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UNILATERAL ELASTIC SUBSOIL OF WINKLER'S TYPE: SEMI-COERCIVE BEAM PROBLEM*

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Abstract. The mathematical model of a beam on a unilateral elastic subsoil of Winkler's type and with free ends is considered. Such a problem is non-linear and semi-coercive. The additional assumptions on the beam load ensuring the problem solvability are formulated and the existence, the uniqueness of the solution and the continuous dependence on the data are proved. The cases for which the solutions need not be stable with respect to the small changes of the load are described. The problem is approximated by the finite element method and the relation between the original problem and the family of approximated problems is analyzed. The error estimates are derived in dependence on the smoothness of the solution, the load and the discretization parameter of the partition.

Keywords: non-linear subsoil of Winkler's type, semi-coercive beam problem, existence, uniqueness, continuous dependence on data, finite element method, numerical quadrature

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1. Introduction

Linear models of beams and plates on elastic subsoil of Winkler's type are used in civil engineering and other mechanical applications. Some of them are described in [3]. However, the models are not always suitable. For example, when the beam or plate is only laid on the subsoil, then the subsoil is active only if the beam or plate deflects against it. In such cases, non-linear models of the subsoil are more precise. The non-linear ones have been studied in [7], [6] and [12]. Some related problems were treated in [5].

In this article we shall study a one-dimensional model of a beam on a unilateral elastic subsoil (see Fig. 1). A two-dimensional model of a thin plate has been studied

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in [7] and a one-dimensional rotary-symmetric model of a thin intercircular plate has been considered in [12]. We will assume that the beam has free ends. Thus, the problem of finding the beam deflection will be only semi-coercive. Therefore, the solvability of the problem and the stability of the problem solution with respect to small changes of data are dependent on the load, in particular on the load resultant and "the balance point of the load".

In Section 2, some preliminaries about function spaces, solvability of a minimization problem and a numerical quadrature are summarized. In Section 3, we set the problem and analyze the existence, the uniqueness and the continuous dependence on data of the problem solution. In Section 4, we approximate the problem by the finite element method, where the subsoil is replaced by insulated "springs", and analyze the relation between the original problem and the family of approximated problems.

2. Some preliminaries

2.1. Function spaces

In the paper we will use the Lebesgue spaces $L^p(\Omega)$, $p=1,2,+\infty$, Sobolev spaces $W^{k,p}(\Omega)$, $p=1,2$, $k=0,1,2,3,4$, and the spaces of continuously differentiable functions $C^k(\overline{\Omega})$, where Ω is an open, bounded and non-empty interval in $\mathbb{R}^1$. The spaces are described in the book [1]. Their standard norms are denoted by $\|\cdot\|_{p,\Omega}$, $\|\cdot\|_{k,p,\Omega}$ and $\|\cdot\|_{C^k(\overline{\Omega})}$, respectively. The ith seminorm, $i=0,1,\ldots,k$, of the spaces $W^{k,p}(\Omega)$ are denoted by $|\cdot|_{i,p,\Omega}$. The spaces $W^{k,2}(\Omega)$, which are also Hilbert spaces, are denoted by $H^k(\Omega)$. The space of polynomials of the kth degree is denoted by P_k.

Now, we summarize some useful properties of the Sobolev spaces $W^{k,p}(\Omega)$. Their proofs can be found for more general cases in the book [1].

Theorem 2.1 (The Sobolev Imbedding Theorem). *Let Ω be a bounded non-empty interval in $\mathbb{R}^1$. Then the Sobolev space $W^{k+1,p}(\Omega)$, $p=1,2$, $k=0,1,\ldots$, can be continuously imbedded into the space $C^k(\overline{\Omega})$, i.e. there exists a positive constant $c_{p,k}$ such that*

$$\|v\|_{C^k(\overline{\Omega})} \leqslant c_{p,k}\|v\|_{k+1,p,\Omega} \quad \forall\, v \in W^{k+1,p}(\Omega). \tag{2.1}$$

In addition, the space $H^{k+1}(\Omega)$, $k=0,1,\ldots$, can be continuously imbedded into the space $C^{k,1/2}(\overline{\Omega})$, i.e.

$$|v^{(k)}(x)-v^{(k)}(y)| \leqslant |v|_{k+1,2,\Omega}|x-y|^{1/2} \quad \forall\, x,y \in \overline{\Omega},\ \forall\, v \in H^{k+1}(\Omega), \tag{2.2}$$

where $v^{(k)}$ is the kth generalized derivative of the function v.

Notice that the values $v^{(i)}(x)$, $i = 0, 1, \dots k$, are correctly defined for any $x \in \overline{\Omega}$ and $v \in W^{k+1,p}(\Omega)$ in the sense of equivalence classes in the space $W^{k+1,p}(\Omega)$. Except for the parameters p, k, the constant $c_{p,k}$ in the estimate (2.1) depends only on the length of the interval Ω.

Theorem 2.2 (The Rellich Theorem). *Let $\{v_n\}_{n=1}^{+\infty}$ be any sequence of functions belonging to the space $H^{k+1}(\Omega)$ such that there exists a function $v \in H^{k+1}(\Omega)$, $v_n \rightharpoonup v$ in $H^{k+1}(\Omega)$. Then $v_n \to v$ in $H^k(\Omega)$.*

Lemma 2.1. *Let $\{v_n\}_{n=1}^{+\infty}$ be a bounded sequence of functions belonging to the space $H^{k+1}(\Omega)$. Let*

$$|v_n|_{k+1,2,\Omega} \to 0, \quad n \to +\infty,$$

where $|\cdot|_{k+1,2}$ is the $(k+1)$st seminorm in $H^{k+1}(\Omega)$. Then there exist a subsequence $\{v_{n_j}\}_j \subset \{v_n\}_{n=1}^{+\infty}$ and a polynomial $p \in P_k$ such that

$$v_{n_j} \to p \ \text{ in } \ H^{k+1}(\Omega), \quad j \to +\infty.$$

P r o o f. Since the sequence $\{v_n\}_{n=1}^{+\infty}$ is bounded in $H^{k+1}(\Omega)$, there exist its subsequence $\{v_{n_j}\}_j$ and a function $v \in H^{k+1}(\Omega)$ such that $v_{n_j} \rightharpoonup v$ in $H^{k+1}(\Omega)$. By Theorem 2.2, $v_{n_j} \to v$ in $H^k(\Omega)$. Then

$$\begin{aligned} \|v_{n_i} - v_{n_j}\|_{k+1,2,\Omega} &\leqslant |v_{n_i} - v_{n_j}|_{k+1,2,\Omega} + \|v_{n_i} - v_{n_j}\|_{k,2,\Omega} \\ &\leqslant |v_{n_i}|_{k+1,2,\Omega} + |v_{n_j}|_{k+1,2,\Omega} + \|v_{n_i} - v_{n_j}\|_{k,2,\Omega} \\ &\to 0, \quad i, j \to +\infty. \end{aligned}$$

Hence, $v_{n_j} \to v$ also in $H^{k+1}(\Omega)$ and $|v|_{k+1,2,\Omega} = 0$. Therefore $v = p \in P_k$. □

Lemma 2.2 (Equivalent norms). *Let $\|\cdot\|_{k,2,\Omega}$ be the standard norm in the Sobolev space $H^k(\Omega)$. Let $\Omega_s \subset \Omega$ be a non-empty open interval and define*

$$[v]_{k,2,\Omega} := (|v|^2_{k,2,\Omega} + |v|^2_{0,2,\Omega_s})^{1/2}, \quad v \in H^k(\Omega).$$

Then the formula $[\cdot]_{k,2,\Omega}$ is a norm in the space $H^k(\Omega)$ which is equivalent to the standard norm, i.e. there exists a positive constant c such that

$$c\|v\|_{k,2,\Omega} \leqslant [v]_{k,2,\Omega} \leqslant \|v\|_{k,2,\Omega} \quad \forall\, v \in H^k(\Omega). \tag{2.3}$$

The proof of Lemma 2.2 can be found for more general case in the book [9].

Lemma 2.3. *Let Ω be a bounded, non-empty interval in $\mathbb{R}^1$ and let $\Omega_h \subset \Omega$ be any interval whose length is h. Let $k \geqslant 0$ be an integer. Then there exists a positive constant $c = c(k, \Omega)$ such that*

$$|v|_{k,2,\Omega_h} \leqslant ch^{1/2}\|v\|_{k+1,2,\Omega} \quad \forall\, v \in H^{k+1}(\Omega). \tag{2.4}$$

P r o o f. By the well-known Mean Value Theorem and Theorem 2.1, there exists $\xi \in \Omega_h$ such that

$$\begin{aligned} |v|_{k,2,\Omega_h} &= \left(\int_{\Omega_h} (v^{(k)}(x))^2 \,\mathrm{d}x\right)^{1/2} = h^{1/2}|v^{(k)}(\xi)| \\ &\leqslant h^{1/2}\|v\|_{C^k(\overline{\Omega})} \leqslant ch^{1/2}\|v\|_{k+1,2,\Omega}, \quad \forall\, v \in H^{k+1}(\Omega). \end{aligned}$$

Since the constant c does not depend on the choice of Ω_h, Lemma 2.3 is proven. $\square$

Lemma 2.4. *Let Ω be a non-empty interval in $\mathbb{R}^1$ and $u, v \in H^k(\Omega)$. Then $uv \in W^{k,1}(\Omega)$ and there exists a positive constant $c = c(k)$ such that*

$$\|uv\|_{k,1,\Omega} \leqslant c\|u\|_{k,2,\Omega}\|v\|_{k,2,\Omega} \quad \forall\, u, v \in H^k(\Omega). \tag{2.5}$$

The proof is based on an application of the well-known Cauchy-Schwarz inequality.

Lemma 2.5. *Let Ω be a bounded and non-empty interval in $\mathbb{R}^1$ and $v \in H^2(\Omega)$. Then the negative part*

$$v^-(x) := \min\{0, v(x)\}, \quad x \in \Omega,$$

of the function v belongs to the space $H^1(\Omega)$ and $\|v^-\|_{1,2,\Omega} \leqslant \|v\|_{1,2,\Omega}$.

In addition, the following inequality holds:

$$|u^-(x) - v^-(x)| \leqslant |u(x) - v(x)| \quad \forall\, u, v \in C(\overline{\Omega}),\ \forall\, x \in \overline{\Omega}. \tag{2.6}$$

P r o o f. Let $v \in H^2(\Omega)$. By Theorem 2.1 we can also assume that $v \in C^1(\overline{\Omega})$. Therefore the set

$$M := \{x \in \Omega\colon\ v(x) < 0\}$$

can be expressed as a countable union of open intervals belonging to Ω. Thus the first generalized derivative of v^- can be defined in the following way:

$$(v^-)'(x) := \begin{cases} v'(x), & x \in M, \\ 0, & x \in \Omega \setminus M. \end{cases}$$

Since $|(v^-)'| \leqslant |v'|$ almost everywhere in Ω, we have $\|v^-\|_{1,2,\Omega} \leqslant \|v\|_{1,2,\Omega}$.

The inequality (2.6) follows from the inequality

$$||s| - |t|| \leqslant |s - t| \quad \forall\, s, t \in \mathbb{R},$$

since $s^- = \frac{1}{2}(s - |s|)$ for any $s \in \mathbb{R}$. $\square$

2.2. Minimization of convex functionals

The main goal of the subsection is to formulate a solvability criterion of the minimization problem

$$\text{find } u \in V\colon\ J(u) \leqslant J(v) \quad \forall\, v \in V,$$

where V is a reflexive Banach space and $J\colon\ V \to \mathbb{R}$ is a functional defined on V. We summarize some basic results, which can be found for example in the book [4].

