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ON QUASI-STATIONARY MODELS OF MIXTURES OF COMPRESSIBLE FLUIDS*

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Abstract. We consider mixtures of compressible viscous fluids consisting of two miscible species. In contrast to the theory of non-homogeneous incompressible fluids where one has only one velocity field, here we have two densities and two velocity fields assigned to each species of the fluid. We obtain global classical solutions for quasi-stationary Stokes-like system with interaction term.

Keywords: compressible viscous fluids, miscible mixtures, quasi-stationary

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1. Statement of the problem and main results

In this article we deal with mixtures of compressible viscous fluids consisting of two miscible species. In literature one may find several contributions to the mathematical theory of incompressible density dependent fluids which can be interpreted as mixtures, cf. [14], [15], [12], [19], [18]. In these contributions physical models using only one velocity field and one density are studied. In the present work we consider an alternative model of a mixture where densities and velocity fields are assigned to each species of the fluid. For the derivation of the constitutive equations from the physical model see the books of Rajagopal [20] and Haupt [10]. We study the quasi-stationary model which is a reasonable approximation of the general case if accelerations are small. Furthermore, the convective term is neglected, which is justified for small velocities.

The one component quasi-stationary model as an approximation of the Navier-Stokes system has been considered in the works [1], [13], [17], [16]. The stationary

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Stokes-like case with two components has been considered in [5], [6], and [8]. In these papers the existence of weak solutions with additional L^p-properties of the densities has been proved. The analytical tools there are based on techniques developed in e.g. [2], [3], [4], [15], [16].

In the present article we establish existence of global *classical* solutions to the initial-boundary value problem of quasi-stationary mixtures with two species. The main idea is to establish new a priori estimates which then imply the existence result. Note that the system of equations is nonlinear and of first order with respect to the densities. Obviously, we have performed considerable simplifications of the physical model. However, existence of global classical solutions with a general non-monotone pressure law is a result which is unlikely to be ever achieved in the general case. (Furthermore, having classical solutions for small data, the convective term may be treated in a secondary step using perturbation arguments.)

The partial differential equations of the quasi-stationary model which describe the motion of the mixture in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, read:

Balance of momentum for the ith species $(i = 1, 2)$:

$$\sum_{j=1}^{2}(\mu_{ij}\Delta u^{(j)} + (\mu_{ij} + \lambda_{ij})\nabla \operatorname{div} u^{(j)}) + (-1)^{i+1}(u^{(2)} - u^{(1)})g - \nabla p^{(i)} = 0. \tag{1}$$

Conservation of mass for the ith species $(i = 1, 2)$:

$$\frac{\partial}{\partial t}\varrho^{(i)} + \operatorname{div}(\varrho^{(i)} u^{(i)}) = 0. \tag{2}$$

The equations (1) *and* (2) *have to hold in* $Q_T = \Omega \times (0, T)$, $T = \text{const} > 0$.

The quantities in equations (1) and (2) have the following meaning:

- $\varrho^{(i)}(x,t)$—mass density for the ith component of the mixture, $i = 1, 2$;
- $p^{(i)}(\varrho^{(1)}, \varrho^{(2)})$—pressure for the ith component of the mixture, $i = 1, 2$;
- $u_j^{(i)}(x,t)$—the jth component of the ith velocity field, $j = 1, \ldots, N$;
- $u^{(i)} = (u_1^{(i)}, \ldots, u_N^{(i)})$, $x = (x_1, \ldots, x_N)$, t—time;
- μ_{ij}, λ_{ij}—viscosity constants;
- Δ—Laplacian in $\mathbb{R}^N$, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$—gradient operator, $\operatorname{div} u = \sum_{i=1}^{N} \partial u_i/\partial x_i$.

For simplicity we start with the case when the flow domain is taken to be the N-dimensional parallelepiped Ω:

$$\Omega = \prod_{k=1}^{N}(0, d_k) = \{x \in \mathbb{R}^n \colon\ 0 < x_k < d_k,\ k = 1, \ldots, N\}.$$

The boundary $\partial\Omega$ of Ω consists of the parts

$$S_k = (\{x_k = 0\} \cup \{x_k = d_k\}) \cap \bar{\Omega}, \quad \partial\Omega = \bigcup_{k=1}^{N} S_k.$$

For the *viscosity constants* we require

$$(3) \qquad \begin{cases} \mu_{11} > 0,\ \mu_{22} > 0,\ \nu_{11} > 0,\ \nu_{22} > 0,\ \nu_{ij} = 2\mu_{ij} + \lambda_{ij},\ 1 \leqslant i,\ j \leqslant 2, \\ 4\mu_{11}\mu_{22} - (\mu_{12} + \mu_{21})^2 > 0,\ 4\nu_{11}\nu_{22} - (\nu_{12} + \nu_{21})^2 > 0. \end{cases}$$

The *pressure law* has the form

$$(4) \qquad \begin{cases} p^{(i)}(\varrho^{(1)}, \varrho^{(2)}) = k^{(i)}\varrho^{(i)} p(\varrho^{(1)}, \varrho^{(2)}),\ k^{(i)} = \text{const} > 0,\ i = 1, 2, \\ p(\varrho^{(1)}, \varrho^{(2)}) = (\varrho^{(1)}/\varrho^{(1)}_{\text{ref}} + \varrho^{(2)}/\varrho^{(2)}_{\text{ref}})^{\gamma-1},\ \gamma = \text{const} > 1. \end{cases}$$

The factor g in the interaction term is assumed to satisfy

$$(5) \qquad \begin{cases} g(t, x, \varrho^{(1)}, \varrho^{(2)}, u^{(2)} - u^{(1)}) = a_0 + a_1(\varrho^{(1)} + \varrho^{(2)})^{\theta_1} \\ \qquad\qquad\qquad + a_2(1 + |u^{(2)} - u^{(1)}|^2)^{\theta_2}, \\ \quad \text{a)}\ a_0 = \text{const} \geqslant 0,\ a_1 = \text{const} \geqslant 0,\ a_2 = \text{const} \geqslant 0, \\ \quad \text{b)}\ \theta_1 = \text{const} > 0,\ \theta_1 \in [0, 2/N), \\ \quad \text{c)}\ \theta_2 = \text{const} > 0,\ \theta_2 \in [0, 1/(N\gamma - 1)) \text{ for } N \geqslant 2, \\ \qquad \theta_2 \in [0, 1/(2\gamma - 2)) \text{ for } N = 1. \end{cases}$$

We have the initial condition

$$(6) \qquad \varrho^{(i)}(x, t)|_{t=0} = \varrho_0^{(i)}(x), \quad x \in \bar{\Omega},\ i = 1, 2.$$

We consider the following mixed boundary condition for the velocity fields $u^{(i)}$:

$$(7) \qquad \begin{cases} u_k^{(i)} = 0 \text{ on } S_k \times [0, T], & k = 1, \dots, N, \\ \partial u_m^{(i)}/\partial x_k = 0 \text{ on } S_k \times [0, T], & m \neq k,\ m, k = 1, 2, \dots, N. \end{cases}$$

Remark 1.1.

a) If $N = 1$, then $k = 1$, $S_1 = \partial\Omega$, and we have just the Dirichlet boundary condition

$$u^{(i)} = 0 \quad \text{on } \partial\Omega \times [0, T].$$

b) If $N = 2$, the boundary conditions (7) have the form

$$\begin{cases} u^{(i)} \cdot \vec{n} = 0 & \text{on } \partial\Omega \times [0, T], \\ \operatorname{curl} u^{(i)} = 0 & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

c) If $N=3$, the boundary conditions (7) read

$$\begin{cases} u^{(i)}\cdot\vec{n}=0 & \text{on } \partial\Omega\times[0,T],\\ \vec{n}\times\operatorname{curl}u^{(i)}=0 & \text{on } \partial\Omega\times[0,T].\end{cases}$$

Here $\vec{n}$ is the outer normal vector at the boundary.

