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GLOBAL SUPERCONVERGENCE OF FINITE ELEMENT
METHODS FOR PARABOLIC INVERSE PROBLEMS*

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Dedicated to Ivan Hlaváček on the occasion of his 75th birthday

Abstract. In this article we transform a large class of parabolic inverse problems into a nonclassical parabolic equation whose coefficients consist of trace type functionals of the solution and its derivatives subject to some initial and boundary conditions. For this nonclassical problem, we study finite element methods and present an immediate analysis for global superconvergence for these problems, on basis of which we obtain a posteriori error estimators.

Keywords: inverse problem, global superconvergence, finite element method

MSC 2010: 35R30, 65M32, 76R50, 65M06

1. INTRODUCTION

In the present work we study the numerical solutions of some inverse problems, i.e., the determination of some unknown function $p(t)$ in a parabolic equation. The classical example is that one needs to find the temperature distribution $u(x, t)$ as well as the thermal coefficient $a(t)$ which simultaneously satisfy

$$(1.1) \quad u_t = a(t)u_{xx}, \quad 0 < x < 1, \quad 0 < t < T,$$

$$(1.2) \quad u(x, 0) = u_0(x), \quad 0 \leq x \leq 1,$$

$$(1.3) \quad u(0, t) = f_1(t), \quad 0 \leq t \leq T,$$

$$(1.4) \quad u(1, t) = f_2(t), \quad 0 \leq t \leq T,$$

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and the over-specification

$$(1.5) \quad a(t)u_x(0, t) = g(t), \quad 0 \leq t \leq T.$$

The well-posedness of the problem is studied in [9]. Moreover, in [8] the numerical solution by the finite difference method is also discussed. However, we investigate the problem (1.1)–(1.5) from a rather different point of view. If we solve equation (1.5) for $a(t)$ and substitute it into the equation (1.1), then the equation (1.1) becomes

$$u_t = \frac{g(t)}{u_x(0, t)} u_{xx}$$

which is nonclassical since a functional of the derivative of the solution with respect to the variable x is involved in the equation. This example motivates us to consider the following general nonclassical problem:

$$(1.6) \quad u_t = a(x, t, u, u_x, u_x(x_0, t))u_{xx} + b(x, t, u, u_x, u_x(x_0, t)),$$

$$(1.7) \quad u(x, 0) = u_0(x), \quad 0 \leq x \leq 1,$$

$$(1.8) \quad u(0, t) = f_1(t), \quad 0 \leq t \leq T,$$

$$(1.9) \quad u(1, t) = f_2(t), \quad 0 \leq t \leq T,$$

where $x_0 \in [0, 1]$ is a fixed point.

The problem (1.6)–(1.9) represents a large class of parabolic inverse problems in which an unknown function $p(t)$ as well as the solution itself are to be determined. As another example of the motivation of our study, let us consider the problem of finding $(u(x, t), a(t))$ such that

$$(1.10) \quad u_t = u_{xx} + a(t)u_x, \quad 0 < x < 1, \quad 0 < t < T,$$

subject to the initial-boundary conditions (1.2)–(1.4) and an additional condition

$$\int_0^1 u(x, t) dx = g(t), \quad 0 \leq t \leq T.$$

If one differentiates with respect to the variable t in the above equation and uses the resulting equation, then one obtains

$$a(t) = \frac{g'(t) - [u_x(1, t) - u_x(0, t)]}{f_2(t) - f_1(t)}$$

provided that $f_1(t) \neq f_2(t)$. Consequently, the above inverse problem is of the form of our problem (1.6)–(1.9) with the specific equation

$$u_t = u_{xx} + \frac{g'(t) - [u_x(1, t) - u_x(0, t)]}{f_2(t) - f_1(t)} u_x.$$

When an unknown function $a(t)$ appears as the coefficient of a lower order term in a parabolic equation, it is not difficult to see that one is able to transform the problem into our nonclassical form under reasonable conditions. The reader can find many more such examples in [5] and [2].

Recently, the study of parabolic inverse problems has received much attention. For example, finite element methods and finite difference methods have been investigated in [10], [11], and [19], respectively. Here and throughout this paper “parabolic inverse problem” means that an unknown coefficient that is assumed to be a function of only the time variable and the solution of a parabolic equation subject to suitable initial-boundary conditions is to be determined. For the existence and uniqueness of solutions of such problems, the reader can refer to [1], [7], [15], [4], [16], [17], [8], [9], [3], [18], [14] etc.

