Mei-Lan Tang; Xin-Ge Liu; Xin-Bi Liu

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NEW RESULTS ON PERIODIC SOLUTIONS FOR A KIND OF RAYLEIGH EQUATION*

Mei-Lan Tang, Xin-Ge Liu, Xin-Bi Liu, Changsha

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Abstract. The paper deals with the existence of periodic solutions for a kind of non-autonomous time-delay Rayleigh equation. With the continuation theorem of the coincidence degree and a priori estimates, some new results on the existence of periodic solutions for this kind of Rayleigh equation are established.

Keywords: Rayleigh equations, existence, periodic solution, a priori estimate

MSC 2010: 34C25

1. Introduction

Liénard equations have been used to describe fluid mechanical and nonlinear elastic mechanical phenomena. Rayleigh equation arises as a model including the delay Duffing equation and the delay Liénard equation. The existence of periodic solutions of Rayleigh equation has been extensively investigated (see [1]–[4], [6]–[9], [11]–[12]). In [3]–[4], continuation theorems are introduced and applied to the existence of solutions of differential equations. In the course of derivation of the existence of solutions of differential equations, if appropriate a priori bounds for the periodic solutions of the auxiliary equations for the differential equations can be obtained, then standard procedures allow these continuation theorems to establish the existence of periodic solutions of differential equations. Employing this approach, Peng et al. [10] and Huang et al. [5] considered the existence of periodic solutions of a class of Rayleigh

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equation with two deviating arguments of the form

$$x''(t) + f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_1(t))) = p(t)$$

where $f, \tau_1, \tau_2, p\colon \mathbb{R} \to \mathbb{R}$ and $g_1, g_2\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are real continuous functions, $f(0) = 0$, τ_1, τ_2, p are T-periodic, and g_1, g_2 are T-periodic in the first argument.

Wang and Yan [13] considered the non-autonomous Rayleigh equation of retarded type

$$x''(t) + f(t, x'(t - \sigma)) + g(t, x(t - \tau)) = p(t) \tag{1}$$

where $\sigma \geqslant 0$, $\tau \geqslant 0$, $f, g \in C(\mathbb{R}^2, \mathbb{R})$, and $f(t, x)$, $g(t, x)$ are 2π-periodic in the first argument t, $p \in C(\mathbb{R}, \mathbb{R})$ is periodic with period 2π. Under the assumptions $f(t, 0) = 0$ for $t \in \mathbb{R}$ and $\int_0^{2\pi} p(s)\,\mathrm{d}s = 0$, they got the following result:

Theorem 1. *Assume that there exist constants $K > 0$, $M > 0$ and $d > 0$ such that*

(A1) $|f(t, x)| \leqslant K$ *for* $(t, x) \in \mathbb{R}^2$;

(A2) $xg(t, x) > 0$ *and* $|g(t, x)| > K$ *for* $t \in \mathbb{R}$, $|x| \geqslant d$;

(A3) $g(t, x) \geqslant -M$ *for* $t \in \mathbb{R}$, $x \leqslant -d$;

(A4) $\sup\limits_{(t,x) \in \mathbb{R} \times [-d,d]} |g(t, x)| < +\infty$.

Then Eq. (1) *has at least one periodic solution with period* 2π.

Recently, Zhou and Tang [14] studied the non-autonomous Rayleigh equation with time-varying delay

$$x''(t) + f(t, x'(t - \sigma)) + g(t, x(t - \tau(t))) = p(t) \tag{2}$$

where $\sigma \geqslant 0$, $\tau, p \in C(\mathbb{R}, \mathbb{R})$ are periodic with period 2π, $f, g \in C(\mathbb{R}^2, \mathbb{R})$ and $f(t, x)$, $g(t, x)$ are 2π-periodic in the first argument t, $f(t, 0) = 0$ for $t \in \mathbb{R}$ and $\int_0^{2\pi} p(s)\,\mathrm{d}s = 0$. They generalized and improved the corresponding results of [13] and proved

Theorem 2. *Assume that there exist constants $r_1 \geqslant 0$, $r_2 \geqslant 0$, $d > 0$, $K > 0$ and $M > 0$ such that*

(H1) $|f(t, x)| \leqslant r_1|x| + K$ *for* $(t, x) \in \mathbb{R}^2$;

(H2) $xg(t, x) > 0$ *and* $|g(t, x)| > r_1|x| + K$ *for* $t \in \mathbb{R}$, $|x| \geqslant d$;

(H3) $g(t, x) \geqslant r_2 x - M$ *for* $t \in \mathbb{R}$, $x \leqslant -d$.