Remark 1.2. We treat all dimensions $N\geqslant 1$. The results and methods of proof hold and work analogously in the case of periodic boundary conditions and can be easily extended to the case of a mixture of l species, $l\geqslant 3$.

Definition 1.1. A classical solution to problem (1)–(7) is a quadruple of functions $(u^{(1)},u^{(2)},\varrho^{(1)},\varrho^{(2)})$ such that

$$u^{(1)},u^{(2)}\in C^{2,1}(\bar\Omega\times[0,T]);\quad \varrho^{(1)},\varrho^{(2)}\in C^{1}(\bar\Omega\times[0,T]);$$
$$\varrho^{(1)}(x,t)>0,\quad \varrho^{(2)}(x,t)>0\quad\text{in }\bar\Omega\times[0,T].$$

The main results of the article are contained in

Theorem 1.1. *Let the initial data* $\varrho_0^{(1)}$, $\varrho_0^{(2)}$ *satisfy* $\varrho_0^{(1)},\varrho_0^{(2)}\in W^{l,r}(\Omega)$, $r>1$, $l>1$, $r(l-1)>N$, $0<m_0\leqslant\varrho_0^{(i)}\leqslant M_0$, $i=1,2$, *where* m_0, M_0 *are constants. Then there exists a global unique classical solution* $(u^{(1)},u^{(2)},\varrho^{(1)},\varrho^{(2)})$ *of the boundary-initial-value problem* (1)–(7), *and we have*

$$\text{a)}\ \ \frac{\partial^k\varrho^{(i)}}{\partial t^k}\in L^\infty(0,T;W^{l-k,r}(\Omega)),\quad i=1,2,$$
$$\text{b)}\ \ \frac{\partial^k u^{(i)}}{\partial t^k}\in L^\infty(0,T;W^{l+1-k,r}(\Omega)),\quad i=1,2$$

for $0\leqslant k\leqslant l$.

Furthermore, there exist numbers m_1 *and* M_1 *such that*

$$0<m_1\leqslant\varrho^{(i)}(x,t)\leqslant M_1<\infty,\quad (x,t)\in\bar\Omega\times[0,T],\ \ i=1,2.$$

Strategy of the proof

The existence and uniqueness for classical solutions in a sufficiently small time interval is well known and follows from the theory of [21], [22], [23]. Therefore, the main difficulty in studying the "global in time" problem is connected with *a priori estimates* where the constants depend only on the data of the problem and the duration T of the time interval, but are independent of the interval for which one can show existence of local solutions. Such estimates imply that local solutions can be extended to the whole interval $[0, T]$.

In Section 2 the system for the effective viscous fluxes is established. Section 3 contains first estimates for the velocities and densities. In Section 4 we prove a global L^∞-bound for the densities from above and from below. In the last section we establish $W^{2,p}$-estimates for the velocities and $W^{1,p}$-estimates for the densities, using an approach for obtaining $W^{1,\infty}$-estimates for linear elliptic systems due to Yudovich [25], [26].

2. Auxiliary results

We state some assertions that are used later. Lemmas (2.2)–(2.5) are simple inequalities for real numbers which are used for the proof of the boundedness assertions in Section 4. The consideration concerning the effective viscous fluxes start with (15).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ be an arbitrary bounded domain satisfying the cone condition.*

1) *Then the following inequality is valid for every function $u \in W^{l,p}(\Omega) \cap L^q(\Omega)$, $l \geqslant 1$, $p > 1$, $q > 1$:*

$$\|u\|_{W^{k,r}(\Omega)} \leqslant c_1 \|u\|^{\alpha}_{W^{l,p}(\Omega)} \cdot \|u\|^{1-\alpha}_{L^q(\Omega)}, \tag{8}$$

where $1/r = k/N + \alpha \cdot (1/p - l/N) + (1-\alpha)/q$, $k/L \leqslant \alpha \leqslant 1$.

If $l - k - N/p$ is an integer, $l - k - N/p \geqslant 0$ and $1 < p < \infty$, then $0 \leqslant \alpha < 1$.

2) *Furthermore, the following inequality is valid for every function $u \in \overset{\circ}{W}{}^{1,m}(\Omega)$ or $u \in W^{1,m}(\Omega)$, $\int_\Omega u \,\mathrm{d}x = 0$ or $u \in W^{1,m}(\Omega)$, $u|_{S_0} = 0$, $S_0 \subset \partial\Omega$, $\mathrm{mes}_{\partial\Omega} S_0 > 0$:*

$$\|u\|_{L^q(\Omega)} \leqslant C_2 \cdot \|\nabla u\|^{\alpha}_{L^m(\Omega)} \cdot \|u\|^{1-\alpha}_{L^r(\Omega)}, \tag{9}$$

where $\alpha = (1/r - 1/q)(1/r - 1/m + 1/N)^{-1}$; moreover, if $m < N$ then $q \in [r, mN/(N-m)]$ for $r \leqslant mN/(N-m)$ and $q \in [mN/(N-m), r]$ for $r > Nm \times (N-m)^{-1}$. If $m \geqslant N$ then $q \in [r, \infty)$ is arbitrary; moreover, if $m > N$ then inequality (9) is also valid for $q = \infty$.

The positive constants C_1, C_2 in inequalities (8), (9) are independent of the function $u(x)$. Inequalities (8) and (9) are particular cases of the more general multiplicative inequalities proved in [7], [11], [9].

Lemma 2.2. *Let ν_{ij} $(i,j=1,2)$ be constants such that*

$$\nu_{11}>0,\ \ \nu_{22}>0,\quad 4\nu_{11}\nu_{22}-(\nu_{12}+\nu_{21})^2>0.$$

Then there exists a number $\nu_{00}>0$ such that

$$1-\frac{\nu_{12}\,\nu_{00}}{\nu_{22}}>0,\quad 1-\frac{\nu_{21}}{\nu_{00}\,\nu_{11}}>0.$$

P r o o f. We consider the following four cases:

(i) If $\nu_{12},\nu_{21}\leqslant 0$, then choose $\nu_{00}=1$.

(ii) If $\nu_{12}\leqslant 0$, $\nu_{21}>0$, choose $\nu_{00}=2\nu_{21}/\nu_{11}$.

(iii) If $\nu_{12}>0$, $\nu_{21}\leqslant 0$, choose $\nu_{00}=\frac{1}{2}(\nu_{22}/\nu_{12})$.