Theorem 2.3. *Let J be a convex and Gâteaux differentiable[1] functional on V. Then a function u minimizes J in V if and only if*

$$J'(u; v) = 0 \quad \forall\, v \in V,$$

where the symbol $J'(u; v)$ denotes the Gâteaux differential at the point u and in the direction v.

Theorem 2.4 (The fundamental theorem)**.** *Let J be a convex, coercive and Gâteaux differentiable functional on the reflexive Banach space V. Then there exists at least one element u minimizing J in the space V.*

Lemma 2.6 (Criterion of convexity)**.** *Let J be a Gâteaux differentiable functional on the space V. If the inequality*

$$J'(u; u - v) - J'(v; u - v) \geqslant 0 \quad \forall\, u, v \in V$$

holds, then the functional J is convex on V.

2.3. Numerical quadrature

Numerical quadrature will be used to approximate the problem, see Section 4.2. In this subsection, we summarize some basic properties.

First, we define a numerical quadrature on the reference interval $[-1, 1]$. Let $\hat{\varphi}$ be any function belonging to $W^{1,1}((-1, 1))$ and let $\hat{y}_i$, $i = 1, 2, \ldots, m$, be points

[1] The definition of a Gâteaux differentiable functional can be found in [8] or [10].

belonging to the interval $[-1,1]$. An m-points numerical quadrature of the function $\hat{\varphi}$ with positive weights $\hat{\omega}_i$ has the form

$$\int_{-1}^{1} \hat{\varphi}(\xi)\,\mathrm{d}\xi \approx \sum_{i=1}^{m} \hat{\omega}_i \hat{\varphi}(\hat{y}_i). \tag{2.7}$$

Note that the values $\hat{\varphi}(\hat{y}_i)$, $i=1,\ldots,m$, are correctly defined by Theorem 2.1.

We say that the reference numerical quadrature (2.7) is exact for polynomials of the kth degree if

$$\int_{-1}^{1} \hat{p}(\xi)\,\mathrm{d}\xi = \sum_{i=1}^{m} \hat{\omega}_i \hat{p}(\hat{y}_i) \quad \forall\, \hat{p} \in P_k.$$

We will assume that the numerical quadrature is exact at least for polynomials of the zeroth degree.

Secondly, we define a numerical quadrature on any interval $[s,t]$ with the length $h := t-s > 0$. Let φ be any function belonging to $W^{1,1}((s,t))$ and Φ the transformation of the interval $[s,t]$ onto the interval $[-1,1]$ such that

$$\Phi(x) := \xi = \frac{2}{h}(x-s) - 1, \quad x \in [s,t]. \tag{2.8}$$

Since

$$\int_{s}^{t} \varphi(x)\,\mathrm{d}x = \frac{h}{2}\int_{-1}^{1} \hat{\varphi}(\xi)\,\mathrm{d}\xi, \quad \hat{\varphi}(\xi) := \varphi(\Phi^{-1}(\xi)),$$

the numerical quadrature of φ corresponding to the numerical quadrature (2.7) has the form

$$\int_{s}^{t} \varphi(x)\,\mathrm{d}x \approx \sum_{i=1}^{m} \omega_i \varphi(y_i), \tag{2.9}$$

where $y_i := \Phi^{-1}(\hat{y}_i) = \frac{1}{2}h(\hat{y}_i + 1) + s$ and $\omega_i := \frac{1}{2}h\hat{\omega}_i$.

Clearly, if the numerical quadrature (2.7) is exact for polynomials of the kth degree then the numerical quadrature (2.9) is also exact for polynomials of the kth degree.

Theorem 2.5. *Let (s,t) be any interval with the length $h := t-s > 0$. If the numerical quadrature* (2.9) *is exact for polynomials of the kth degree then there exists a constant $c > 0$, independent of h, such that*

$$\left| \int_{s}^{t} \varphi(x)\,\mathrm{d}x - \sum_{i=1}^{m} \omega_i \varphi(y_i) \right| \leqslant c h^{k+1} |\varphi|_{k+1,1,(s,t)} \quad \forall\, \varphi \in W^{k+1,1}((s,t)). \tag{2.10}$$

The proof can be found in the book [2].

3. Setting and analyzing the problem

The main goal of the section is to analyze solvability of the problem and its continuous dependence on the data. First, we set the problem and its variational formulation. Then, we derive necessary and sufficient conditions for the existence and uniqueness of the problem solution, and finally we prove the continuous dependence of the problem solutions on the data and consequently describe some situations for which the solutions need not be stable with respect to a small change of the load.

Since we will mainly use the interval $\Omega := (0, l)$ in the remaining parts of the paper, we will denote the norms and seminorms of the Sobolev spaces $H^k(\Omega)$, $k = 0, 1, 2, 3, 4$, without the symbol Ω for this particular choice of the interval.

3.1. Setting of the problem

We consider a beam of the length l with free ends which is situated in the interval $\Omega = (0, l)$, and assume that the beam is supported by a unilateral elastic subsoil in the interval $\Omega_s := (x_l, x_r)$, $0 \leqslant x_l < x_r \leqslant l$. Such subsoil is active only if the beam deflects against it. Let E, I, q and f denote functions that represent, respectively, Young's modulus of the beam material, the inertia moment of the cross-section of the beam, the stiffness coefficient of the subsoil and the beam load density. The aim is to find the deflection ω of the axes of the beam caused by the beam load. The situation is depicted in Fig. 1.

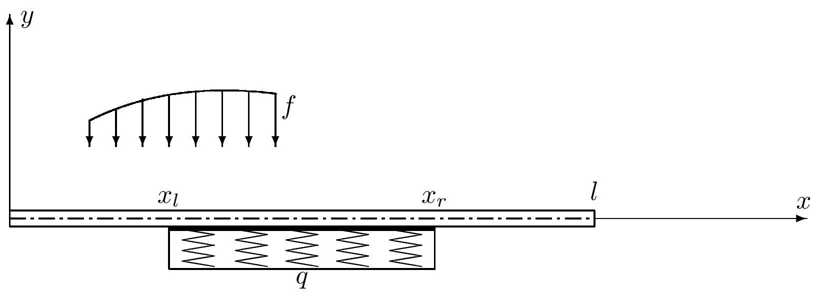

Figure 1. Scheme of the subsoiled beam with axes orientation.

If we assume that the functions E, I, q and f are sufficiently smooth, then we can give (see [10] and [3]) the classical formulation of the problem:

$$\begin{cases} (E(x)I(x)w''(x))'' + q(x)w^-(x) = f(x), & x \in \Omega, \\ w''(0) = w'''(0) = w''(l) = w'''(l) = 0, \end{cases} \tag{3.1}$$

where $q = 0$ in $\Omega \setminus \Omega_s$ and

$$w^-(x) := \min\{0, w(x)\}, \quad x \in \Omega_s,$$

is the negative part of w. Thus, the formulation has the form of a non-linear differential equation of the fourth order with homogeneous Neumann boundary conditions.

In fact, the functions E, I, q and f need not be sufficiently smooth to use the classical formulation. Therefore, we will work with the variational formulation of the problem.

3.2. Variational formulation of the problem

We will assume that the functions E, I, q belong to the Lebesgue space $L^\infty(\Omega)$. Then we can define forms

$$a(v_1, v_2) := \int_\Omega EIv_1''v_2''\,\mathrm{d}x, \quad \text{and} \quad b(v_1, v_2) := \int_{\Omega_s} qv_1v_2\,\mathrm{d}x, \quad v_1, v_2 \in H^2(\Omega),$$

to represent the work of the inner forces and the subsoil, respectively. The forms a, b are bilinear and bounded on the space $H^2(\Omega)$ by Lemma 2.4.

From the mechanical point of view, we will also assume that there exist positive constants E_0, I_0 and q_0 such that

$$E(x) \geqslant E_0, \;\; I(x) \geqslant I_0, \;\text{ a.e. in } \Omega, \quad \text{and} \quad q(x) \geqslant q_0 \;\text{ a.e. in } \Omega_s.$$

Therefore, the following inequalities hold:

$$a(v, v) \geqslant E_0I_0|v|^2_{2,2} \geqslant 0 \quad \forall\, v \in H^2(\Omega), \tag{3.2}$$

$$b(v^-, v) = b(v^-, v^-) \geqslant q_0|v^-|^2_{0,2,\Omega_s} \geqslant 0 \quad \forall\, v \in L^2(\Omega), \tag{3.3}$$

$$b(v_1^- - v_2^-, v_1 - v_2) \geqslant b(v_1^- - v_2^-, v_1^- - v_2^-) \geqslant 0 \quad \forall\, v_1, v_2 \in L^2(\Omega). \tag{3.4}$$

The load density can be expressed in the form

$$f = f_1 + f_2,$$

where $f_1 \in L^1(\Omega)$ and f_2 represents the generalized forces, i.e.

$$f_2 = \sum_{y \in X_P} P_y\delta_y + \sum_{y \in X_M} M_y\delta'_y,$$

where X_P, X_M are finite sets of points belonging to $\overline{\Omega}$, $P_y, M_y \in \mathbb{R}$, δ_y, δ'_y denote the Dirac distribution and its first generalized derivative at a point y, and $P_y\delta_y$, $M_y\delta'_y$ represent respectively the point load and the moment at a point y.

The space of all continuous and linear functionals defined on $H^2(\Omega)$ will be denoted by V^* and its corresponding norm by $\|\cdot\|_*$. The beam load will be represented by a functional $L \in V^*$ in the form

$$L(v) := \langle f, v\rangle = \int_\Omega f_1 v \,\mathrm{d}x + \sum_{y\in X_P} P_y v(y) + \sum_{y\in X_M} M_y v'(y),$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing on $V^* \times H^2(\Omega)$.

The total potential energy functional for the problem has the form

$$J(v) := \frac{1}{2}\big(a(v,v) + b(v^-,v^-)\big) - L(v), \quad v \in H^2(\Omega). \tag{3.5}$$

The variational formulation of the problem can be written as the minimization problem

$$\text{find } w \in H^2(\Omega)\colon\ J(w) \leqslant J(v) \quad \forall\, v \in H^2(\Omega). \tag{P}$$

To analyze the problem (P), we derive some properties of the functional J.

Lemma 3.1. *The functional J is Gâteaux differentiable and convex on the space $H^2(\Omega)$. Its Gâteaux derivative at any point $w \in H^2(\Omega)$ and direction $v \in H^2(\Omega)$ has the form*

$$J'(w;v) = a(w,v) + b(w^-,v) - L(v). \tag{3.6}$$

Proof. Let s, t be any real numbers. Then it can be easily shown that

$$\lim_{\varepsilon\to 0} \frac{[(s+\varepsilon t)^-]^2 - [s^-]^2}{\varepsilon} = 2s^- t.$$

Hence

$$\lim_{\varepsilon\to 0} \frac{b\big((w+\varepsilon v)^-, w+\varepsilon v\big) - b(w^-,w)}{\varepsilon} = 2b(w^-,v).$$

This easily implies the relation (3.6). Since $J'(w;\cdot)$ is a continuous linear functional on $H^2(\Omega)$ for any $w \in H^2(\Omega)$, the functional J is Gâteaux differentiable on $H^2(\Omega)$. Since the relation (3.6) and the inequalities (2.3)–(2.4) imply

$$\begin{aligned} J'(v_1; v_1 - v_2) - J'(v_2; v_1 - v_2) &= a(v_1 - v_2, v_1 - v_2) + b(v_1^- - v_2^-, v_1 - v_2) \\ &\geqslant E_0 I_0 |v_1 - v_2|_{2,2}^2 + q_0 |v_1^- - v_2^-|_{0,2,\Omega_s}^2 \geqslant 0 \end{aligned}$$

for all $v_1, v_2 \in H^2(\Omega)$, the functional J is convex on $H^2(\Omega)$ by Lemma 2.6. $\square$

By Theorem 2.3 and the relation (3.6), a function $w \in H^2(\Omega)$ solves the problem (P) if and only if it solves the nonlinear variational equation

$$a(w, v) + b(w^-, v) = L(v) \quad \forall\, v \in H^2(\Omega). \tag{3.7}$$

If the functions w, E, I, q and f are sufficiently smooth, then it is possible (see [11]) to derive the classical formulation (3.1) of the problem from the variational equation (3.7).