If

$$2\pi(r_1 + (\pi+1)r_2) < 1, \tag{3}$$

then Eq. (2) *has at least one* 2π*-periodic solution.*

The purpose of this paper is to reconsider the existence of 2π-periodic solution to Eq. (2). We will change the conditions imposed on f and g in [14]. Under some new conditions for f and g, we shall employ a new a priori estimate of the periodic solution to establish new sufficient conditions for the existence of a 2π-periodic solution to Eq. (2).

For the sake of convenience, we denote by $C_{2\pi}$ the Banach space of continuous 2π-periodic functions, endowed with the norm $\|x\|_0 = \max\limits_{t\in[0,2\pi]} |x(t)|$.

2. Main results

Theorem 3. *Assume that there exist constants* $r_1 \geqslant 0$, $r_2 > 0$, $r_3 \geqslant 0$, $d > 0$, $K > 0$, $M > 0$ *and* $0 < \beta < 1$ *such that*

(H1) $|f(t,x)| \leqslant r_1|x|^\beta + K$ *for* $(t,x) \in \mathbb{R}^2$;

(H2) $xg(t,x) > 0$ *and* $|g(t,x)| > r_2|x|^\beta + K$ *for* $t \in \mathbb{R}$, $|x| \geqslant d$;

(H3) $g(t,x) \geqslant r_3 x - M$ *for* $t \in \mathbb{R}$, $x \leqslant -d$.

If

$$2\pi r_3 \Big[\pi + \Big(\frac{r_1}{r_2}\Big)^{1/\beta}\Big] < 1, \tag{4}$$

then Eq. (2) *has at least one* 2π*-periodic solution.*

Proof. Consider the auxiliary equation

$$x''(t) + \lambda f(t, x'(t-\sigma)) + \lambda g(t, x(t-\tau(t))) = \lambda p(t). \tag{5}$$

From the results (degree theory) in [3]–[4] (see also the proof in [13]), it is sufficient to show that there are positive constants M_0 and M_1, independent of λ, such that if $x(t)$ is a 2π-periodic solution of Eq. (5), then $\|x\|_0 < M_0$ and $\|x'\|_0 < M_1$.

Now, let $x = x(t)$ be any 2π-periodic solution of Eq. (5). Integrating both sides of Eq. (5) on $[0, 2\pi]$, we have

$$\int_0^{2\pi} f(s, x'(s-\sigma)) + g(s, x(s-\tau(s)))\,\mathrm{d}s = 0. \tag{6}$$

It follows that there exists a $t_1 \in [0, 2\pi]$ such that

$$f(t_1, x'(t_1 - \sigma)) + g(t_1, x(t_1 - \tau(t_1))) = 0. \tag{7}$$

We assert that there exists a $t^* \in [0, 2\pi]$ such that

$$|x(t^*)| \leqslant \left[\frac{r_1}{r_2}\right]^{1/\beta} \|x'\|_0 + d. \tag{8}$$

Indeed, if $|x(t_1 - \tau(t_1))| \leqslant d$, then obviously, $|x(t_1 - \tau(t_1))| \leqslant [r_1/r_2]^{1/\beta} \|x'\|_0 + d$. If $|x(t_1 - \tau(t_1))| > d$, it follows from Eq. (7), (H1) and (H2) that

$$\begin{aligned} r_2 |x(t_1 - \tau(t_1))|^\beta + K &\leqslant |g(t_1, x(t_1 - \tau(t_1)))| \\ &= |-f(t_1, x'(t_1 - \sigma))| \leqslant r_1 |x'(t_1 - \sigma)|^\beta + K. \end{aligned} \tag{9}$$

So

$$|x(t_1 - \tau(t_1))| \leqslant \left[\frac{r_1}{r_2}\right]^{1/\beta} |x'(t_1 - \sigma)| \leqslant \left[\frac{r_1}{r_2}\right]^{1/\beta} \|x'\|_0 \leqslant \left[\frac{r_1}{r_2}\right]^{1/\beta} \|x'\|_0 + d. \tag{10}$$

Since $x(t)$ is periodic, there exists a $t^* \in [0, 2\pi]$ such that Inequality (8) holds.

