphi_{\varepsilon_0}}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t \\
&= \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4.
\end{aligned}
$$

For any given $\delta > 0$, by (H_1), (H_4), (4.24), and the absolute continuity of the Lebesgue integral, we can take $\tilde{\Omega}'$ so small that

$$
(4.33)\qquad |\tilde{I}_1| + |\tilde{I}_2| \leqslant \frac{1}{2}\delta.
$$

Once such $\tilde{\Omega}'$ is chosen, by (H_5) and (4.24)–(4.25), there exists $0 < \varepsilon_1 < \varepsilon_0$ such that for any ε with $0 < \varepsilon < \varepsilon_1$,

$$
(4.34)\qquad |\tilde{I}_3| + |\tilde{I}_4| \leqslant \frac{1}{2}\delta.
$$

From (4.32)–(4.34), the validity of (4.31) follows.

Letting $\varepsilon \to 0$ in (4.26), (4.27)–(4.31) yields

$$
\begin{aligned}
(4.35)\qquad & \int_0^T \int_\Omega u \varphi_{\varepsilon_0 tt} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \sum_{i,j=1}^N \tilde{a}_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi_{\varepsilon_0}}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega F(x,t) \varphi_{\varepsilon_0} \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \psi_0(x) \varphi_{\varepsilon_0 t}(x,0) \,\mathrm{d}x \\
&\quad + \int_\Omega \psi_1(x) \varphi_{\varepsilon_0}(x,0) \,\mathrm{d}x + \int_0^T A(t) \varphi_{\varepsilon_0}|_{\tilde{\Sigma}_{\varepsilon_0}} \,\mathrm{d}t.
\end{aligned}
$$

The convergence (4.25) and Lemma 1.2 in [12] imply that as $\varepsilon_0 \to 0$

$$
(4.36)\qquad \varphi_{\varepsilon_0} \to \varphi \quad \text{strongly in } C([0,T]; L^2(\Omega))
$$

and

$$
(4.37)\qquad \varphi'_{\varepsilon_0} \to \varphi' \quad \text{strongly in } C([0,T]; L^2(\Omega)).
$$

Hence as $\varepsilon_0 \to 0$, we also have

$$\varphi_{\varepsilon_0}(x,0) \to \varphi(x,0) \quad \text{strongly in } L^2(\Omega) \tag{4.38}$$

and

$$\varphi'_{\varepsilon_0}(x,0) \to \varphi'(x,0) \quad \text{strongly in } L^2(\Omega). \tag{4.39}$$

By (4.25) and the trace theorem, we get

$$\varphi_{\varepsilon_0} \to \varphi \quad \text{strongly in } L^2(\tilde{\Sigma}_0), \text{ as } \varepsilon_0 \to 0. \tag{4.40}$$

Hence

$$\varphi_{\varepsilon_0}|_{\tilde{\Sigma}_{\varepsilon_0}} = \varphi_{\varepsilon_0}|_{\tilde{\Sigma}_0} \to \varphi|_{\tilde{\Sigma}_0} \quad \text{strongly in } L^2(0,T). \tag{4.41}$$

Letting $\varepsilon_0 \to 0$ in (4.35), by (4.25), (4.38)–(4.39), and (4.41) we deduce that u satisfies (3.4). By the uniqueness of the weak solution to the problem (P), (4.24) holds for the whole sequence $\{u_\varepsilon\}$. This completes the proof of Theorem 4.3. □

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Authors' address: *F. Li* (corresponding author), *W. Sun*, Department of Applied Mathematics, Dalian University of Technology, Dalian, 116024, P. R. China, e-mail: `fqli@dlut.edu.cn`, `xiesunweiwei@126.com`.