3.3. Solvability of the problem

Since the beam does not have fixed ends (it is only laid on the subsoil), the problem solvability depends on the beam load. This fact will be demonstrated in the following lemmas and theorem.

Lemma 3.2 (Necessary condition for existence of the solution). *Let the problem* (P) *have a solution. Then the condition*

$$L(p) \leqslant 0 \quad \forall\, p \in P_1, \quad p \geqslant 0 \ \text{ in } \ \Omega_s, \tag{3.8}$$

is fulfilled.

Proof. Let $w \in H^2(\Omega)$ be a solution of the problem (P) and $v = p \in P_1$, $p \geqslant 0$ in Ω_s. Then, from the variational equation (3.7), we obtain

$$L(p) = b(w^-, p) \leqslant 0.$$

□

Thus additional assumptions for the beam load must be considered. Therefore, the problem (P) is only semi-coercive (the functional J is not coercive in $H^2(\Omega)$ in general).

Lemma 3.3 (Necessary condition for uniqueness of the solution). *Let the problem* (P) *have a unique solution. Then the condition*

$$L(p) < 0 \quad \forall\, p \in P_1, \quad p > 0 \ \text{ in } \ \Omega_s, \tag{3.9}$$

is fulfilled. In addition, if (3.9) *is not fulfilled then the problem* (P) *has a solution if and only if the beam load satisfies*

$$L(p) = 0 \quad \forall\, p \in P_1. \tag{3.10}$$

Such a solution $w \in H^2(\Omega)$ is almost everywhere non-negative in Ω_s and also solves the Neumann problem

$$a(w, v) = L(v) \quad \forall\, v \in H^2(\Omega). \tag{3.11}$$

P r o o f. Suppose that the condition (3.9) is not fulfilled and there exists a solution w of the problem (P). Then, by Lemma 3.2, there exists a polynomial $\tilde{p} \in P_1$, $\tilde{p} > 0$ in Ω_s, such that $L(\tilde{p}) = 0$. If we substitute $\tilde{p}$ in the variational equation (3.7), we obtain $b(w^-, \tilde{p}) = 0$. Hence

$$w \geqslant 0 \quad \text{a.e. in } \Omega_s.$$

Therefore, if the solution w exists, it also solves the Neumann problem (3.11). The Neumann problem (3.11) has a solution if and only if the condition (3.10) is fulfilled. Therefore, the solution w exists if and only if

$$L(p) = 0 \quad \forall\, p \in P_1$$

is fulfilled. In such a case, the function $w + \overline{p}$, $\overline{p} \in P_1$, $\overline{p} \geqslant 0$ in Ω_s, also solves (3.11) and consequently (P). Therefore only the condition (3.9) can ensure the uniqueness of the solution of (P). $\square$

Theorem 3.1 (Necessary and sufficient condition for existence and uniqueness of a solution). *The problem* (P) *has a unique solution if and only if the condition* (3.9) *is fulfilled.*

P r o o f. We know that the condition (3.9) is necessary for the existence and uniqueness of the solution. Its sufficiency remains to be shown.

Existence. The existence of a solution of (P) can be proven by Theorem 2.4. To use this theorem, we must show that the functional J is coercive in $H^2(\Omega)$ if the beam load satisfies (3.9). Suppose that the functional J is not coercive in $H^2(\Omega)$. Then there exist a sequence $\{v_n\}_{n=1}^{+\infty} \subset H^2(\Omega)$, $\|v_n\|_{2,2} \to +\infty$, and a constant $c > 0$ such that

$$J(v_n) \leqslant c \quad \forall\, n \geqslant 1$$

or

$$0 \leqslant \tfrac{1}{2} a(v_n, v_n) + \tfrac{1}{2} b(v_n^-, v_n^-) \leqslant L(v_n) + c \quad \forall\, n \geqslant 1. \tag{3.12}$$

First, let us divide (3.12) by $\|v_n\|_{2,2}^2$. Then

$$E_0 I_0 |w_n|_{2,2}^2 + q_0 |w_n^-|_{0,2,\Omega_s}^2 \leqslant a(w_n, w_n) + b(w_n^-, w_n^-) \to 0, \quad n \to +\infty \tag{3.13}$$

where the sequence $\{w_n\}_{n=1}^{+\infty}$, $w_n := v_n/\|v_n\|_{2,2}$, is bounded in $H^2(\Omega)$. By Lemma 2.1, there exist a subsequence $\{w_{n_k}\}_{k=1}^{+\infty} \subset \{w_n\}_{n=1}^{+\infty}$ and $p \in P_1$ such that

$$w_{n_k} \to p \quad \text{in} \quad H^2(\Omega) \quad \text{for } k \to +\infty.$$

Moreover, the equality $|p^-|_{0,2,\Omega_s} = 0$ yields $p \geqslant 0$ in Ω_s.

Secondly, let us divide (3.12) by $\|v_n\|_{2,2}$. Then

$$0 \leqslant L(w_{n_k}) + c/\|v_{n_k}\|_{2,2} \to L(p).$$

Thus the condition (3.9) implies that $p = 0$. However, this is a contradiction with $1 = \|w_{n_k}\|_{2,2} \to \|p\|_{2,2} = 0$. Therefore, the functional J is coercive in $H^2(\Omega)$.

Uniqueness. Let $w_1, w_2 \in H^2(\Omega)$ solve the problem (P), i.e.

$$a(w_1, v) + b(w_1^-, v) = L(v) \quad \forall\, v \in H^2(\Omega), \tag{3.14}$$

$$a(w_2, v) + b(w_2^-, v) = L(v) \quad \forall\, v \in H^2(\Omega). \tag{3.15}$$

The choice $v = w_1 - w_2$ and subtraction of the equations (3.14) and (3.15) yield

$$a(w_1 - w_2, w_1 - w_2) + b(w_1^- - w_2^-, w_1 - w_2) = 0.$$

Hence and by the inequalities (3.2)–(3.4),

$$w_1 - w_2 = p,\;\; p \in P_1, \quad \text{and} \quad w_1^- - (w_1 - p)^- = 0 \;\text{ a.e. in } \Omega_s. \tag{3.16}$$

If there exists a set $M \subset \Omega_s$ with a positive one-dimensional Lebesgue measure such that $w_1 < 0$ in M, then (3.16) implies $p = 0$ in Ω. On the other hand, the case $w_1 \geqslant 0$ a.e. in Ω_s contradicts the condition (3.9)—it is enough to choose $v = 1$ in the variational equation (3.14). Therefore $w_1 = w_2$ a.e. in Ω. □

Theorem 3.1 can be generalized to the problem with more parts of the subsoils and also to the 2D case of thin elastic plates (see [7]).

Now, we rewrite the condition 3.9 equivalently for easier verification of the admissible loads.

Definition 1. Let L be a beam load which satisfies the condition (3.9). Then we can define *the load resultant*

$$F := L(1) = \int_\Omega f_1(x)\,\mathrm{d}x + \sum_{y \in X_P} P_y$$

and *the balance point*

$$T := L(x)/L(1) = \frac{\int_\Omega f_1(x)x\,\mathrm{d}x + \sum_{y \in X_P} P_y y + \sum_{y \in X_M} M_y}{\int_\Omega f_1(x)\,\mathrm{d}x + \sum_{y \in X_P} P_y}$$

of the load.

Lemma 3.4. *The condition* (3.9) *is fulfilled if and only if*

$$F < 0 \quad \text{and} \quad x_l < T < x_r. \tag{3.17}$$

P r o o f. Let $p_1(x) := 1$, $p_2(x) := x - x_l$ and $p_3(x) := x_r - x$. Then the inequalities $L(p_i) < 0$, $i = 1, 2, 3$, imply the condition (3.17). Conversely, let $p \in P_1$, $p > 0$ in Ω_s. Then there exist constants $c_2, c_3 \geqslant 0$, at least one of which is positive, such that $p = c_2 p_2 + c_3 p_3$. Hence,

$$L(p) = L(1)[c_2(T - x_l) + c_3(x_r - T)] < 0.$$

$\square$

The condition (3.17) means that the load resultant is situated in Ω_s and oriented against the subsoil. First degree polynomials p, $p > 0$ in Ω_s, represent the rigid beam motions for which the subsoil is not active.

Lemma 3.5 (Solution characterization)**.** *Let the condition* (3.9) *or* (3.17) *be fulfilled and let* $w \in H^2(\Omega)$ *be a solution of the problem* (P)*. Then the set*

$$M := \{x \in \Omega_s \colon\ w(x) < 0\}$$

has a positive one-dimensional Lebesgue measure. In addition, let $(x_l^M, x_r^M) \subset \Omega_s$ *be the smallest interval (convex closure) containing almost all* $x \in M$*. Then the balance point* T *belongs to* (x_l^M, x_r^M)*.*

P r o o f. The first part of the assertion has been shown in the proof of Theorem 3.1. Let (P_M) be an auxiliary beam problem with the unilateral subsoil in the interval (x_l^M, x_r^M). Since the solution w is non-negative almost everywhere in $\Omega_s \setminus (x_l^M, x_r^M)$, it also solves (P_M). Since $w < 0$ in $M \cap (x_l^M, x_r^M)$, the problem (P_M) has a unique solution by Lemma 3.3. Therefore, by Theorem 3.1, the condition (3.17) for the problem (P_M) holds. $\square$

3.4. Stability of the problem

The aim of the subsection is to analyze the continuous dependence of the solution w on the data E, I, q and L, and consequently describe the stability of the problem with respect to small changes in the data. We introduce the following notation for sets with addmissible data:

$$\begin{aligned}
\mathcal{D} &:= \{(E, I, q) \in L^\infty(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega_s)\colon\\
&\qquad E(x)I(x) \geqslant E_0 I_0 > 0 \text{ a.e. in } \Omega,\ q(x) \geqslant q_0 > 0 \text{ a.e. in } \Omega_s\},\\
\mathcal{S} &:= \{L \in V^*\colon\ F(L) < 0,\ x_l < T(L) < x_r\},\\
\mathcal{S}_\delta &:= \{L \in \mathcal{S}\colon\ T(L) \in (x_l + \delta, x_r - \delta)\},\\
\mathcal{S}_{\delta,\xi,\eta} &:= \{L \in \mathcal{S}_\delta\colon\ F(L) < -\xi < 0,\ \|L\|_* \leqslant \eta\},
\end{aligned}$$

where δ, ξ, η are positive constants and $F(L)$, $T(L)$ are the load resultant and the balance point corresponding to a load L. If $(E, I, q) \in \mathcal{D}$ and $L \in \mathcal{S}$ then we know that the problem (P) has a unique solution $w = w(E, I, q, L)$. The set of all such solutions will be denoted by $\mathcal{W}$, i.e.

$$\mathcal{W} := \{w \in H^2(\Omega)\colon\ \exists\,(E, I, q) \in \mathcal{D},\ \exists\, L \in \mathcal{S}\colon\ w = w(E, I, q, L) \text{ solves (P)}\}.$$

By analogy, we define sets $\mathcal{W}_\delta$ and $\mathcal{W}_{\delta,\xi,\eta}$ where the set $\mathcal{S}$ is replaced by $\mathcal{S}_\delta$ and $\mathcal{S}_{\delta,\xi,\eta}$, respectively.