(iv) If $\nu_{12}>0$, $\nu_{21}>0$, choose $\nu_{00}=\frac{1}{2}(\nu_{21}/\nu_{11}+\nu_{22}/\nu_{12})$.

With these choices the statement of Lemma 2.2 is satisfied in all cases. □

For further use we define

$$M=\min\Big\{1,1-\frac{\nu_{12}\,\nu_{00}}{\nu_{22}},1-\frac{\nu_{21}}{\nu_{00}\,\nu_{11}}\Big\},\quad M\in(0,1]. \tag{10}$$

Lemma 2.3. *Let ν_{ij}, $i,j=1,2$, be constants such that*

$$\nu_{11}>0,\ \ \nu_{22}>0,\quad 4\nu_{11}\nu_{22}-(\nu_{12}+\nu_{21})^2>0,$$

and let $m>1$, $k^{(1)}>0$, $k^{(2)}>0$ be constants. Then there exist numbers $D^{(1)}>0$, $D^{(2)}>0$ such that for all $x\geqslant 0$, $y\geqslant 0$ the inequality

$$\begin{aligned}D^{(1)}k^{(1)}\nu_{22}x^{m+1}+D^{(2)}k^{(2)}\nu_{11}y^{m+1}-D^{(1)}k^{(2)}\nu_{12}x^m y-D^{(2)}k^{(1)}\nu_{21}y^m x\\ \geqslant M(D^{(1)}k^{(1)}\nu_{22}x^{m+1}+D^{(2)}k^{(2)}\nu_{11}y^{m+1})\end{aligned}$$

holds, where M is the constant from (10).

P r o o f. To prove the lemma we define

$$\begin{cases} D^{(1)}=\dfrac{\nu_{11}}{\nu_{22}}\Big(\dfrac{k^{(1)}}{k^{(2)}}\Big)^m\nu_{00}^{m+1},\quad D^{(2)}=1,\\ a=x(D^{(1)}k^{(1)}\nu_{22})^{1/(m+1)},\quad b=y(D^{(2)}k^{(2)}\nu_{11})^{1/(m+1)}\end{cases} \tag{11}$$

where ν_{00} comes from Lemma 2.2.

The left-hand side of the inequality stated in Lemma 2.3 has the form

$$\begin{aligned}F(a,b)&=a^{m+1}+b^{m+1}-\frac{\nu_{00}\,\nu_{12}}{\nu_{22}}a^m b-\frac{\nu_{21}}{\nu_{00}\,\nu_{11}}ab^m\\ &=a^{m+1}+b^{m+1}-a^m b-ab^m+\Big(1-\frac{\nu_{00}\,\nu_{12}}{\nu_{22}}\Big)a^m b+\Big(1-\frac{\nu_{21}}{\nu_{00}\,\nu_{11}}\Big)ab^m\\ &\geqslant M(a^{m+1}+b^{m+1})+(1-M)(a^m-b^m)(a-b),\end{aligned}$$

where M has been defined in (10). Thus

$$F(a,b) \geqslant M(a^{m+1} + b^{m+1}),$$

and the lemma is proved. $\square$

Lemma 2.4. *Let $\gamma > 1$, $m > 1$ and $D^{(1)}, D^{(2)} > 0$ be constants. Then for all $x, y \geqslant 0$ we have the inequality*

$$(x+y)^{\gamma} \leqslant K_{00}\big[(D^{(1)}x^{m+1} + D^{(2)}y^{m+1})(x+y)^{\gamma-1}\big]^{\delta_1}(x+y)^{1-\delta_1},$$

where

$$\delta_1 = \frac{\gamma-1}{m+\gamma-1}, \quad K_{00} = \left[\frac{\big((D^{(1)})^{1/m} + (D^{(2)})^{1/m}\big)^m}{D^{(1)}D^{(2)}}\right]^{\delta_1}.$$

P r o o f. It is easy to see that the statement follows from the inequality

$$(x+y)^{m+1} \leqslant K_{00}^{1/\delta_1}(D^{(1)}x^{m+1} + D^{(2)}y^{m+1}), \quad x \geqslant 0,\ y \geqslant 0.$$

To prove this, one considers the minimization problem: Find (a,b) such that

$$D^{(1)}a^{m+1} + D^{(2)}b^{m+1} = \min!, \quad a+b=1,\ a \geqslant 0,\ b \geqslant 0.$$

One finds that

$$\min_{\substack{a+b=1\\ a\geqslant 0,\, b\geqslant 0}} (D^{(1)}a^{m+1} + D^{(2)}b^{m+1}) = \frac{D^{(1)}D^{(2)}}{\big((D^{(1)})^{1/m} + (D^{(2)})^{1/m}\big)^m}$$

and Lemma 2.4 follows. $\square$

Lemma 2.5. *Let $\gamma > 1$, $m > 1$ and $D^{(1)} > 0$, $D^{(2)} > 0$ be constants. Then the inequality*

$$D^{(1)}x^m + D^{(2)}y^m \leqslant K_{01}\big[(D^{(1)}x^{m+1} + D^{(2)}y^{m+1})(x+y)^{\gamma-1}\big]^{\delta_3}(x+y)^{\delta_4} \tag{12}$$

holds for all $x \geqslant 0$, $y \geqslant 0$.

Here δ_3, δ_4 are positive constants, $0 < \delta_3 < m/(m+\gamma)$, $\delta_4 = m - (m+\gamma)\delta_3$ and

$$K_{01} = 2^{1/(m+1)}(D^{(1)} + D^{(1)})^{1-\delta_3}.$$

P r o o f. By homogeneity, it suffices to prove (12) for all $x \geqslant 0$, $y \geqslant 0$ such that $x+y=1$. By a convexity argument we have

$$\left(\frac{r_1^m + r_2^m}{2}\right)^{1/m} \leqslant \left(\frac{r_1^{m+1} + r_2^{m+1}}{2}\right)^{1/(m+1)}, \quad r_1 \geqslant 0,\ r_2 \geqslant 0,\ m > 1.$$

Hence we conclude that

$$\left(\frac{D^{(1)}a^m + D^{(2)}b^m}{2}\right)^{1/m} \leqslant \left(\frac{(D^{(1)})^{1+1/m}a^{m+1} + (D^{(2)})^{1+1/m}b^{m+1}}{2}\right)^{1/(m+1)}$$

and continue to estimate

$$\begin{aligned} D^{(1)}a^m + D^{(2)}b^m &\leqslant 2^{1/(m+1)}(D^{(1)} + D^{(2)})^{1/(m+1)}(D^{(1)}a^{m+1} + D^{(2)}b^{m+1})^{m/(m+1)} \\ &\leqslant 2^{1/(m+1)}(D^{(1)} + D^{(2)})^{1/(m+1)}(D^{(1)}a^{m+1} + D^{(2)}b^{m+1})^{\delta_3} \\ &\quad \times (D^{(1)}a^{m+1} + D^{(2)}b^{m+1})^{(m/(m+1))-\delta_3} \\ &\leqslant 2^{1/(m+1)}(D^{(1)} + D^{(2)})^{1-\delta_3}(D^{(1)}a^{m+1} + D^{(2)}b^{m+1})^{\delta_3}. \end{aligned}$$

Here we have used that $m/(m+1) - \delta_3 \geqslant 0$, $a \geqslant 0$, $b \geqslant 0$, $a+b=1$. Thus (12) is proved and the lemma follows. □

R e m a r k 2.1. Lemma 2.5 yields

(13) $$(D^{(1)}x^m + D^{(2)}y^m) \leqslant K_{02}[(D^{(1)}x^{m+1} + D^{(2)}y^{m+1})(x+y)^{\gamma-1}]^{\delta_2}(x+y)^{1-\delta_2},$$

where $\delta_2 = (m-1)/(m+\gamma-1)$ and $K_{02} = 2^{1/(m+1)}(D^{(1)} + D^{(2)})^{\gamma/(m+\gamma-1)}$.