For any $t \in [t^*, t^* + 2\pi]$,

$$|x(t)| = \left| x(t^*) + \int_{t^*}^{t} x'(s)\,\mathrm{d}s \right| \leqslant |x(t^*)| + \int_{t^*}^{t} |x'(s)|\,\mathrm{d}s. \tag{11}$$

Again

$$|x(t)| = \left| x(t^* + 2\pi) + \int_{t^*+2\pi}^{t} x'(s)\,\mathrm{d}s \right| \leqslant |x(t^*)| + \int_{t}^{t^*+2\pi} |x'(s)|\,\mathrm{d}s. \tag{12}$$

Combining Inequalities (11) and (12) gives

$$|x(t)| \leqslant |x(t^*)| + \frac{1}{2} \int_0^{2\pi} |x'(s)|\,\mathrm{d}s. \tag{13}$$

From Inequalities (8) and (13) we have

$$\begin{aligned} \|x\|_0 &\leqslant |x(t^*)| + \frac{1}{2} \int_0^{2\pi} |x'(s)|\,\mathrm{d}s \\ &\leqslant \left[\frac{r_1}{r_2}\right]^{1/\beta} \|x'\|_0 + d + \pi \|x'\|_0 \leqslant \left[\pi + \left(\frac{r_1}{r_2}\right)^{1/\beta}\right] \|x'\|_0 + d. \end{aligned} \tag{14}$$

As $x(t)$ is a 2π-periodic solution of Eq. (5), there exists $t_0 \in [0, 2\pi]$ such that

$$x(t_0) = \max_{s \in [0,2\pi]} x(s).$$

It follows that

$$x'(t_0) = 0.$$

For any $t \in [t_0, t_0 + 2\pi]$,

$$|x'(t)| = \left| x'(t_0) + \int_{t_0}^{t} x''(s)\,\mathrm{d}s \right| \leqslant |x'(t_0)| + \int_{t_0}^{t} |x''(s)|\,\mathrm{d}s. \tag{15}$$

Again

$$|x'(t)| = \left| x'(t_0 + 2\pi) + \int_{t_0+2\pi}^{t} x''(s)\,\mathrm{d}s \right| \leqslant |x'(t_0)| + \int_{t}^{t_0+2\pi} |x''(s)|\,\mathrm{d}s. \tag{16}$$

Combining Inequalities (15) and (16) gives

$$|x'(t)| \leqslant \frac{1}{2} \int_0^{2\pi} |x''(s)|\,\mathrm{d}s. \tag{17}$$

Let

$$\begin{aligned}
E_1 &= \{t\colon\ t \in [0, 2\pi],\ x(t - \tau(t)) > d\},\\
E_2 &= \{t\colon\ t \in [0, 2\pi],\ x(t - \tau(t)) < -d\},\\
E_3 &= \{t\colon\ t \in [0, 2\pi],\ |x(t - \tau(t))| \leqslant d\}.
\end{aligned}$$

From Eq. (6) and (H2) we obtain

$$\begin{aligned}
&\int_{E_1} |g(s, x(s - \tau(s)))|\,\mathrm{d}s = \int_{E_1} g(s, x(s - \tau(s)))\,\mathrm{d}s = \left| \int_{E_1} g(s, x(s - \tau(s)))\,\mathrm{d}s \right| \\
&\quad \leqslant \int_0^{2\pi} |f(s, x'(s - \sigma))|\,\mathrm{d}s + \left(\int_{E_2} + \int_{E_3} \right) |g(s, x(s - \tau(s)))|\,\mathrm{d}s.
\end{aligned} \tag{18}$$