Lemma 3.6. *Let δ be a positive parameter. Then there exists a constant $c > 0$, $c = c(\delta)$, such that*

$$c\|w\|_{2,2}^2 \leqslant |w|_{2,2}^2 + |w^-|_{0,2,\Omega_s}^2 \quad \forall\, w \in \mathcal{W}_\delta. \tag{3.18}$$

Proof. Suppose that the estimate (3.18) does not hold. Then there exists a sequence $\{w_n\}_{n=1}^{+\infty} \subset \mathcal{W}_\delta$ such that

$$|w_n|_{2,2}^2 + |w_n^-|_{0,2,\Omega_s}^2 < \frac{1}{n}\|w_n\|_{2,2}^2, \quad n \geqslant 1.$$

In the same manner as in the proof of Theorem 3.1, it can be shown that there exist a subsequence $\{w_{n_k}\}_{k=1}^{+\infty} \subset \{w_n\}_{n=1}^{+\infty}$ and a polynomial $p \in P_1$, $p > 0$, in Ω_s such that

$$v_{n_k} \to p \quad \text{in } H^2(\Omega), \quad v_{n_k} := w_{n_k}/\|w_{n_k}\|_{2,2}.$$

Since every solution w_{n_k}, $k \geqslant 1$, is negative somewhere in Ω_s by Lemma 3.5, the polynomial p satisfies $p(x_l) = 0$ or $p(x_r) = 0$. Suppose for example $p(x_l) = 0$ and $p > 0$ in Ω_s. Then the solutions w_{n_k} can only be negative in the interval $(x_l, x_l + \varepsilon_{n_k})$, where $0 < \varepsilon_{n_k} \to 0$. However, by Lemma 3.5, $T(L_{n_k}) \in (x_l, x_l + \varepsilon_{n_k})$. This is in contradiction with the definition of the set $\mathcal{S}_\delta$. Therefore the estimate (3.18) holds. $\square$

Theorem 3.2 (Boundedness of the solution). *Let δ be a positive parameter. Then there exists a constant $c > 0$, $c = c(\delta, E_0, I_0, q_0)$, such that*

$$\|w(E, I, q, L)\|_{2,2} \leqslant c\|L\|_* \quad \forall\, (E, I, q, L) \in \mathcal{D} \times \mathcal{S}_\delta, \tag{3.19}$$

where $w = w(E, I, q, L)$ is the solution of the problem (P) *with the data E, I, q, L.*

Proof. By Lemma 3.6, the inequalities (3.2), (3.3) and the variational equation (3.7), we obtain

$$\begin{aligned}\|w\|_{2,2}^2 &\leqslant c(E_0 I_0 |w|_{2,2}^2 + q_0 |w^-|_{0,2,\Omega_s}^2) \\ &\leqslant c\big(a(w, w) + b(w^-, w^-)\big) \\ &= cL(w) \leqslant c\|L\|_* \|w\|_{2,2} \quad \forall\, (E, I, q, L) \in \mathcal{D} \times \mathcal{S}_\delta.\end{aligned}$$

□

Lemma 3.7. *Let δ, ξ, η be positive parameters. Let $\{w_n\}_{n=1}^{+\infty}$ be a sequence of solutions belonging to the set $\mathcal{W}_{\delta,\xi,\eta}$. Then there exist its subsequence $\{w_{n'}\}_{n'}$, a function $w \in H^2(\Omega)$ and a set $M \subset \Omega_s$ with a positive one-dimensional Lebesgue measure such that $w_{n'} \rightharpoonup w$ weakly in $H^2(\Omega)$ for $n' \to +\infty$ and $w_{n'}, w < 0$ in M for all n'.*

Proof. The sequence $\{w_n\}_{n=1}^{+\infty}$, is bounded in $H^2(\Omega)$ by Theorem 3.2 and by the definition of the set $\mathcal{W}_{\delta,\xi,\eta}$. Therefore there exists a subsequence $\{w_{n'}\}_{n'} \subset \{w_n\}_n$ which has a weak limit w in the space $H^2(\Omega)$. By Theorems 2.2 and 2.1, $w_{n'} \to w$ in $H^1(\Omega)$ and consequently in $C(\overline{\Omega})$. Then the choice $v = 1$ in the variational equation (3.7) and the inequality (2.6) yield

$$0 > -\xi \geqslant L_{n'}(1) = b(w_{n'}^-, 1) \to b(w^-, 1).$$

Hence, we can find a set $M \subset \Omega_s$ with a positive Lebesgue one-dimensional measure, such that $w_{n'}, w < 0$ in M for sufficiently large n'. □

Lemma 3.8. *Let δ, ξ, η be positive parameters. Then there exists a positive constant $c = c(\delta, \xi, \eta)$ such that*

$$c\|w_1 - w_2\|_{2,2}^2 \leqslant |w_1 - w_2|_{2,2}^2 + |w_1^- - w_2^-|_{0,2,\Omega_s}^2 \quad \forall\, w_1, w_2 \in \mathcal{W}_{\delta,\xi,\eta}. \tag{3.20}$$

Proof. Suppose that the inequality (3.20) does not hold. Then there exist sequences $\{w_{i,n}\}_{n=1}^{+\infty} \subset \mathcal{W}_{\delta,\xi,\eta}$, $i = 1, 2$, such that

$$|w_{1,n} - w_{2,n}|_{2,2}^2 + |w_{1,n}^- - w_{2,n}^-|_{0,2,\Omega_s}^2 < \frac{1}{n}\|w_{1,n} - w_{2,n}\|_{2,2}^2 \quad \forall\, n \geqslant 1. \tag{3.21}$$

By virtue of Lemma 3.7 we can assume that there exist $w_{1,0}, w_{2,0} \in H^2(\Omega)$ and sets $M_1, M_2 \subset \Omega_s$ with positive one-dimensional Lebesgue measures such that $w_{i,n} \rightharpoonup w_{i,0}$ in $H^2(\Omega)$ for $n \to +\infty$ and $w_{i,n}, w_{i,0} < 0$ in M_i for all n, $i = 1,2$. Let $v_{i,n} := w_{i,n}/\|w_{1,n} - w_{2,n}\|_{2,2}$, $i = 1,2$. By Lemma 2.1 there exist subsequences $\{w_{i,n'}\}_{n'} \subset \{w_{i,n}\}_n$, $i = 1,2$, and a polynomial $p \in P_1$ such that the inequality (3.21) yields

$$v_{1,n'} - v_{2,n'} \to p \in P_1 \text{ in } H^2(\Omega) \quad \text{and} \quad v^-_{1,n'}(x) - v^-_{2,n'}(x) \to 0 \text{ a.e in } \Omega_s. \tag{3.22}$$

First, suppose

$$\|w_{1,n'} - w_{2,n'}\|_{2,2} \to 0.$$

Then $w_{1,0} = w_{2,0}$ and it is possible to choose M_1, M_2 such that $M_1 = M_2 =: M$. Hence, (3.22) yields

$$v^-_{1,n'} - v^-_{2,n'} = v_{1,n'} - v_{2,n'} \to 0 \quad \text{a.e. in } M$$

for sufficiently large n'. Thus $p = 0$.

Secondly, suppose

$$\exists\, c_1 > 0\colon\ \|w_{1,n'} - w_{2,n'}\|_{2,2} \geqslant c_1 \quad \forall\, n'.$$

Then the sequences $\{v_{i,n'}\}_{n'}$ are bounded in $H^2(\Omega)$ and there exist their subsequences $\{v_{i,n''}\}_{n''}$ with weak limits v_i in $H^2(\Omega)$, $i = 1,2$. Hence, the convergences (3.22) yield

$$v_1^- = v_2^- = (v_1 - p)^- \quad \text{a.e. in } \Omega_s.$$

Since the sequence $\{w_{1,n}\}_n$ is bounded in $H^2(\Omega)$ and $w_{1,0} < 0$ in M_1, also $v_1 < 0$ in M_1. Thus $p = 0$.

However, the case $p = 0$ contradicts

$$1 = \|v_{1,n'} - v_{2,n'}\|_{2,2} \to \|p\|_{2,2} = 0.$$

Therefore, the estimate (3.20) holds. $\square$

Theorem 3.3. *Let δ, ξ, η be positive parameters. Then there exists a positive constant $c = c(\delta, \xi, \eta, E_0, I_0, q_0)$ such that*

$$\|w_1 - w_2\|_{2,2} \leqslant c(\|L_1 - L_2\|_* + \|q_1 - q_2\|_{\infty,\Omega_s} + \|E_1 I_1 - E_2 I_2\|_{\infty,\Omega}) \tag{3.23}$$

for all $(E_i, I_i, q_i, L_i) \in \mathcal{D} \times \mathcal{S}_{\delta,\xi,\eta}$, where $w_i = w_i(E_i, I_i, q_i, L_i)$ solve the problem (P) *with the parameters E_i, I_i, q_i, L_i, $i = 1, 2$.*

P r o o f. Let a_i, b_i denote the forms a, b with respect to the parameters E_i, I_i, q_i and let $w_i = w_i(E_i, I_i, q_i, L_i) \in \mathcal{W}_{\delta,\xi,\eta}$, $i = 1, 2$, solve the problem (P), i.e.

$$\begin{aligned} a_1(w_1, v) + b_1(w_1^-, v) &= L_1(v) \quad \forall\, v \in H^2(\Omega), \\ a_2(w_2, v) + b_2(w_2^-, v) &= L_2(v) \quad \forall\, v \in H^2(\Omega). \end{aligned}$$

The choice $v = w_1 - w_2$ and subtraction of the two equations yield

$$\begin{aligned} & a_1(w_1 - w_2, w_1 - w_2) + b_1(w_1^- - w_2^-, w_1 - w_2) \\ &\quad = (a_2 - a_1)(w_2, w_1 - w_2) + (b_2 - b_1)(w_2^-, w_1 - w_2) + (L_1 - L_2)(w_1 - w_2) \\ &\quad \leqslant \|w_1 - w_2\|_{2,2}(\|E_1 I_1 - E_2 I_2\|_{\infty,\Omega}\|w_2\|_{2,2} \\ &\qquad\qquad + \|q_1 - q_2\|_{\infty,\Omega_s}\|w_2\|_{2,2} + \|L_1 - L_2\|_*) \\ &\quad \leqslant c_1\|w_1 - w_2\|_{2,2}(\|E_1 I_1 - E_2 I_2\|_{\infty,\Omega} + \|q_1 - q_2\|_{\infty,\Omega_s} + \|L_1 - L_2\|_*), \end{aligned}$$

where the constant $c_1 > 0$ depends on δ, η, E_0, I_0, q_0 by Theorem 3.2 and the definition of the set $\mathcal{W}_{\delta,\xi,\eta}$. Since Lemma 3.8 and the inequalities (3.2)–(3.4) yield

$$\begin{aligned} \|w_1 - w_2\|_{2,2}^2 &\leqslant c_2(E_0 I_0 |w_1 - w_2|_{2,2}^2 + q_0 |w_1^- - w_2^-|_{0,2,\Omega_s}^2) \\ &\leqslant c_2[a_1(w_1 - w_2, w_1 - w_2) + b_1(w_1^- - w_2^-, w_1 - w_2)] \end{aligned}$$

with $c_2 = c_2(\delta, \xi, \eta, E_0, I_0, q_0) > 0$, the estimate (3.23) holds. □

By Theorem 3.3, the stability of the problem solution depends on the constant c in the estimate (3.23), i.e. on the parameters δ, ξ, η. Since

$$\mathcal{S}_{\delta_1,\xi_1,\eta_1} \subset \mathcal{S}_{\delta_2,\xi_2,\eta_2} \subset \mathcal{S}, \quad \forall\, \delta_i, \xi_i, \eta_i,\ i = 1, 2,\ \delta_1 \geqslant \delta_2,\ \xi_1 \geqslant \xi_2,\ \eta_1 \leqslant \eta_2,$$

we conclude that $c(\delta_1, \xi_1, \eta_1) \leqslant c(\delta_2, \xi_2, \eta_2)$.