R e m a r k 2.2. In our consideration we use that the differential equation (1) and the boundary conditions (7) imply the additional natural boundary conditions (in the generalized sense)

(14) $$\begin{cases} \dfrac{\partial}{\partial \vec{n}}(\nu_{11}\operatorname{div} u^{(1)} + \nu_{12}\operatorname{div} u^{(2)} - p^{(1)} + p_1^{(1)})|_{\partial\Omega} = 0, \\ \dfrac{\partial}{\partial \vec{n}}(\nu_{21}\operatorname{div} u^{(1)} + \nu_{22}\operatorname{div} u^{(2)} - p^{(2)} + p_1^{(2)})|_{\partial\Omega} = 0, \\ p_1^{(i)} = \dfrac{1}{\operatorname{mes}\Omega}\displaystyle\int_\Omega p^{(i)}(\varrho^{(1)}, \varrho^{(2)})\,\mathrm{d}x, \quad i = 1,2, \quad \text{for all } t \in [0,T]. \end{cases}$$

We now derive an "algebraic" equation between the quantities $\operatorname{div} u^{(i)}$, $\varrho^{(i)}$ which corresponds to the equation of the effective viscous flux in the one component case.

We introduce a function φ defined by

$$\text{(15)} \qquad \begin{cases} \Delta\varphi = \operatorname{div}((u^{(2)} - u^{(1)})g), \\ \dfrac{\partial\varphi}{\partial\vec{n}}\Big|_{\partial\Omega} = 0, \quad \int_\Omega \varphi \,\mathrm{d}x = 0 \ \text{ for all } \ t \in [0,T]. \end{cases}$$

From (1) and (15) we deduce

$$-\Delta\Big(\sum_{j=1}^{2} \nu_{ij} \operatorname{div} u^{(j)}\Big) + \Delta((-1)^i\varphi + p^{(i)} - p_1^{(i)}) = 0, \quad i = 1,2.$$

Then we find from (14) and (15) after some calculation

$$\text{(16)} \qquad \begin{cases} \nu_{11} \operatorname{div} u^{(1)} + \nu_{12} \operatorname{div} u^{(2)} = -\varphi + p^{(1)} - p_1^{(1)}, \\ \nu_{21} \operatorname{div} u^{(1)} + \nu_{22} \operatorname{div} u^{(2)} = \varphi + p^{(2)} - p_1^{(2)}. \end{cases}$$

From (16) we eliminate $\operatorname{div} u^{(i)}$ using the number $D^{(0)} = \nu_{11}\nu_{22} - \nu_{12}\nu_{21} > 0$. Then we find equations for the *effective viscous fluxes*:

$$\text{(17)} \qquad \begin{cases} D^{(0)} \operatorname{div} u^{(1)} = -(\nu_{22} + \nu_{12})\varphi + \nu_{22}(p^{(1)} - p_1^{(1)}) - \nu_{12}(p^{(2)} - p_1^{(2)}), \\ D^{(0)} \operatorname{div} u^{(2)} = (\nu_{11} + \nu_{21})\varphi + \nu_{11}(p^{(2)} - p_1^{(2)}) - \nu_{21}(p^{(1)} - p_1^{(1)}). \end{cases}$$

3. The first a priori estimate for the velocities and densities

Contrary to the usual procedure in compressible flow theory, we do not start with the usual energy estimate coming from the momentum equation by testing with $u^{(i)}$, $i = 1,2$, but we establish in the first step L^q-bounds for the densities via the equation of the effective viscous fluxes.

Let $(u^{(1)}, u^{(2)}, \varrho^{(1)}, \varrho^{(2)})$ be a classical solution of the problem under consideration.

1) From (2) and (7) we obtain

$$\text{(18)} \qquad \int_\Omega \varrho^{(i)}(x,t)\,\mathrm{d}x = \int_\Omega \varrho_0^{(i)}(x)\,\mathrm{d}x, \quad i = 1,2, \quad \text{for all } \ t \in [0,T].$$

2) Let $m = \text{const} \geqslant \gamma > 1$. From (2) and (7) we obtain equations $(i = 1,2)$

$$\text{(19)} \qquad \frac{1}{m-1} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (\varrho^{(i)})^m \,\mathrm{d}x + \int_\Omega (\varrho^{(i)})^m \cdot \operatorname{div} u^{(i)} \,\mathrm{d}x = 0, \quad \text{for all } \ t \in [0,T].$$

This will be used for a certain sequence of numbers $m \to \infty$; the aim is to obtain an L^∞-bound $\varrho^{(i)}$ (which reminds us to Moser's iteration technique).

Let $D^{(2)} = 1$ and $D^{(1)} = \nu_{11}\,\nu_{00}/\nu_{22}\cdot((k^{(1)}\,\nu_{00})/k^{(2)})^m$ (see Lemma 2.3 where these constants have been introduced).

Now, we replace $\operatorname{div} u^{(i)}$ $(i = 1, 2)$ in formula (19) by the expressions in (17). Then we obtain the following identities:

$$
(20)\qquad \left\{
\begin{aligned}
&\frac{1}{m-1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega D^{(1)}(\varrho^{(1)})^m + D^{(2)}(\varrho^{(2)})^m\,\mathrm{d}x\\
&\quad+\frac{1}{D^{(0)}}\int_\Omega D^{(1)}(\varrho^{(1)})^m\big[\nu_{22}(p^{(1)}-p_1^{(1)})-\nu_{12}(p^{(2)}-p_1^{(2)})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -(\nu_{22}+\nu_{12})\varphi\big]\,\mathrm{d}x\\
&\quad+\frac{1}{D^{(0)}}\int_\Omega D^{(2)}(\varrho^{(2)})^m\big[\nu_{11}(p^{(2)}-p_1^{(2)})-\nu_{21}(p^{(1)}-p_1^{(1)})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad +(\nu_{11}+\nu_{21})\varphi\big]\,\mathrm{d}x = 0,\\
&D^{(0)} = \nu_{11}\,\nu_{22}-\nu_{12}\,\nu_{21} > 0,\\
&\quad p_1^{(1)} = \frac{1}{\operatorname{mes}\Omega}\cdot\int_\Omega p^{(1)}\,\mathrm{d}x,\quad p_1^{(2)} = \frac{1}{\operatorname{mes}\Omega}\cdot\int_\Omega p^{(2)}\,\mathrm{d}x.
\end{aligned}\right.
$$