However, the theory of numerical solution of these problems is far from satisfactory. For the nonclassical problem (1.6)–(1.9), the paper [5] established the global solvability as well as the continuous dependence of the solution upon the data. In the present paper we continue the work of [5], [6] and investigate the numerical calculation for the solution of our problem. We will introduce a new function and transform the problem into a variational form. The classical Galerkin procedure is then applied to our problem, and the global superconvergence of finite element methods is derived in this article. In addition, as a by-product of the superconvergence, a posteriori error estimates are obtained.

2. SUPERCLOSE ESTIMATES

For the sake of simplicity, we assume that $f_1(t) = f_2(t) = 0$ and $x_0 = 0$ in problem (1.6)–(1.9). We shall use the following notation throughout the paper.

Set $I = (0, 1)$. For $u, v \in L^2(I)$ we define

$$\begin{aligned} \langle u, v \rangle &= \int_0^1 u(x)v(x) \, dx, \\ \|u\|_0^2 &= \langle u, u \rangle = \int_0^1 u^2(x) \, dx, \\ \tilde{H}_0^1(I) &= \left\{ u \in H^1(I), \int_0^1 u(x) \, dx = 0 \right\}, \end{aligned}$$

and

$$\|u\|_1 = \|u\|_{H_0^1(I)} = \|u_x\|_0.$$

Evidently, $\tilde{H}_0^1(I)$ is a Banach space with the $H_0^1(I)$ -norm. To transform the problem (1.6)–(1.9) into a variational form, we introduce a new function

$$v(x, t) = u_x(x, t), \quad (x, t) \in \overline{Q}_T,$$

where $u(x, t)$ is the solution of problem (1.6)–(1.9) and $\overline{Q}_T = I \times [0, T]$. Then $v(x, t) \in \tilde{H}_0^1(I)$ and

$$u(x, t) = \int_0^x v(y, t) \, dy, \quad (x, t) \in \overline{Q}_T.$$

Furthermore, using (1.6) we see that $v(x, t)$ satisfies

$$(2.1) \quad \left(\int_0^x v(y, t) \, dy \right)_t = av_x + b \quad \text{in } Q_T,$$

and

$$(2.2) \quad u_t(0, t) = u_t(1, t) = 0, \quad 0 \leq t \leq T.$$

Let $w \in \tilde{H}_0^1(I)$ be arbitrary. We multiply the equation (2.1) by $w_x(x)$ and integrate it over $(0, 1)$ with respect to x . Integrating by parts the term on the left-hand side of the equation (2.1) and using the boundary condition (2.2), we obtain

$$(2.3) \quad \langle v_t, w \rangle + \langle av_x + b, w_x \rangle = 0, \quad t \in (0, T],$$

$$(2.4) \quad \langle v, w \rangle = \langle u'_0, w \rangle, \quad t = 0,$$

where

$$(2.5) \quad a = a \left(x, t, \int_0^x v(y, t) \, dy, v(x, t), v(0, t) \right), \quad (x, t) \in \overline{Q}_T,$$

$$(2.6) \quad b = b \left(x, t, \int_0^x v(y, t) \, dy, v(x, t), v(0, t) \right), \quad (x, t) \in \overline{Q}_T.$$

It is easy to see that if $v(x, t)$ is a variational solution of (2.3)–(2.6) and sufficiently smooth, then $u(x, t) = \int_0^x v(y, t) \, dy$ is a solution of the problem (1.6)–(1.9). Hence if one can obtain an approximate solution for the variational problem (2.3)–(2.6), one also has an approximate solution for the original problem (1.6)–(1.9). Therefore, we shall concentrate on the approximation of the solution of the variational problem (2.3)–(2.6) in the rest of this paper.

For convenience, the following conditions are assumed in the sequel:

- (H1) The functions $a(x, t, u, p, q)$ and $b(x, t, u, p, q)$ are twice differentiable with respect to all their arguments.
- (H2) (i) $a(x, t, u, p, q) \geq a_0 > 0$;
(ii) $|\nabla_{u,p,q}a(\dots)| + |\nabla_{u,p,q}b(\dots)| \leq A_0$ for $(x, t, u, p, q) \in \overline{Q}_T \times \mathbb{R}^3$, where $\nabla_{u,p,q}a = (\partial a/\partial u, \partial a/\partial p, \partial a/\partial q)$.

Next, we will approximate the problem (2.3)–(2.4) by the finite element method. To this purpose, we let $T_h: 0 \leq x_0 < x_1 < \dots < x_N \leq 1$ be a division of the interval $[0, 1]$, and $e_i := [x_i, x_{i+1}]$ ($i = 0, 1, \dots, N-1$) represents the $(i+1)$ st element.