Now, we describe two unstable cases of the load in the problem (P). In the first, suppose that the balance point T is close to the ends of the subsoil. Then a small change of the load can cause a displacement of the balance point beyond the interval Ω_s and a subsequent overturn of the beam from the subsoil.

The second unstable case can occur if the load resultant F has too small size in comparison with the V^*-norm of the load. In such a case, a small change of the load can cause a large change of the balance point T. For example, let $\Omega = (0,1)$, $\Omega_s = (0.1, 0.9)$, $EI = \mathrm{const}_1 > 0$, $q = \mathrm{const}_2 > 0$ and

$$L_1(v) := \int_0^{0.1} 100v(x)\,\mathrm{d}x + \int_{0.4}^{0.6} -101v(x)\,\mathrm{d}x + \int_{0.9}^{1} 100v(x)\,\mathrm{d}x,$$
$$L_2(v) := \int_0^{0.1} 100v(x)\,\mathrm{d}x + \int_{0.4}^{0.6} -101v(x)\,\mathrm{d}x + \int_{0.9}^{1} 101v(x)\,\mathrm{d}x.$$

Then $F_1 = -0.2$, $T_1 = 0.5$, $F_2 = -0.1$ and $T_2 = 0.05$. Though the difference between the loads L_1 and L_2 is relative small, the problem (P) has no solution for the load L_2, since $T_2 \notin \Omega_s$.

Corollary 3.1. *Let* $\{(E_n, I_n, q_n, L_n)\}_{n=1}^{+\infty} \subset \mathcal{D} \times \mathcal{S}$ *and let* $(E, I, Q, L) \in \mathcal{D} \times \mathcal{S}$ *be an admissible data such that*

$$E_n \to E,\ \ I_n \to I,\ \ q_n \to q \ \text{ in } \ L^\infty(\Omega), \quad L_n \to L \ \text{ in } \ V^*.$$

Let w_n, w *be solutions of the problem* (P) *with the data* (E_n, I_n, q_n, L_n) *and* (E, I, Q, L)*. Then*

$$w_n \to w \quad \text{in } \ H^2(\Omega).$$

4. Approximation of the problem

The aim of the section is to set a suitable family of problem approximations and analyze their solvability and relation to the original problem (P). First, we define families of subspaces and bilinear forms, which approximate the space $H^2(\Omega)$ and the bilinear form b, respectively. Then, we set a finite element approximation of the problem (P) and summarize its properties. Finally, we analyze the relation between the original problem and its approximations, subject to the smoothness of the problem solution.

4.1. Finite element approximations of the space $H^2(\Omega)$

Let us define a partition τ_h,

$$0 = x_0 < x_1 < x_2 < \ldots < x_n = l,$$

of the interval $\overline{\Omega} = [0, l]$, with the nodal points x_j, $j = 0, 1, \ldots, n$, and with the discretization parameter $h > 0$,

$$h := \max_{j=1,\ldots,n} H_j, \quad H_j := x_j - x_{j-1}.$$

The partition τ_h can also be characterized by the parameter

$$h_{\min} := \min_{j=1,\ldots,n} H_j.$$

The system of all partitions $\{\tau_h\}_{h>0}$ of the interval $\overline{\Omega}$ will be denoted by $\mathcal{T}$. The system of all "strong" regular partitions of the interval $\overline{\Omega}$ with respect to a parameter $\theta > 0$ will be defined by

$$\mathcal{T}_\theta := \{\tau_h \in \mathcal{T} \colon\ h \leqslant \theta h_{\min}\}.$$

The interval $(x_{j-1}, x_j) \in \tau_h$, $j = 1, 2, \ldots, n$, will be denoted by Ω_j.

With respect to a partition $\tau_h \in \mathcal{T}$ with $n+1$ nodal points, we will define the function space

$$V_h \subset H^2(\Omega), \quad V_h := \{v_h \in C^1(\overline{\Omega}) \colon\ v_h|_{\Omega_j} \in P_3,\ \Omega_j \in \tau_h,\ j = 1, 2, \ldots, n\},$$

i.e. the space of continuously differentiable and piecewise cubic functions.

Let v be any function belonging to $H^2(\Omega)$. Let $\tau_h \in \mathcal{T}$ be a partition of the interval Ω and V_h the corresponding function space. By Theorem 2.1 we can define the interpolation

$$r_h \colon\ H^2(\Omega) \mapsto V_h, \quad (r_h(v))^{(i)}(x_j) = v^{(i)}(x_j),\ \ i = 0, 1, \tag{4.1}$$

at the nodal points x_j, $j = 0, 1, \ldots, n$, of the partition τ_h.

Theorem 4.1. *Let $\mathcal{T}$ be the system of all partitions of the interval $\overline{\Omega}$. Then there exist constants $c_1, c_2 > 0$ such that the following estimates hold:*

$$\|v - r_h(v)\|_{2,2} \leqslant c_1 h^2 |v|_{4,2} \quad \forall\, v \in H^4(\Omega),\ \forall \tau_h \in \mathcal{T}, \tag{4.2}$$

$$\|v - r_h(v)\|_{2,2} \leqslant c_2 h |v|_{3,2} \quad \forall\, v \in H^3(\Omega),\ \forall\, \tau_h \in \mathcal{T}, \tag{4.3}$$

$$\|v - r_h(v)\|_{2,2} \to 0 \quad \forall\, v \in H^2(\Omega),\ h \to 0 \tag{4.4}$$

where r_h is the interpolation defined by (4.1) *of the space $H^2(\Omega)$ onto V_h, which corresponds to a partition τ_h.*

Theorem 4.1 is proven for a more general case in the book [2]. The theorem says the system $\{V_h\}_h$ approximates the space $H^2(\Omega)$.

4.2. Approximation of the bilinear form b

The evaluation of the term $b(w_h^-, v_h)$, $w_h, v_h \in V_h$, cannot be computed directly due to the non-linear term w_h^-. Therefore, an approximation of the form b must be used.

We will apply the reference numerical quadrature (2.7) to approximate the form b. For the sake of simplicity, we will assume that the function q, which represents the stiffness coefficient of the subsoil, is piecewise constant in the interval Ω_s. Therefore, we introduce the notation $\mathcal{T}'$ and $\mathcal{T}'_\theta$, $\theta > 0$, for all partitions and strong regular partitions, respectively, of the interval $\overline{\Omega}$ that include the points x_l, x_r and the points where the function q is not continuous.

Let $\tau_h \in \mathcal{T}'$ be a partition with nodal points

$$0 = x_0 < x_1 < \ldots < x_l \equiv x_{j_l - 1} < \ldots < x_r \equiv x_{j_r} < \ldots < x_n = l.$$

Let Φ_j, $j = j_l, j_l{+}1, \ldots, j_r$, be the transformation of the interval $\overline{\Omega}_j = [x_{j-1}, x_j]$ onto the interval $[-1, 1]$ defined by (2.8), with $s = x_{j-1}$ and $t = x_j$. Let $y_{j,i} := \Phi_j^{-1}(\hat{y}_i)$ and $\omega_{j,i} := H_j \hat{\omega}_i / 2$, $H_j = x_j - x_{j-1}$, $i = 1, 2, \ldots, m$, be the points and weights corresponding to $\hat{y}_i$ and $\hat{\omega}_i$ of the reference numerical quadrature (2.7). The situation is depicted in Fig. 2 for a 1-point numerical quadrature.

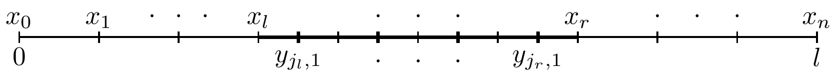

Figure 2. Scheme of the partition.

Then the bilinear form b can be approximated by the bilinear form

$$b_h(v_1, v_2) := \sum_{j=j_l}^{j_r} \left(q_j \sum_{i=1}^{m} \omega_{j,i} v_1(y_{j,i}) v_2(y_{j,i}) \right), \tag{4.5}$$

defined on $H^2(\Omega)$ and associated with the partition τ_h and the reference numerical quadrature. If the reference numerical quadrature is exact for polynomials of the kth degree, we introduce the notation $b_h \in \mathcal{B}^k_{\tau_h}$, $k = 0, 1, 2, \ldots$.

In fact, we approximate the subsoil by insulated springs at the quadrature points $y_{j,i}$, $j = j_l, j_l + 1, \ldots, j_r$, $i = 1, 2, \ldots, m$. The set of these points will be denoted by Q_h.

Lemma 4.1 (Uniform boundedness of approximated bilinear forms b_h). *Let $\tau_h \in \mathcal{T}'$ and $b_h \in \mathcal{B}^0_{\tau_h}$. Then there exists a positive constant c depending only on the subsoil length $x_r - x_l$ such that*

$$|b_h(u,v)| \leqslant c\|q\|_{\infty,\Omega_s}\|u\|_{1,2}\|v\|_{1,2} \quad \forall\, u,v \in H^1(\Omega),$$

where $q \in L^\infty(\Omega_s)$ is the function representing the stiffness coefficient of the subsoil.