Let us define (for all $t\in[0,T]$)

$$
(21)\qquad \left\{
\begin{aligned}
&I_1 = \frac{1}{D^{(0)}}\int_\Omega D^{(1)}(\varrho^{(1)})^m\big[\nu_{22}p_1^{(1)}-\nu_{12}p_1^{(2)}\big]\\
&\qquad\qquad\qquad\qquad +D^{(2)}(\varrho^{(2)})^m\big[p_1^{(2)}\nu_{11}-p_1^{(1)}\nu_{21}\big]\,\mathrm{d}x,\\
&I_2 = \frac{1}{D^{(0)}}\int_\Omega\big[D^{(1)}(\varrho^{(1)})^m(\nu_{22}+\nu_{12})\\
&\qquad\qquad\qquad\qquad -D^{(2)}(\varrho^{(2)})^m(\nu_{11}+\nu_{21})\big]\cdot\varphi\,\mathrm{d}x,\\
&I_3 = \frac{1}{D^{(0)}}\int_\Omega D^{(1)}(\varrho^{(1)})^m\big[\nu_{22}p^{(1)}-\nu_{12}p^{(2)}\big]\\
&\qquad\qquad\qquad\qquad +D^{(2)}(\varrho^{(2)})^m\big[p^{(2)}\nu_{11}-p^{(1)}\nu_{21}\big]\,\mathrm{d}x,\\
&y(t) = \int_\Omega D^{(1)}(\varrho^{(1)})^m + D^{(2)}(\varrho^{(2)})^m\,\mathrm{d}x,\\
&A(t) = \int_\Omega\big[D^{(1)}(\varrho^{(1)})^{m+1}+D^{(2)}(\varrho^{(2)})^{m+1}\big](\varrho^{(1)}+\varrho^{(2)})^{\gamma-1}\,\mathrm{d}x.
\end{aligned}\right.
$$

In the rest of this section we confine ourselves to the case $m=\gamma$.

2a) By (4) we have the estimate

$$
I_1 \leqslant Cy(t)\int_\Omega(\varrho^{(1)}+\varrho^{(2)})^\gamma\,\mathrm{d}x.
$$

Furthermore, Lemma 2.4, Lemma 2.5 and Hölder's inequality yield

$$\int_\Omega (\varrho^{(1)}+\varrho^{(2)})^\gamma \,\mathrm{d}x \leqslant C(A(t))^{(\gamma-1)/(2\gamma-1)}\Big[\int_\Omega (\varrho^{(1)}+\varrho^{(2)})\,\mathrm{d}x\Big]^{1-(\gamma-1)/(2\gamma-1)},$$
$$y(t) \leqslant C(A(t))^{(\gamma-1)/(2\gamma-1)}\Big[\int_\Omega (\varrho^{(1)}+\varrho^{(2)})\,\mathrm{d}x\Big]^{1-(\gamma-1)/(2\gamma-1)}.$$

Hence we obtain the inequality

$$I_1 \leqslant C(A(t))^{(2\gamma-2)/(2\gamma-1)} \tag{22}$$

with a positive constant C.

2b) Now, the term I_2 can be estimated in the following way:

$$\begin{aligned} I_2 &\leqslant C\int_\Omega (D^{(1)}\cdot(\varrho^{(1)})^\gamma + D^{(2)}\cdot(\varrho^{(2)})^\gamma)\cdot|\varphi|\,\mathrm{d}x \\ &\leqslant C\Big[\int_\Omega (\varrho^{(1)}+\varrho^{(2)})^{\gamma\cdot((q_1)/(q_1-1))}\,\mathrm{d}x\Big]^{1-1/q_1}\cdot\|\varphi_1\|_{L^{q_1}(\Omega)} \\ &\quad + C\Big[\int_\Omega (\varrho^{(1)}+\varrho^{(2)})^\gamma\cdot\frac{q_2}{q_2-1}\,\mathrm{d}x\Big]^{1-1/q_2}\cdot\|\varphi_2\|_{L^{q_2}(\Omega)} \\ &\leqslant C\Big[\int_\Omega (\varrho^{(1)}+\varrho^{(2)})^{2\gamma}\,\mathrm{d}x\Big]^{(\gamma-1+1/q_1)/(2\gamma-1)}\cdot\|\varphi_1\|_{L^{q_1}(\Omega)} \\ &\quad + C\Big[\int_\Omega (\varrho^{(1)}+\varrho^{(2)})^{2\gamma}\,\mathrm{d}x\Big]^{(\gamma-1+1/q_2)/(2\gamma-1)}\cdot\|\varphi_2\|_{L^{q_2}(\Omega)} \end{aligned} \tag{23}$$

with positive constants C, $q_1=\mathrm{const}>2$, $q_2=\mathrm{const}>2$.

The functions φ_1 and φ_2 are defined in the following way: We write $\varphi=\varphi_1+\varphi_2$ and define φ_1,φ_2 as solutions to the problems

$$\begin{cases} \Delta\varphi_1 = \operatorname{div}((a_0+a_1(\varrho^{(1)}+\varrho^{(2)})^{\theta_1})(u^{(2)}-u^{(1)})), \\ \dfrac{\partial\varphi_1}{\partial\vec{n}}\Big|_{\partial\Omega}=0, \quad \displaystyle\int_\Omega \varphi_1\,\mathrm{d}x=0, \quad \forall\, t\in[0,T], \end{cases} \tag{24}$$

$$\begin{cases} \Delta\varphi_2 = \operatorname{div}(a_2(1+|u^{(2)}-u^{(1)}|^2)^{\theta_2}\cdot(U^{(2)}-U^{(1)})), \\ \dfrac{\partial\varphi_2}{\partial\vec{n}}\Big|_{\partial\Omega}=0, \quad \displaystyle\int_\Omega \varphi_2\,\mathrm{d}x=0, \quad \forall\, t\in[0,T]. \end{cases} \tag{25}$$

By virtue of Lemma 2.3 we find

$$\begin{aligned} I_3 &\geqslant \frac{M}{D^{(0)}}\int_\Omega \big[D^{(1)}k^{(1)}\nu_{22}(\varrho^{(1)})^{\gamma+1}+D^{(2)}k^{(2)}\nu_{11}(\varrho^{(2)})^{\gamma+1}\big]\Big(\frac{\varrho^{(1)}}{\varrho^{(1)}_{\mathrm{ref}}}+\frac{\varrho^{(2)}}{\varrho^{(2)}_{\mathrm{ref}}}\Big)^{\gamma-1}\,\mathrm{d}x \\ &\geqslant C\int_\Omega \big[D^{(1)}(\varrho^{(1)})^{\gamma+1}+D^{(2)}(\varrho^{(2)})^{\gamma+1}\big](\varrho^{(1)}+\varrho^{(2)})^{\gamma-1}\,\mathrm{d}x \end{aligned} \tag{26}$$

where $M = \text{const} > 0$ comes from Lemma 2.3, $C = \text{const} > 0$. So, we conclude from (20) in the case $m = \gamma$ the inequality

$$\begin{aligned}\frac{1}{\gamma-1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}y(t)+CA(t)&\\ \leqslant C(A(t))^{(2\gamma-2)/(2\gamma-1)}&+C\|\varrho^{(1)}+\varrho^{(2)}\|_{L^{2\gamma}(\Omega)}^{2\gamma(\gamma-1+1/q_1)/(2\gamma-1)}\cdot\|\varphi_1\|_{L^{q_1}(\Omega)}\\ +C\|\varrho^{(1)}+\varrho^{(2)}\|_{L^{2\gamma}(\Omega)}^{2\gamma(\gamma-1+1/q_2)/(2\gamma-1)}&\cdot\|\varphi_2\|_{L^{q_2}(\Omega)}.\end{aligned}$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+CA(t)\leqslant C\big(1+\|\varphi_1\|_{L^{q_1}(\Omega)}^{(2\gamma-1)/(\gamma-1/q_1)}+\|\varphi_2\|_{L^{q_2}(\Omega)}^{(2\gamma-1)/(\gamma-1/q_2)}\big) \tag{27}$$

with C a positive constant.