Furthermore, let $V_h \subset \tilde{H}_0^1(I)$ consist of piecewise linear polynomials with respect to the partition T_h . Thus, the semi-discrete finite element scheme of (2.3)–(2.4) is defined as follows: Find $U(x, t) \in V_h$ such that

$$(2.7) \quad \langle U_t, W \rangle + \langle AU_x + B, W_x \rangle = 0, \quad t \in (0, T], \quad W \in V_h,$$

$$(2.8) \quad U(x, 0) = i_h u'_0(x),$$

where i_h is the piecewise linear interpolation operator, and

$$(2.9) \quad A = a\left(x, t, \int_0^x U(y, t) dy, U(x, t), U(0, t)\right), \quad (x, t) \in \overline{Q}_T,$$

$$(2.10) \quad B = b\left(x, t, \int_0^x U(y, t) dy, U(x, t), U(0, t)\right), \quad (x, t) \in \overline{Q}_T.$$

From (2.3) and (2.7) we can derive the following error equation:

$$(2.11) \quad \langle v_t - U_t, W \rangle + \langle av_x - AU_x + b - B, W_x \rangle = 0, \quad W \in V_h.$$

First of all, from [12] we recall the following lemma.

Lemma 2.1. *For any $W \in V_h$ we have*

$$|\langle A(v - i_h v)_x, W_x \rangle| \leq Ch^2 \|v\|_2 \|W\|_1,$$

where A is given by (2.9).

Theorem 2.2. Under the conditions (H1) and (H2) we have the supercloseness estimate

$$\|U - i_h v\|_0 + \|U - i_h v\|_{L^2(0,T;H^1(I))} \leq Ch^2.$$

Proof. Let

$$\theta(x, t) = U(x, t) - i_h v(x, t).$$

Then from (2.11) we find for any $W \in V_h$ that

$$\begin{aligned} & \langle \theta_t, W \rangle + \langle A\theta_x + B, W_x \rangle \\ &= \langle (a - A)v_x + b, W_x \rangle + \langle v_t - i_h v_t, W \rangle + \langle A(v - i_h v)_x, W_x \rangle \end{aligned}$$

or

$$\begin{aligned} (2.12) \quad & \langle \theta_t, W \rangle + \langle A\theta_x, W_x \rangle \\ &= \langle (a - A)v_x + b - B, W_x \rangle + \langle v_t - i_h v_t, W \rangle + \langle A(v - i_h v)_x, W_x \rangle \\ &:= I + II, \quad W \in V_h. \end{aligned}$$

From Lemma 2.1 we obtain

$$\begin{aligned} (2.13) \quad & |II| = |\langle v_t - i_h v_t, W \rangle + \langle A(v - i_h v)_x, W_x \rangle| \\ & \leq Ch^2(\|v_t\|_2 + \|v\|_2)|W|_1. \end{aligned}$$

Since

$$\begin{aligned} & (a - A)v_x + b - B \\ &= \left[a \left(x, t, \int_0^t v(y, t) dy, v(x, t), v(0, t) \right) \right. \\ & \quad \left. - a \left(x, t, \int_0^t U(y, t) dy, U(x, t), U(0, t) \right) \right] v_x \\ & \quad + b \left(x, t, \int_0^t v(y, t) dy, v(x, t), v(0, t) \right) \\ & \quad - b \left(x, t, \int_0^t U(y, t) dy, U(x, t), U(0, t) \right) \\ &= b_1 Z + c_1 Z(0, t) + d_1 \int_0^x Z(y, t) dy, \end{aligned}$$

where

$$\begin{aligned}
b_1 &= \left[\int_0^1 a_p(x, t, \alpha(\tau), \beta(\tau), \gamma(\tau)) \, d\tau \right] v_x + \int_0^1 b_p(x, t, \alpha(\tau), \beta(\tau), \gamma(\tau)) \, d\tau, \\
c_1 &= \left[\int_0^1 a_q(x, t, \alpha(\tau), \beta(\tau), \gamma(\tau)) \, d\tau \right] v_x + \int_0^1 b_q(x, t, \alpha(\tau), \beta(\tau), \gamma(\tau)) \, d\tau, \\
d_1 &= \left[\int_0^1 a_u(x, t, \alpha(\tau), \beta(\tau), \gamma(\tau)) \, d\tau \right] v_x + \int_0^1 b_u(x, t, \alpha(\tau), \beta(\tau), \gamma(\tau)) \, d\tau, \\
\alpha(\tau) &= \tau \left(\int_0^\tau U(y, t) \, dy \right) + (1 - \tau) \left(\int_0^\tau v(y, t) \, dy \right), \\
\beta(\tau) &= \tau U + (1 - \tau)v, \\
\gamma(\tau) &= \tau U(0, t) + (1 - \tau)v(0, t), \\
Z(x, t) &= v(x, t) - U(x, t),
\end{aligned}$$