P r o o f. By Theorem 2.1 we obtain

$$\begin{aligned}
|b_h(u,v)| &\leqslant \|q\|_{\infty,\Omega_s} \sum_{j=j_l}^{j_r}\sum_{i=l}^{m} \omega_{j,i}|u(y_{j,i})||v(y_{j,i})| \\
&\leqslant c_1\biggl(\sum_{j=j_l}^{j_r}\sum_{i=l}^{m}\omega_{j,i}\biggr)\|q\|_{\infty,\Omega_s}\|u\|_{1,2}\|v\|_{1,2} \\
&= c_1\biggl(\int_{\Omega_s} 1\,\mathrm{d}x\biggr)\|q\|_{\infty,\Omega}\|u\|_{1,2}\|v\|_{1,2} \\
&= c\|q\|_{\infty,\Omega_s}\|u\|_{1,2}\|v\|_{1,2} \quad \forall\, u,v \in H^1(\Omega),
\end{aligned}$$

with $c = c_1(x_r - x_l)$. $\square$

Theorem 4.2. *Let $\tau_h \in \mathcal{T}'$ be any partition and $b_h \in \mathcal{B}^1_{\tau_h}$ an approximated bilinear form. Let $0 \leqslant N < +\infty$ and*

$$\mathcal{V}_N := \Bigl\{ v \in H^2(\Omega)\colon \exists\, p \leqslant N,\ \exists\, z_1, z_2, \ldots, z_{2p} \in \overline{\Omega}_s\colon \\ \{x \in \overline{\Omega}_s\colon\ v^-(x) = 0\} = \bigcup_{i=1}^{p}[z_{2i-1}, z_{2i}] \Bigr\}.$$

Then there exist positive constants c_1 and $c_2 = c_2(N)$, independent of the partition τ_h, such that

$$|b(v^-,u) - b_h(v^-,u)| \leqslant c_1 h\|v\|_{1,2}\|u\|_{1,2} \quad \forall\, u,v \in H^1(\Omega), \tag{4.6}$$

$$|b(v^-,u) - b_h(v^-,u)| \leqslant c_2 h^2\|v\|_{2,2}\|u\|_{2,2} \quad \forall\, u \in H^2(\Omega),\ \forall\, v \in \mathcal{V}_N. \tag{4.7}$$

P r o o f. We prove only the estimate (4.7). The proof of the estimate (4.6) is easier and does not contain anything new in comparison with (4.7). Let $\tau_h \in \mathcal{T}'$ be a partition with nodal points

$$0 = x_0 < x_1 < \ldots < x_l = x_{j_l-1} < \ldots < x_r = x_{j_r} < \ldots < x_n.$$

Let $u \in H^2(\Omega)$, $v \in \mathcal{V}_N$ and

$$\mathcal{I}_v := \{j \in \{j_l, j_l+1, \ldots, j_r\}\colon\ v^- \in H^1(\Omega_j) \setminus H^2(\Omega_j)\}, \quad \Omega_j = (x_{j-1}, x_j).$$

Clearly, $\operatorname{card}(\mathcal{I}_v) \leqslant 2N$ and $v^- = v$ in Ω_j, $j \notin \mathcal{I}_v$. Hence, by Theorem 2.5, the Cauchy-Schwarz inequality and Lemmas 2.4, 2.5 and 2.3 we obtain

$$\begin{aligned}
|b(v^-, u) - b_h(v^-, u)| &\leqslant \sum_{j=j_l}^{j_r} q_j \left| \int_{\Omega_j} v^- u \,\mathrm{d}x - \sum_{i=1}^{m} \omega_{j,i} v^-(y_{j,i}) u(y_{j,i}) \right| \\
&\leqslant c_1 \sum_{j \in \mathcal{I}_v} H_j |v^- u|_{1,1,\Omega_j} + c_2 \sum_{j \notin \mathcal{I}_v} H_j^2 |vu|_{2,1,\Omega_j} \\
&\leqslant c_1 \sum_{j \in \mathcal{I}_v} H_j (|v|_{0,2,\Omega_j} |u|_{1,2,\Omega_j} + |v|_{1,2,\Omega_j} |u|_{0,2,\Omega_j}) \\
&\qquad\qquad + c_2 \sum_{j \notin \mathcal{I}_v} H_j^2 \|v\|_{2,2,\Omega_j} \|u\|_{2,2,\Omega_j} \\
&\leqslant (N\tilde{c}_1) h^2 \|v\|_{2,2} \|u\|_{2,2} + \tilde{c}_2 h^2 \|v\|_{2,2} \|u\|_{2,2} \\
&\leqslant c(N) h^2 \|v\|_{2,2} \|u\|_{2,2},
\end{aligned}$$

where $H_j = x_j - x_{j-1}$, $j = j_l, \ldots, j_r$. □

R e m a r k 4.1. The set $\mathcal{V}_N$ is sufficiently wide to contain mechanically reasonable functions from $H^2(\Omega)$ which can represent the beam deflection. Since the number N does not depend on the partition $\tau_h \in \mathcal{T}'$, the order of convergence b_h to b is two.

The following lemmas will be useful for convergence analysis of approximated solutions to the solution of the problem (P), see subsection 4.4.

Lemma 4.2. *Let $\{\tau_{h_k}\}_{k=1}^{+\infty} \subset \mathcal{T}'$ be a sequence of partitions and let $\{u_k\}_k, \{v_k\}_k$ be sequences, defined and bounded in $H^1(\Omega)$, such that*

$$b_{h_k}(u_k, v_k) \to 0, \quad k \to +\infty,$$

where the bilinear form b_{h_k} belongs to $\mathcal{B}^0_{\tau_{h_k}}$. If $h_k \to 0$ then also

$$b(u_k, v_k) \to 0, \quad k \to +\infty.$$

P r o o f. By the inequality (4.6) in Theorem 4.2 we obtain

$$\begin{aligned}
|b(u_k, v_k)| &\leqslant |b_{h_k}(u_k, v_k) - b(u_k, v_k)| + |b_{h_k}(u_k, v_k)| \\
&\leqslant c_1 h_k \|u_k\|_{1,2} \|v_k\|_{1,2} + |b_{h_k}(u_k, v_k)| \\
&\leqslant c_2 h_k + |b_{h_k}(u_k, v_k)| \to 0.
\end{aligned}$$

□

Lemma 4.3. *Let $\{\tau_{h_k}\}_{k=1}^{+\infty} \subset \mathcal{T}'$ be a sequence of partitions and $\{b_{h_k}\}_{k=1}^{+\infty}$ the corresponding sequence of the approximated forms $b_{h_k} \in \mathcal{B}^0_{\tau_{h_k}}$. Let $\{v_k\}_{k=1}^{+\infty}$ be a sequence of functions belonging to $H^1(\Omega)$ such that $v_k \to p$ in $H^1(\Omega)$, where $p \in P_1$. Let $0 \leqslant h < \frac{1}{2}(x_r - x_l)$ be a parameter such that $h_k \to h$. If $b_{h_k}(v_k^-, v_k^-) \to 0$ for $k \to +\infty$ then*

$$b_{h_k}(p^-, p^-) \to 0, \quad k \to +\infty \quad \text{and} \quad p \geqslant 0 \ \text{ in } \ [x_l + h, x_r - h].$$

P r o o f. By the inequality (2.6) and Lemma 4.1, the convergence $b_{h_k}(v_k^-, v_k^-) \to 0$ implies $b_{h_k}(p^-, p^-) \to 0$, since

$$\begin{aligned} b_{h_k}(p^-, p^-) &= b_{h_k}(p^- - v_k^-, v_k^-) + b_{h_k}(p^- - v_k^-, p^-) + b_{h_k}(v_k^-, v_k^-) \\ &\leqslant c_1 \|v_k - p\|_{1,2} + b_{h_k}(v_k^-, v_k^-) \to 0. \end{aligned}$$

Suppose that the inequality $p \geqslant 0$ in $[x_l + h, x_r - h]$ does not hold. Then there exists $\varepsilon > 0$ such that $p < 0$ in $[x_l, x_l + h + \varepsilon]$ or $p < 0$ in $[x_r - h - \varepsilon, x_r]$. We will assume the former case. The proof of the latter is similar. Then there exist a positive constant c and an index k_0 such that $p^2 \geqslant c$ in $[x_l, x_l + h + \varepsilon]$ and $h_k - h < \varepsilon/2$ for any $k \geqslant k_0$, since $h_k \to h$. Let

$$0 = x_0^k < x_1^k < \ldots < x_l = x_{j_l(k)-1}^k < \ldots < x_r = x_{j_r(k)}^k < \ldots < x_{n(k)}^k = l$$

be the nodal points of the partition τ_{h_k} and let $y_{j,i}^k \in Q_{h_k}$, $j = j_l(k), \ldots, j_r(k)$, $i = 1, \ldots, m$, be the corresponding points of the numerical quadrature. Let $j(k) \in \{j_l(k), \ldots, j_r(k)\}$ be a maximal index such that $x_{j(k)}^k \leqslant x_l + h + \varepsilon$. Then

$$x_{j(k)}^k \geqslant x_l + h + \varepsilon - h_k > x_l + \varepsilon/2$$

and

$$\begin{aligned} b_{h_k}(p^-, p^-) &= \sum_{j=j_l(k)}^{j_r(k)} \Big(q_j^k \sum_{i=1}^m \omega_{j,i}^k (p^-(y_{j,i}^k))^2 \Big) \\ &\geqslant q_0 \sum_{j=j_l(k)}^{j(k)} \sum_{i=1}^m \omega_{j,i}^k p^2(y_{j,i}^k) \\ &\geqslant q_0 c \int_{x_l}^{x_{j(k)}^k} 1 \, \mathrm{d}x > q_0 c \frac{\varepsilon}{2} > 0. \end{aligned}$$

This contradicts $b_{h_k}(p^-, p^-) \to 0$. $\square$

4.3. Setting of approximated problems

For the sake of simplicity, we will not consider numerical quadrature of the forms a and L. Let $\tau_h \in \mathcal{T}'$ be a partition of the interval Ω with the discretization parameter $h > 0$ and let V_h, b_h be the corresponding approximations of the space $H^2(\Omega)$ and the bilinear form b, respectively. The approximated problem corresponding to the partition τ_h has the form

$$(\mathrm{P}_h) \qquad \begin{cases} \text{find } w_h \in V_h\colon\ J_h(w_h) \leqslant J_h(v_h) \ \ \forall\, v_h \in V_h, \\ J_h(v_h) := \frac{1}{2}a(v_h, v_h) + \frac{1}{2}b_h(v_h^-, v_h^-) - L(v_h). \end{cases}$$

Since V_h is a closed subspace of $H^2(\Omega)$, the approximated problem (P_h) has properties similar to the original problem (P), except small differences caused by the numerical quadrature. Therefore, the properties of (P_h) will be summarized more briefly.

Lemma 4.4. *The functional J_h is convex and has the Gâteaux derivative on the space V_h. In addition, a function $w_h \in V_h$ solves the problem (P_h) if and only if it solves the nonlinear variational equation*

$$a(w_h, v_h) + b_h(w_h^-, v_h) = L(v_h) \quad \forall\, v_h \in V_h. \tag{4.8}$$

Lemma 4.5 (Necessary condition for existence of a solution). *Let the problem (P_h) have a solution. Then the condition*

$$L(p) \leqslant 0 \quad \forall\, p \in P_1,\ \ p \geqslant 0 \ \text{ in } \ (y_{j_l,1}, y_{j_r,m}) \tag{4.9}$$

is fulfilled, where $y_{j_l,1} \in Q_h$ is the first point of the numerical quadrature (2.9) *in the interval (x_{j_l-1}, x_{j_l}) and $y_{j_r,m} \in Q_h$ is the last point of the numerical quadrature* (2.9) *in the interval (x_{j_r-1}, x_{j_r}) (see Fig.* 2*).*

In general, the condition (4.9) is more restrictive than (3.8) which must hold for the original problem (P). This is caused by the fact that the subsoil is situated in the interval Ω_s, whilst the "springs" which approximate the subsoil are situated only in the interval $[y_{j_l,1}, y_{j_r,m}] \subset \overline{\Omega}_s$.

Lemma 4.6 (Necessary condition for uniqueness of the solution). *Let the problem* (P_h) *have a unique solution. Then the condition*

$$L(p) < 0 \quad \forall\, p \in P_1,\ \ p > 0 \ \ \textit{in}\ \ (y_{j_l,1}, y_{j_r,m}) \tag{4.10}$$

is fulfilled.