3) From (1) and the boundary condition (7) one obtains via (3)

$$\begin{aligned}\|\nabla u^{(1)}\|_{L^2(\Omega)}^2+\|\nabla u^{(2)}\|_{L^2(\Omega)}^2+\int_\Omega g|u^{(2)}-u^{(1)}|^2\,\mathrm{d}x&\\ \leqslant C\big(\|p^{(1)}-p_1^{(1)}\|_{L^2(\Omega)}^2+\|p^{(2)}-p_1^{(2)}\|_{L^2(\Omega)}^2\big)&\\ \leqslant C\big(1+\|\varrho^{(1)}+\varrho^{(2)}\|_{L^{2\gamma}(\Omega)}^{2\gamma}\big).&\end{aligned} \tag{28}$$

4) Let $q_1 > 2$, $q_2 > 2$. Then the problems (24), (25) are solvable and by the usual L^p-theory for elliptic operators (see also Lemma 2.1) we have the following estimates:

$$\begin{aligned}\|\varphi_1\|_{L^{q_1}(\Omega)}&\leqslant C\|\nabla\varphi_1\|_{L^{r_1}(\Omega)}\\ &\leqslant C\|(a_0+a_1\cdot(\varrho^{(1)}+\varrho^{(2)})^{\theta_1})\cdot|u^{(2)}-u^{(1)}|\|_{L^{r_1}(\Omega)},\end{aligned} \tag{29}$$

$$\begin{aligned}\|\varphi_2\|_{L^{q_2}(\Omega)}&\leqslant C\|\nabla\varphi_2\|_{L^{r_2}(\Omega)}\\ &\leqslant C\|a_2\cdot(1+|u^{(2)}-u^{(1)}|^2)^{\theta_2}\cdot|u^{(2)}-u^{(1)}|\|_{L^{r_2}(\Omega)},\end{aligned} \tag{30}$$

where $r_1, r_2 \in (1,+\infty)$ for $N = 1$ and $r_1 \in [Nq_1/(N+q_1),+\infty)$, $r_2 \in [Nq_2/(N+q_2),+\infty)$ for $N \geqslant 2$. The numbers satisfy $Nq_1/(N+q_1) > 1$, $Nq_2/(N+q_2) > 1$ for $N \geqslant 2$, since $q_1, q_2 \in (2,+\infty)$.

Now, we estimate the terms on the right-hand side of (29), (30):

4a) In the case $N = 1$ we set (observing that here $0 < \theta_1 < 2$)

$$\begin{cases} q_1 = \dfrac{4(6-\theta_1)}{(2-\theta_1)(4-\theta_1)} > 2, & \text{since } 0 < \theta_1 < 2;\\[2mm] r_1 = \dfrac{2}{1+\theta_1} & \text{if } 0 < \theta_1 < 1;\\[2mm] r_1 = \dfrac{6-\theta_1}{4} & \text{if } 1 \leqslant \theta_1 < 2.\end{cases}$$

Consequently,

$$
\text{(31)} \qquad \begin{cases} 1)\ 1 < r_1 < 2; & \\ 2)\ \theta_1 r_1 \leqslant 2 - r_1 & \text{if } 0 < \theta_1 < 1; \\ 3)\ 2 - r_1 < \theta_1 r_1 < 2\gamma(2 - r_1) & \text{if } 1 \leqslant \theta_1 < 2; \\ 4)\ \dfrac{2}{q_1} + \theta_1 < \dfrac{2}{r_1} & \text{if } 1 \leqslant \theta_1 < 2. \end{cases}
$$

In the case $N \geqslant 2$, we set (observe $0 < \theta_1 < 2/N$)

$$
\begin{cases} r_1 = \dfrac{2N}{N+2} + \varepsilon, \ \ 0 < \varepsilon \leqslant \dfrac{2N}{N+2} \cdot \dfrac{2/N - \theta_1}{1 + \theta_1}; \\ q_1 = 2 + \delta, \ \ \delta = \varepsilon \cdot \Bigl(1 + \dfrac{2}{N}\Bigr)^2. \end{cases}
$$

This yields

$$
\text{(32)} \qquad \begin{cases} 5)\ 1 \leqslant r_1 \cdot \Bigl(\dfrac{1}{N} + \dfrac{1}{q_1}\Bigr), \quad r_1 < 1; \\ 6)\ \theta_1 r_1 \leqslant 2 - r_1. \end{cases}
$$

Furthermore, from (29) and Hölder's inequality ($1 < r_1 < 2$) we have

$$
\begin{aligned} \|\varphi_1\|_{L^{q_1}(\Omega)} \leqslant{}& C a_0 \|u^{(2)} - u^{(1)}\|_{L^2(\Omega)} \\ &+ C a_1 \Bigl(\int_\Omega (\varrho^{(1)} + \varrho^{(2)})^{\theta_1} |u^{(2)} - u^{(1)}|^2 \,\mathrm{d}x\Bigr)^{1/2} \\ &\times \Bigl(\int_\Omega (\varrho^{(1)} + \varrho^{(2)})^{(\theta_1 r_1)/(2 - r_1)} \,\mathrm{d}x\Bigr)^{(2-r_1)/(2r_1)}. \end{aligned}
$$

If $N \geqslant 2$ or $N = 1$ and $0 < \theta_1 < 1$ we conclude from (31) and (32) the inequality $\theta_1 r_1 \leqslant 2 - r_1$. Thus, using also the estimate (18) and the inequality (28) we obtain

$$
\text{(33)} \qquad \|\varphi_1\|_{L^{q_1}(\Omega)} \leqslant C\bigl(1 + \|\varrho^{(1)} + \varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\bigr)^{1/2}.
$$

If $N = 1$ and $1 \leqslant \theta_1 < 2$, then we have the representation

$$
\frac{\theta_1 \cdot r_1}{2 - r_1} = 2\gamma \cdot \tau + 1 - \tau, \quad \tau = \frac{\theta_1 r_1/(2 - r_1) - 1}{2\gamma - 1} \in (0, 1).
$$

This gives

$$
\int_\Omega (\varrho^{(1)} + \varrho^{(2)})^{\theta_1 r_1/(2 - r_1)} \,\mathrm{d}x \leqslant C\Bigl[\int_\Omega (\varrho^{(1)} + \varrho^{(2)})^{2\gamma} \,\mathrm{d}x\Bigr]^{\eta},
$$

where

$$\eta = \frac{\theta_1/2 - (2 - r_1)/(2r_1)}{2\gamma - 1}.$$

Hence we conclude via inequality (28) that

$$\|\varphi_1\|_{L^{q_1}(\Omega)} \leqslant C\bigl(1 + \|\varrho^{(1)} + \varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\bigr)^{1/2+\eta}. \tag{34}$$

It is important to note that

$$\Bigl(\frac{1}{2} + \eta\Bigr) \cdot \frac{2\gamma - 1}{\gamma - 1/q_1} = \frac{1}{\gamma - 1/q_1}\Bigl(\gamma - \frac{1}{2} + \frac{\theta_1}{2} - \frac{1}{r_1} + \frac{1}{2}\Bigr) < 1.$$

In fact, this estimate follows from (31).