Thus

$$\begin{aligned}
\|x'\|_0 &\leqslant \frac{1}{2} \int_0^{2\pi} |x''(s)|\,\mathrm{d}s \\
&\leqslant \frac{1}{2} \left[\int_0^{2\pi} |f(s, x'(s - \sigma))|\,\mathrm{d}s + \int_0^{2\pi} |g(s, x(s - \tau(s)))|\,\mathrm{d}s + \int_0^{2\pi} |p(s)|\,\mathrm{d}s \right] \\
&\leqslant \frac{1}{2} \left[\int_0^{2\pi} |f(s, x'(s - \sigma))|\,\mathrm{d}s \right. \\
&\quad \left. + \left(\int_{E_1} + \int_{E_2} + \int_{E_3} \right) |g(s, x(s - \tau(s)))|\,\mathrm{d}s + 2\pi \|p\|_0 \right]
\end{aligned} \tag{19}$$

$$
\begin{aligned}
&\leqslant \int_0^{2\pi} |f(s, x'(s-\sigma))|\,\mathrm{d}s + \left(\int_{E_2} + \int_{E_3}\right)|g(s, x(s-\tau(s)))|\,\mathrm{d}s + \pi\|p\|_0 \\
&\leqslant 2\pi[r_1\|x'\|_0^\beta + K] + 2\pi[r_3\|x\|_0 + M] + 2\pi g_d + \pi\|p\|_0 \\
&\leqslant 2\pi r_1\|x'\|_0^\beta + 2\pi r_3\|x\|_0 + 2\pi(K + M + g_d) + \pi\|p\|_0 \\
&\leqslant 2\pi r_1\|x'\|_0^\beta + 2\pi r_3\left\{\left[\pi + \left(\frac{r_1}{r_2}\right)^{1/\beta}\right]\|x'\|_0 + d\right\} + 2\pi(K + M + g_d) + \pi\|p\|_0 \\
&\leqslant 2\pi r_1\|x'\|_0^\beta + 2\pi r_3\left[\pi + \left(\frac{r_1}{r_2}\right)^{1/\beta}\right]\|x'\|_0 + 2\pi(K + M + r_3 d + g_d) + \pi\|p\|_0,
\end{aligned}
$$

where $g_d = \max\limits_{t\in[0,2\pi],\,|x|\leqslant d} |g(t,x)|$.

Inequality (19) is equivalent to

$$
\left\{1 - 2\pi r_3\left[\pi + \left(\frac{r_1}{r_2}\right)^{1/\beta}\right]\right\}\|x'\|_0 \leqslant 2\pi r_1\|x'\|_0^\beta + 2\pi(K + M + r_3 d + g_d) + \pi\|p\|_0. \tag{20}
$$

Since $2\pi r_3[\pi + (r_1/r_2)^{1/\beta}] < 1$ and $0 < \beta < 1$, there exists a positive constant $M_1 > 0$ such that

$$
\|x'\|_0 < M_1. \tag{21}
$$

Let $M_0 = \left[\pi + (r_1/r_2)^{1/\beta}\right]M_1 + d$.

It follows from Inequality (21) that

$$
\|x\|_0 \leqslant \left[\pi + \left(\frac{r_1}{r_2}\right)^{1/\beta}\right]\|x'\|_0 + d \leqslant \left[\pi + \left(\frac{r_1}{r_2}\right)^{1/\beta}\right]M_1 + d = M_0. \tag{22}
$$

This completes the proof. $\square$

Similarly, we have the following

Theorem 4. *Assume that there exist constants $r_1 \geqslant 0, r_2 > 0$, $r_3 \geqslant 0$, $d > 0$, $K > 0$, $M > 0$ and $0 < \beta < 1$ such that*

(H1) *$|f(t,x)| \leqslant r_1|x|^\beta + K$ for $(t,x) \in \mathbb{R}^2$;*

(H2) *$xg(t,x) > 0$ and $|g(t,x)| > r_2|x|^\beta + K$ for $t \in \mathbb{R}$, $|x| \geqslant d$;*

(H3) *$g(t,x) \leqslant r_3 x + M$ for $t \in \mathbb{R}$, $x \geqslant d$.*

If

$$
2\pi r_3\left[\pi + \left(\frac{r_1}{r_2}\right)^{1/\beta}\right] < 1, \tag{23}
$$

then Eq. (2) has at least one 2π-periodic solution.

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Authors' addresses: *Mei-Lan Tang*, *Xin-Ge Liu* (corresponding author), School of Mathematical Science and Computing Technology, Central South University, Changsha, Hunan 410083, P. R. China, e-mail: `liuxgliuhua@163.com`; *Xin-Bi Liu*, School of Materials Science and Engineering, Central South University, Changsha, Hunan 410083, P. R. China.