If the problem (P_h) has a solution and the condition (4.10) does not hold then the equilibrium condition $L(p) = 0$ for all $p \in P_1$ need not be fulfilled in comparison with Lemma 3.3. For example, if $L(v) = -v(y_{j_l,1})$ then the problem (P_h) is solved by every polynomial $p \in P_1$ such that $p(y_{j_l,1}) = -1/q_1$ and $p(y) \geqslant 0$ for any $y \in Q_h$, $y \neq y_{j_l,1}$.

Theorem 4.3 (Necessary and sufficient condition for existence and uniqueness of the solution). *Let the discretization parameter* $h > 0$ *be sufficiently small. Then the problem* (P_h) *has a unique solution if and only if the condition* (4.10) *is fulfilled.*

It is possible to prove existence of a solution without the assumption on h. The assumption of sufficiently small h ensures the uniqueness of the solution. For larger h, it is possible that a solution w_h activates only one spring $y \in Q_h$. If the spring y is situated at the balance point T, then there exists a polynomial $p \in P_1$ such that $p(T) = 0$ and $w_h(z) + p(z) \geqslant 0$ for any $z \in Q_h$, $z \neq y$. Clearly, the function $w_h + p$ is also a solution of (P_h), since $J_h(w_h) = J_h(w_h + p)$. The size of the parameter h, which ensures the uniqueness of the solution, depends on the beam load. This assertion will be justified by Lemmas 4.8 and 4.9.

Lemma 4.7. *The condition* (4.10) *is fulfilled if and only if*

$$F < 0 \quad \textit{and} \quad y_{j_l,1} < T < y_{j_r,m}, \tag{4.11}$$

where $F = L(1)$ *is the load resultant and* $T = L(x)/L(1)$ *is the balance point of the load.*

Notice that if the condition (3.17) holds and the discretization parameter h is sufficiently small, then the condition (4.11) also holds.

Lemma 4.8 (Solution characteristic). *Let the condition* (4.10) *or* (4.11) *be fulfilled and let* $w_h \in V_h$ *be any solution of the problem* (P_h). *Then the set*

$$\mathcal{A} := \{y \in Q_h \colon\ w_h(y) < 0\}$$

of active springs is non-empty. In addition, let $x_l^{\mathcal{A}}, x_r^{\mathcal{A}} \in \mathcal{A}$ be the outer springs of the set $\mathcal{A}$, i.e. $x_l^{\mathcal{A}} \leqslant y \leqslant x_r^{\mathcal{A}}$ for any $y \in \mathcal{A}$. Then the balance point $T \in [x_l^{\mathcal{A}}, x_r^{\mathcal{A}}]$. If the solution w_h is not unique then $x_l^{\mathcal{A}} = x_r^{\mathcal{A}}$.

Notice that if the discretization parameter h is sufficienly small then the set $\mathcal{A}$ contains more than one active spring and the problem (P_h) has a unique solution. The smaller the parameter h, the more active springs.

4.4. Convergence analysis

In this subsection we will analyze the relation between the solution w of the problem (P) and its approximation w_h solving the problem (P_h) in dependence on the beam load L and the partition τ_h. The dependence on the data E, I, q will not be considered. However, it can be shown, similarly to subsection 3.4, that the relation depends on the data E, I, q only through the parameters E_0, I_0, q_0. Further, we will assume that the reference numerical quadrature, which determines the form b_h, is the same for all partitions $\tau_h \in \mathcal{T}'$ and is exact for polynomials of the first degree.

We recall the notation $\mathcal{S}$, $\mathcal{S}_\delta$ and $\mathcal{S}_{\delta,\xi,\eta}$, with positive parameters δ, ξ, η, introduced in subsection 3.4 for the classes of the beam loads. The solutions of the problems (P) and (P_h), which depend on the load $L \in \mathcal{S}$, will be denoted by $w = w(L)$ and $w_h = w_h(L)$, respectively.

Theorem 4.4 (Uniform boundedness of approximated solutions). *Let δ be a positive parameter. Then there exists a positive constant $c = c(\delta)$ such that*

$$\|w_h(L)\|_{2,2} \leqslant c\|L\|_* \quad \forall\, L \in \mathcal{S}_\delta,\ \forall\, \tau_h \in \mathcal{T}',\ 0 < h \leqslant \delta. \tag{4.12}$$

Notice that the solution $w_h(L)$ exists for any $L \in \mathcal{S}_\delta$ and $\tau_h \in \mathcal{T}'$, since the assumption that the reference numerical quadrature is exact for polynomials of the first degree implies $[x_l + \delta, x_r - \delta] \subset (y_{j_l,1}, y_{j_r,m})$, i.e. the condition (4.11) is fulfilled. The solution $w_h(L)$ need not be unique in general, but it can be characterized by Lemma 4.8.

Proof. Due to the variational equation (4.8), it is sufficient to show the inequality

$$\|w_h\|_{2,2}^2 \leqslant c[a(w_h, w_h) + b_h(w_h^-, w_h^-)] \quad \forall\, L \in \mathcal{S}_\delta,\ \forall\, \tau_h \in \mathcal{T}',\ 0 < h \leqslant \delta. \tag{4.13}$$

Suppose that the inequality (4.13) does not hold. Then there exist sequences $\{L_k\}_{k=1}^{+\infty} \subset \mathcal{S}_\delta$ and $\{\tau_{h_k}\}_{k=1}^{+\infty}$ such that the inequalities

$$a(w_{h_k}, w_{h_k}) + b_{h_k}(w_{h_k}^-, w_{h_k}^-) < \frac{1}{k}\|w_{h_k}\|_{2,2}^2, \quad k \geqslant 1,$$

hold, where $w_{h_k} = w_{h_k}(L_k)$. Let $v_{h_k} := w_{h_k}/\|w_{h_k}\|_{2,2}$. Since the sequences $\{h_k\}_{k=1}^{+\infty}$ and $\{v_{h_k}\}_{k=1}^{+\infty}$ are bounded, there exist subsequences $\{h_{k'}\}_{k'} \subset \{h_k\}_k$ and $\{v_{h_{k'}}\}_{k'} \subset \{v_{h_k}\}_k$ such that

$$\begin{gathered} h_{k'} \to h, \quad 0 \leqslant h \leqslant \delta, \\ v_{h_{k'}} \to p \quad \text{in } H^2(\Omega),\ p \in P_1 \text{ (by Lemma 2.1)}, \\ b_{h_{k'}}(v_{h_{k'}}^-, v_{h_{k'}}^-) \to 0, \quad k' \to +\infty. \end{gathered}$$

By Lemma 4.3,

$$b_{h_{k'}}(p^-, p^-) \to 0 \quad \text{and} \quad p \geqslant 0 \text{ in } [x_l + h, x_r - h].$$

The case $p = 0$ is a contradiction with $1 = \|v_{h_{k'}}\|_{2,2} \to \|p\|_{2,2} = 0$. Therefore, $p > 0$ in the interval $(x_l + h, x_r - h)$. By Lemma 4.8, the solutions $w_{h_{k'}}$ are negative somewhere in $[x_l, x_r]$. Therefore, $p \leqslant 0$ somewhere in $[x_l, x_l + h]$ or in $[x_r - h, x_r]$. Suppose, for example, that $p \leqslant 0$ somewhere in $[x_l, x_l + h]$. Then the solutions $w_{h_{k'}}$ can be only negative somewhere in $[x_l, x_l + h + \varepsilon_{k'}]$, where $\varepsilon_{k'} \to 0^+$. By Lemma 4.8 and the definition of $\mathcal{S}_\delta$,

$$T(L_{k'}) \in [x_l, x_l + h + \varepsilon_{k'}] \cap [x_l + \delta, x_r - \delta].$$

It means that

$$h = \delta, \quad T(L_{k'}) \to x_l + \delta \quad \text{and} \quad p < 0 \text{ in } [x_l, x_l + \delta),\ p(x_l + \delta) = 0.$$

Hence,

$$\exists\, c_1 > 0\colon\ b_{h_{k'}}(p^-, p^-) \geqslant c_1 > 0 \quad \forall\, k',$$

by virtue of the assumptions on the reference numerical quadrature. This is a contradiction with $b_{h_{k'}}(p^-, p^-) \to 0$. Thus the estimates (4.13) and consequently (4.12) hold. $\square$

Lemma 4.9. *Let δ, ξ, η be positive parameters. Then there exists a positive parameter $h_0 = h_0(\delta, \xi, \eta)$ such that for any load $L \in \mathcal{S}_{\delta,\xi,\eta}$ and any partition $\tau_h \in \mathcal{T}'$, $h \leqslant h_0$, the corresponding problem* (P_h) *has a unique solution $w_h = w_h(L)$.*

P r o o f. By Lemma 4.8, it is sufficient to show that the solution w_h activates at least two springs, i.e. there exist points $y_1, y_2 \in Q_h$, where two springs are situated, such that $w_h(y_i) < 0$, $i = 1, 2$. Suppose that the assertion does not hold. Then there exist sequences $\{h_k\}_k$ and $\{w_{h_k}\}_k$ such that $h_k \to 0$ and w_{h_k} activate at most one spring. By Theorem 4.4, the sequence $\{w_{h_k}\}_k$ is bounded in $H^2(\Omega)$. Therefore, there exists its subsequence $\{w_{h_{k'}}\}_{k'}$ with a weak limit w in the space $H^2(\Omega)$. The variational equation (4.8) yields

$$0 > -\xi \geqslant L_{k'}(1) = b_{h_{k'}}(w^-_{h_{k'}}, 1) \to b(w^-, 1)$$

by Theorems 4.2 and 2.2. Hence, we can find an interval $M \subset \Omega_s$ with a positive Lebesgue one-dimensional measure such that $w_{h_{k'}}, w < 0$ in M for sufficiently large k'. Since $h_{k'} \to 0$, the set $Q_{h_{k'}} \cap M$ contains at least two active springs for sufficiently large k'. This contradicts the assumption that w_{h_k} activates only one spring. □

Theorem 4.5. *Let δ, ξ, η be positive parameters. Then there exist positive constants c, h_0 depending on the parameters δ, ξ, η such that*

$$c\|w - w_h\|^2_{2,2} \leqslant a(w - w_h, w - w_h) + b_h(w^- - w^-_h, w - w_h) \qquad \forall\, L \in \mathcal{S}_{\delta,\xi,\eta},\ \forall\, \tau_h \in \mathcal{T}',\ 0 < h \leqslant h_0, \tag{4.14}$$

where $w = w(L) \in H^2(\Omega)$ denotes the solution of the problem (P) *corresponding to a beam load $L \in \mathcal{S}_{\delta,\xi,\eta}$ and $w_h = w_h(L) \in V_h$ denotes its approximation with respect to a partition $\tau_h \in \mathcal{T}'$.*

P r o o f. Suppose that the inequality (4.14) does not hold. Then there exist sequences $\{L_k\}_{k=1}^{+\infty} \subset \mathcal{S}_{\delta,\xi,\eta}$ and $\{\tau_{h_k}\}_{k=1}^{+\infty} \subset \mathcal{T}'$ such that $h_k \to 0$ and

$$a(w_k - w_{k,h_k}, w_k - w_{k,h_k}) + b_{h_k}(w_k^- - w^-_{k,h_k}, w_k - w_{k,h_k}) < \frac{1}{k}\|w_k - w_{k,h_k}\|^2_{2,2}, \quad \forall\, k \geqslant 1, \tag{4.15}$$

where $w_k \in H^2(\Omega)$ denotes the solution of the problem (P_k) corresponding to the beam load L_k and $w_{k,h_k} \in V_{h_k}$ denotes its approximation with respect to the partition τ_{h_k}. Since the sequence $\{L_k\}_k$ is bounded in V^*, by Theorems 4.4 and 3.2 the sequences $\{w_k\}_k$, $\{w_{k,h_k}\}_k$ are bounded in $H^2(\Omega)$. Let

$$v_k := w_k/\|w_k - w_{k,h_k}\|_{2,2} \quad \text{and} \quad v_{k,h_k} := w_{k,h_k}/\|w_k - w_{k,h_k}\|_{2,2}.$$