4b) In the case $N = 1$ we define (taking into account that $0 < \theta_2 < 1/(2\gamma - 2)$)

$$q_2 = \frac{4(1 + \theta_2)}{1 - 2\theta_2(\gamma - 1)} > 2; \quad r_2 = \frac{2\theta_2 + 2}{2\theta_2 + 1} > 1.$$

Then, from (30), we derive the inequality

$$\begin{aligned} (35) \qquad & \|\varphi_2\|_{L^{q_2}(\Omega)} \\ & \leqslant C\Bigl(1 + \int_\Omega \bigl(1 + |u^{(2)} - u^{(1)}|^2\bigr)^{\theta_2} |u^{(2)} - u^{(1)}|^2 \,\mathrm{d}x\Bigr)^{(2\theta_2+1)/(2\theta_2+2)} \\ & \leqslant C\bigl(1 + \|\varrho^{(1)} + \varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\bigr)^{(2\theta_2+1)/(2\theta_2+2)}. \end{aligned}$$

It is important to observe that

$$\frac{2\theta_2 + 1}{2\theta_2 + 2} \cdot \frac{2\gamma - 1}{\gamma - 1/q_2} < 1, \quad \text{since } q_2 = \frac{4(1 + \theta_2)}{1 - 2\theta_2(\gamma - 1)}.$$

In the case $N = 2$ we define (taking into account that $0 < \theta_2 < 1/(2\gamma - 1)$)

$$q_2 = \frac{4(1 + \theta_2)}{1 - \theta_2(2\gamma - 1)} > 2; \quad r_2 = \frac{2q_2}{2 + q_2} > 1, \quad \text{since } q_2 > 2.$$

Then we obtain from (30), (28) and Sobolev's imbedding theorem the estimate

$$\begin{aligned} (36) \quad \|\varphi_2\|_{L^{q_2}(\Omega)} & \leqslant C\|\sqrt{g}\,|u^{(2)} - u^{(1)}|\|_{L^2(\Omega)} \cdot \Bigl[\int_\Omega \bigl(1 + |u^{(2)} - u^{(1)}|^{\theta_2 q_2}\bigr) \,\mathrm{d}x\Bigr]^{1/q_2} \\ & \leqslant C\bigl(1 + \|\varrho^{(1)} + \varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\bigr)^{1/2+\theta/2}. \end{aligned}$$

Here it is important to note that

$$\Big(\frac{1}{2}+\frac{1}{2}\theta_2\Big)\cdot\frac{2\gamma-1}{\gamma-1/q_2}<1, \quad \text{since} \quad q_2=\frac{4(1+\theta_2)}{1-\theta_2\cdot(2\gamma-1)}.$$

In the case $N\geqslant 3$ we define (here we have $0<\theta_2<1/(N\gamma-1)$)

$$\delta=\frac{7}{8}, \quad r_2=\frac{2\theta_2+2}{2\theta_2+1}\cdot\delta+\frac{2N}{N-2}\cdot\frac{1}{2\theta_2+1}\cdot(1-\delta); \tag{37}$$

$$q_2=\frac{Nr_2}{N-r_2}>2, \ \text{since the inequalities}$$

$$\frac{2N}{N-2}<r_2<N, \ \ 0<\theta_2<\frac{1}{N\gamma-1} \ \text{ and } \ N\geqslant 3 \ \text{ are satisfied.}$$

Then we have from (30), (28) and Sobolev's imbedding theorem

$$\begin{aligned}
\|\varphi_2\|_{L^{q_2}(\Omega)} &\leqslant C\Big(1+\int_\Omega |u^{(2)}-u^{(1)}|^{(2\theta_2+1)r_2}\,\mathrm{d}x\Big)^{1/r_2} \\
&\leqslant C\Big(1+\Big[\int_\Omega |u^{(2)}-u^{(1)}|^{2\theta_2+2}\,\mathrm{d}x\Big]^{\delta} \\
&\qquad\qquad \times\Big[\int_\Omega |u^{(2)}-u^{(1)}|^{2N/(N-2)}\,\mathrm{d}x\Big]^{1-\delta}\Big)^{1/r_2} \\
&\leqslant C\Big(1+\|\sqrt{g}\,|u^{(2)}-u^{(1)}|\|_{L^2(\Omega)}^{\delta/r_2} \\
&\qquad\qquad \times\|\nabla(u^{(2)}-u^{(1)})\|_{L^2(\Omega)}^{(2N(1-\delta))/((N-2)r_2)}\Big) \\
&\leqslant C\big(1+\|\varrho^{(1)}+\varrho^{(2)}\|_{L^{2\gamma}(\Omega)}^{2\gamma}\big)^{\delta/r_2+(N/(N-2))(1-\delta)/r_2}.
\end{aligned} \tag{38}$$

It is important that

$$\Big(\frac{\delta}{r_2}+\frac{1-\delta}{r_2}\cdot\frac{N}{N-2}\Big)\cdot\frac{2\gamma-1}{\gamma-1/q_2}<1.$$

5) The above considerations in all cases yield the estimate

$$\begin{aligned}
&\|\varphi_1\|_{L^{q_1}(\Omega)}^{(2\gamma-1)/(\gamma-1/q_1)}+\|\varphi_2\|_{L^{q_2}(\Omega)}^{(2\gamma-1)/(\gamma-1/q_2)} \\
&\qquad\qquad \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|_{L^{2\gamma}(\Omega)}^{2\gamma})^{\nu_1}, \quad \nu_1\in(0,1).
\end{aligned}$$

Hence, we find from (27) the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+C_1A(t)\leqslant C_2,$$

where C_1, C_2 are positive constants.

Consequently,

$$
(39) \qquad \begin{cases} \sup\limits_{0<t<T} \big[\|\varrho^{(1)}(t)\|_{L^\gamma(\Omega)} + \|\varrho^{(2)}(t)\|_{L^\gamma(\Omega)}\big] \leqslant C, \\ \displaystyle\int_0^T \|\varrho^{(1)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)} + \|\varrho^{(2)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)} \,\mathrm{d}t \leqslant C. \end{cases}
$$

Further, we obtain from (39), (28) and the boundary condition (7) that

$$
(40) \qquad \begin{cases} \displaystyle\int_0^T \|u^{(1)}(t)\|^2_{W^{1,2}(\Omega)} + \|u^{(2)}(t)\|^2_{W^{1,2}(\Omega)} \,\mathrm{d}t \leqslant C, \\ \displaystyle\int_0^T \|g^{1/2}|u^{(2)} - u^{(1)}|(t)\|^2_{L^2(\Omega)} \,\mathrm{d}t \leqslant C. \end{cases}
$$

4. Estimates for the densities of the mixture from above and below

In this section we derive L^∞-bounds for the densities and its inverses from the effective viscous flux equations. The technique of proof resembles the method of J. Moser for elliptic equations. In our case, the interaction term needs some additional treatment.