we have

$$\begin{aligned}
|I| &= |\langle (a - A)v_x + b - B, W_x \rangle| \\
&= \left| \left\langle b_1 Z + c_1 Z(0, t) + d_1 \int_0^x Z(y, t) \, dy, W_x \right\rangle \right| \\
&\leq \left| \left\langle b_1(v - i_h v) + c_1(v - i_h v)(0, t) + d_1 \int_0^x (v - i_h v)(y, t) \, dy, W_x \right\rangle \right| \\
&\quad + \left| \left\langle b_1 \theta + c_1 \theta(0, t) + d_1 \int_0^x \theta(y, t) \, dy, W_x \right\rangle \right|,
\end{aligned}$$

which, together with Lemma 2.1, leads to

$$\begin{aligned}
(2.14) \quad |I| &\leq Ch^2 \|v\|_2 |W|_1 + \left| \left\langle b_1 \theta + c_1 \theta(0, t) + d_1 \int_0^x \theta(y, t) \, dy, W_x \right\rangle \right| \\
&\leq Ch^2 \|v\|_2 |W|_1 + C(\|\theta\|_0 + |\theta(0, t)|) |W|_1.
\end{aligned}$$

Thus, from (2.12), (2.13) and (2.14) we have

$$(2.15) \quad \langle \theta_t, W \rangle + \langle A\theta_x, W_x \rangle \leq Ch^2 (\|v_t\|_2 + \|v\|_2) |W|_1 + C(\|\theta\|_0 + |\theta(0, t)|) |W|_1.$$

It follows from taking $W = \theta$ in (2.15) and applying the inequality (see Cannon and Yin [6])

$$|\theta(0, t)| \leq \varepsilon |\theta|_1 + C \|\theta\|_0$$

and the ε -inequality that

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_0^2 + \langle A\theta_x, \theta_x \rangle \leq Ch^4 (\|v_t\|_2 + \|v\|_2)^2 + \varepsilon |\theta|_1^2 + C \|\theta\|_0^2,$$

or by condition (H2),

$$\frac{d}{dt} \|\theta\|_0^2 + |\theta|_1^2 \leq Ch^4 (\|v_t\|_2 + \|v\|_2)^2 + C\|\theta\|_0^2.$$

Since $\|\theta(0)\|_0 = 0$, we obtain by integrating the above inequality from 0 to t that

$$\|\theta\|_0^2 + \int_0^t |\theta|_1^2 d\tau \leq Ch^4 \int_0^t (\|v_t\|_2^2 + \|v\|_2^2) d\tau + C \int_0^t \|\theta\|_0^2 d\tau,$$

which, together with the Gronwall lemma, implies

$$\|\theta\|_0^2 + \int_0^t |\theta|_1^2 d\tau \leq Ch^4 \int_0^t (\|v_t\|_2^2 + \|v\|_2^2) d\tau.$$

This completes the proof of the theorem. \square

Remark 2.3. From Theorem 2.2 we know that $\|U - i_h v\|_{L^2(0,T;H^1(I))}$ converges at a rate of $O(h^2)$, which is referred to as the finite element solution being superclose to the nodal interpolant.

3. GLOBAL SUPERCONVERGENCE ESTIMATES

By means of the interpolation postprocessing technique [12], on the basis of Theorem 2.2 we can obtain the following global superconvergence estimate.

Theorem 3.1. *Assume that $v \in H^3(I)$. Then we have under the conditions of Theorem 2.2 that*

$$\|I_{2h}^2 U - v\|_{L^2(0,T;H^1(I))} \leq Ch^2,$$

where I_{2h}^2 is a piecewise polynomial interpolation operator of degree at most 2 associated with the mesh T_{2h} of mesh size $2h$, from which T_h is assumed to be gained by subdividing each element of T_{2h} into two equal elements so that the number of elements M for T_h is an even number.

As a by-product of Theorem 3.1 we can immediately obtain the following result [12].

Theorem 3.2. Under the assumptions of Theorem 3.1 we have

$$\|v - U\|_{L^2(0,T;H^1(I))} = \|U - I_{2h}^2 U\|_{L^2(0,T;H^1(I))} + O(h^2).$$

In addition, if there exist positive constants C_0 and $\varepsilon \in (0, 1)$ such that

$$\|v - U\|_{L^2(0,T;H^1(I))} \geq C_0 h^{2-\varepsilon},$$

then

$$\lim_{h \rightarrow 0} \frac{\|v - U\|_{L^2(0,T;H^1(I))}}{\|I_{2h}^2 U - U\|_{L^2(0,T;H^1(I))}} = 1.$$

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