Then the sequence $\{v_k - v_{k,h_k}\}_k$ is also bounded in $H^2(\Omega)$. Therefore there exist subsequences $\{w_{k'}\}_{k'}$, $\{w_{k',h_{k'}}\}_{k'}$, $\{v_{k'}\}_{k'}$, $\{v_{k',h_{k'}}\}_{k'}$, $\{h_{k'}\}_{k'}$, $\{\tau_{h_{k'}}\}_{k'}$ such that

$$\begin{aligned} w_{k'} &\to w, && \text{in } H^1(\Omega) \text{ (by Theorem 2.2)},\\ w_{k',h_{k'}} &\to w_h, && \text{in } H^1(\Omega) \text{ (by Theorem 2.2)},\\ v_{k'} - v_{k',h_{k'}} &\to p, && \text{in } H^2(\Omega),\ p \in P_1 \text{ (by (4.15) and Lemma 2.1)}, \end{aligned}$$

and

$$b(v_{k'}^- - v_{k',h_{k'}}^-, v_{k'} - v_{k',h_{k'}}) \to 0 \text{ (by (4.15) and Lemma 4.2).} \tag{4.16}$$

By Lemma 3.7 and the proof of Lemma 4.9 we can find sets $M, M_h \subset \Omega_s$ with positive Lebesgue one-dimensional measure such that $w_{k'}, w < 0$ in M and $w_{h_{k'}}, w_h < 0$ in M_h for sufficiently large $k' \geqslant 1$.

First, suppose

$$\|w_{k'} - w_{h_{k'}}\|_{2,2} \to 0.$$

Then $w = w_h$ and it is possible to choose M, M_h such that $M = M_h$ and the convergence (4.16) implies

$$v_{k'}^- - v_{k',h_{k'}}^- = v_{k'} - v_{k',h_{k'}} \to p = 0 \quad \text{a.e. in } M.$$

Thus $p = 0$.

Secondly, suppose

$$\exists\, c_1 > 0\colon\ \|w_{k'} - w_{h_{k'}}\|_{2,2} \geqslant c_1 \quad \forall\, k' \geqslant 1.$$

Then there exist weak limits v_h, $v = v_h + p$ of the sequences $\{v_{k',h_{k'}}\}_{k'}$, $\{v_{k'}\}_{k'}$ in the space $H^2(\Omega)$. Hence, from (4.16) and (2.6),

$$b\big((v_h + p)^- - v_h^-, (v_h + p) - v_h\big) = 0,$$

and consequently $(v_h + p)^- = v_h^-$ almost everywhere in Ω_s. Hence, $p = 0$ in M_h and consequently in Ω.

However, the case $p = 0$ is a contradiction with

$$1 = \|v_{k'} - v_{h_{k'}}\|_{2,2} \to \|p\|_{2,2} = 0.$$

Therefore the estimate (4.14) holds. $\square$

Remark 4.2. Let us consider the class $\mathcal{T}'_\theta$ of the partitions for any $\theta > 0$ instead of $\mathcal{T}'$ in Theorem 4.5. Then it can be shown that the constant h_0 in Theorem 4.5 can be chosen in the same way as in Lemma 4.9, i.e. the uniqueness of the solution must be ensured. The constant c in Theorem 4.5 can also depend on the parameter θ in that case.

Lemma 4.10. *Let $w \in H^2(\Omega)$ be a solution of the problem* (P) *and $w_h \in V_h$ a solution of an approximated problem* (P_h)*. Then*

$$a(w - w_h, v_h) + b(w^-, v_h) - b_h(w_h^-, v_h) = 0 \quad \forall\, v_h \in V_h. \tag{4.17}$$

Lemma 4.10 immediately follows from the variational equations (3.7) and (4.8).

Lemma 4.11. *Let the assumptions of Theorem 4.5 be fulfilled. Then there exist positive constants c, h_0 depending on δ, ξ, η such that*

$$\begin{aligned} c\|w - w_h\|_{2,2}^2 \leqslant \|w - w_h\|_{2,2}\|w - v_h\|_{2,2} \\ + |b(w^-, w_h - v_h) - b_h(w^-, w_h - v_h)| \\ \forall\, L \in \mathcal{S}_{\delta,\xi,\eta},\ \forall\, \tau_h \in \mathcal{T}',\ h \leqslant h_0,\ \forall\, v_h \in V_h, \end{aligned} \tag{4.18}$$

where $w \in H^2(\Omega)$ denotes the solution of the problem (P) *corresponding to a beam load $L \in \mathcal{S}_{\delta,\xi,\eta}$ and $w_h \in V_h$ denotes its approximation corresponding to a partition $\tau_h \in \mathcal{T}'$.*

P r o o f. By Theorem 4.5, Lemmas 4.10 and 4.1 and the inequality (2.6), we obtain

$$\begin{aligned} c\|w - w_h\|_{2,2}^2 &\leqslant a(w - w_h, w - w_h) + b_h(w^- - w_h^-, w - w_h) \\ &= a(w - w_h, w - w_h) + b(w^-, w - w_h) - b_h(w_h^-, w - w_h) \\ &\quad + b_h(w^-, w - w_h) - b(w^-, w - w_h) \\ &= a(w - w_h, w - v_h) + b(w^-, w - v_h) - b_h(w_h^-, w - v_h) \\ &\quad + b_h(w^-, w - w_h) - b(w^-, w - w_h) \\ &= a(w - w_h, w - v_h) + b_h(w^- - w_h^-, w - v_h) \\ &\quad + b_h(w^-, v_h - w_h) - b(w^-, v_h - w_h) \\ &\leqslant \tilde{c}\|w - w_h\|_{2,2}\|w - v_h\|_{2,2} + |b(w^-, w_h - v_h) - b_h(w^-, w_h - v_h)| \\ &\qquad \forall\, L \in \mathcal{S}_{\delta,\xi,\eta},\ \forall\, v_h \in V_h,\ \forall\, \tau_h \in \mathcal{T}_\theta',\ h \leqslant h_0, \end{aligned}$$

where c, $\tilde{c}$ are positive constants which do not depend on the partition τ_h. $\square$

Theorem 4.6 (Convergence results). *Let δ, ξ, η be positive parameters. Let $L \in \mathcal{S}_{\delta,\xi,\eta}$ be any load. Then there exist positive constants c_1, c_2, h_0, which depend on the load only through the parameters δ, ξ, η, such that*

$$\|w - w_h\|_{2,2} \leqslant c_1 h^2 \|w\|_{4,2}, \quad w \in H^4(\Omega) \cap \mathcal{V}_N,\ \forall\, \tau_h \in \mathcal{T}',\ h \leqslant h_0, \tag{4.19}$$

$$\|w - w_h\|_{2,2} \leqslant c_2 h \|w\|_{3,2}, \quad w \in H^3(\Omega),\ \forall\, \tau_h \in \mathcal{T}',\ h \leqslant h_0, \tag{4.20}$$

$$\|w - w_h\|_{2,2} \to 0, \quad w \in H^2(\Omega),\ h \to 0, \tag{4.21}$$

where $w = w(L)$ is the solution of the problem (P) *and $w_h = w_h(L)$ denotes the solution of the problem* (P_h) *with respect to the partition $\tau_h \in \mathcal{T}'$ and the bilinear form $b_h \in \mathcal{B}^1_{\tau_h}$. The set of functions $\mathcal{V}_N$, $N > 0$, is defined in Theorem* 4.2.

Proof. Let $w \in H^4(\Omega) \cap \mathcal{V}_N$. We start from the estimate (4.18) in Lemma 4.11 and choose $v_h = r_h(w)$. By the estimates (4.2) and (4.7) in Theorems 4.1 and 4.2 we derive

$$\begin{aligned}\|w - w_h\|^2_{2,2} &\leqslant c_1\|w - w_h\|_{2,2}\|w - r_h(w)\|_{2,2} + c_2 h^2\|w\|_{2,2}\|w_h - r_h(w)\|_{2,2}\\ &\leqslant c_1\|w - w_h\|_{2,2}\|w - r_h(w)\|_{2,2}\\ &\quad + c_2 h^2\|w\|_{2,2}(\|w - r_h(w)\|_{2,2} + \|w - w_h\|_{2,2})\\ &\leqslant c_3 h^2\|w\|_{4,2}\|w - w_h\|_{2,2} + c_4 h^4\|w\|^2_{4,2}\\ &\leqslant c_5 \max\{h^2\|w\|_{4,2}\|w - w_h\|_{2,2}, h^4\|w\|^2_{4,2}\} \quad \forall\, \tau_h \in \mathcal{T}'\end{aligned}$$

with positive constants c_i, $i = 1, \ldots, 5$. Hence,

$$\|w - w_h\|_{2,2} \leqslant ch^2\|w\|_{4,2} \quad \forall\, \tau_h \in \mathcal{T}',$$

where $c = \max\{c_5, \sqrt{c_5}\}$. So the estimate (4.19) holds.

The proof of the estimate (4.20) is similar, only instead of the estimates (4.2) and (4.7), the estimates (4.3) and (4.6) are used. To prove the limit (4.21), the limit (4.4) and the estimate (4.6) are used. $\square$

Thus the family of the problems $\{(\mathrm{P}_h)\}_h$ approximates the original problem (P). The assumption $w \in H^4(\Omega)$ can be satisfied as the following lemma shows.

Lemma 4.12. *Let the functions E and I, which represent Young's modulus of the beam and the inertia moment of the cross-section of the beam, be constant in the interval Ω. Let the beam load be represented by the density $f \in L^2(\Omega)$. Then the solution w of the problem* (P) *belongs to the space $H^4(\Omega)$.*

Proof. In the sense of the theory of distributions, we can rewrite the variational equation (3.7) in the following way (see also the classical formulation (3.1) of the problem):

$$EIw''''(x) + q(x)w^-(x) = f(x), \quad x \in \Omega.$$

Since $w \in H^2(\Omega)$ and $q \in L^\infty(\Omega)$, the function qw^- belongs to $L^2(\Omega)$. Therefore, also $w'''' = (EI)^{-1}(f - qw^-) \in L^2(\Omega)$. By Lemma 2.2 there exists a positive constant c such that

$$\|w\|_{4,2} \leqslant c(|w|^2_{4,2} + |w|^2_{0,2})^{1/2} < +\infty.$$

Thus $w \in H^4(\Omega)$. $\square$

5. Conclusion

The model of the unilateral elastic subsoil of Winkler's type and its approximation is studied in the paper. The problem can also be formulated as a contact one with two unknowns: deflection of the beam and compression of the subsoil. The non-penetration condition between the beam and the subsoil is considered. By enforcing this condition by Lagrange multipliers, the dual formulation of the problem can be derived, see [14]. The dual problem is a problem of convex quadratic programming with linear constraints and is suitable for numerical realization of the problem. Other numerical methods suitable for the problem are described in [6] and [13].

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