First, let us present some estimates for the function $\varphi(x,t)$.

1) In the case $N = 1$ we set (observe that $0 < \theta_1 < 2$, $0 < \theta_2 < 1/(2\gamma - 2)$)

$$
\varepsilon_1 = \min\left\{\frac{1}{2\theta_2 + 1}, \frac{2\gamma/\theta_1 - 1}{2\gamma/\theta_1 + 1}\right\}, \quad \varepsilon_1 \in (0,1).
$$

Then we have

$$
1 + \varepsilon_1 \leqslant \frac{2\theta_2 + 2}{2\theta_2 + 1}, \quad \frac{1 + \varepsilon_1}{1 - \varepsilon_1} \leqslant 2\gamma\theta_1.
$$

From the imbedding theorem and equation (15) we find

$$
\begin{aligned}
\|\varphi\|_{L^\infty(\Omega)} &\leqslant C\|\nabla\varphi\|_{L^{1+\varepsilon_1}(\Omega)} \leqslant C\|g|u^{(2)} - u^{(1)}|\|_{L^{1+\varepsilon_1}(\Omega)} \\
&\leqslant C\left(1 + a_0\left[\int_\Omega |u^{(2)} - u^{(1)}|^{1+\varepsilon_1} \,\mathrm{d}x\right]^{1/(1+\varepsilon_1)}\right) \\
&\quad + C\left(a_2\left[\int_\Omega |u^{(2)} - u^{(1)}|^{(2\theta_2+1)(1+\varepsilon_1)} \,\mathrm{d}x\right]^{1/(1+\varepsilon_1)}\right. \\
&\quad \left. + a_1\left[\int_\Omega (\varrho^{(1)} + \varrho^{(2)})^{\theta_1(1+\varepsilon_1)} \cdot |u^{(2)} - u^{(1)}|^{1/(1+\varepsilon_1)} \,\mathrm{d}x\right]^{1/(1+\varepsilon_1)}\right).
\end{aligned}
$$

From the choice of ε_1 and inequality (28) we have

$$a_0\|u^{(2)}-u^{(1)}\|_{L^{1+\varepsilon_1}(\Omega)} \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{1/2},$$
$$a_2\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{(2\cdot\theta_2+1)(1+\varepsilon_1)}} \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{(2\theta_2+1)/(2\theta_2+2)},$$
$$a_1\|(\varrho^{(1)}+\varrho^{(2)})^{\theta_1}|u^{(2)}-u^{(1)}|\|_{L^{1+\varepsilon_1}(\Omega)} \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{1/2+\theta_1/4\gamma}.$$

Therefore

$$\|\varphi\|_{L^\infty(\Omega)} \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\beta_1},$$

where $\beta_1=\max\{(2\theta_2+1)/(2\theta_2+2), \frac{1}{2}+(\theta_1/4\gamma), \frac{1}{2}\}<1$.

From this and (39) we have

$$\|\varphi(t)\|_{L^\infty(\Omega)} \in L^1(0,T). \tag{41}$$

2) In the case $N=2$ we set (observe that $0<\theta_1<1$, $0<\theta_2<1/(2\gamma-1)$)

$$\varepsilon_2=\frac{1}{2}\cdot\min\{1,\gamma-1\},$$
$$r_1=\frac{2\gamma}{2\theta_1(1+\varepsilon_2)}>1,\quad r_2=\frac{r_1}{r_1-1},\quad \frac{1}{2}+\frac{\theta_1}{2\gamma}<1.$$

By Sobolev's imbedding theorem and equation (15) we find the estimates

$$\begin{aligned}\|\varphi\|_{L^\infty(\Omega)} &\leqslant C\|\nabla\varphi\|_{2(1+\varepsilon_2)}\\ &\leqslant C\biggl(1+a_0\biggl[\int_\Omega |u^{(2)}-u^{(1)}|^{2(1+\varepsilon_2)}\,\mathrm{d}x\biggr]^{1/(2+2\varepsilon_2)}\\ &\quad+a_1\biggl[\int_\Omega ((\varrho^{(1)}+\varrho^{(2)})^{\theta_1}|u^{(2)}-u^{(1)}|)^{2(1+\varepsilon_2)}\,\mathrm{d}x\biggr]^{1/(2+2\varepsilon_2)}\\ &\qquad+a_2\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{q_2}(\Omega)}\biggr),\end{aligned}$$

where $q_2=2(2\theta_2+1)(1+\varepsilon_2)$.

By the choice of ε_2, r_1 and by inequality (28) we obtain

$$a_0\|u^{(2)}-u^{(1)}\|_{L^{2+2\varepsilon_2}(\Omega)} \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{1/2},$$
$$a_1\|(\varrho^{(1)}+\varrho^{(2)})^{\theta_1}|u^{(2)}-u^{(1)}|\|_{L^{2+2\varepsilon_2}(\Omega)} \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\theta_1/2\gamma+1/2},$$
$$a_2\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{(2\theta_2+1)(1+\varepsilon_2)\cdot 2}(\Omega)} \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\nu_2},$$

where $\nu_2=(1-\varepsilon_2)/(2(1+\varepsilon_2))+(\theta_2(1+3\varepsilon_2)+2\varepsilon_2)/(2(1+\varepsilon_2))<1$ since $0<\theta_2<1/(2\gamma-1)$ and $\varepsilon_2=\frac{1}{2}\min\{1,\gamma-1\}$. Thus, in this case, we have the estimate

$$\|\varphi\|_{L^\infty(\Omega)} \leqslant C(1+\|\varrho^{(1)}+\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\beta_2},$$

where $\beta_2 = \text{const} > 0$, $\beta_2 \in (0,1)$. From this and (39) we have

$$\|\varphi(t)\|_{L^\infty(\Omega)} \in L^1(0,T). \tag{42}$$

3) In the case $N \geqslant 3$ we set (observe that $0 < \theta_1 < 2/N$, $0 < \theta_2 < 1/(N\gamma - 1)$)

$$\begin{cases} \delta = \min\bigl\{\frac{1}{2}, \frac{1}{2}(\gamma - 1), (N-1)/(N+1)\cdot(1+\theta_2 - N\theta_2)\bigr\}; \\ r_1 = (1+\delta)N; \\ r_2 = \max\{1 + N(1-\theta_1/\gamma)\cdot(1 - N\theta_1/2\gamma)^{-1}, 1 + N(1+\delta)(1-(N-1)\theta_2)^{-1}\}. \end{cases}$$

Then the following estimates hold:

3a) $0 < 2\delta < 1,\ 0 < 2\delta < \gamma - 1;\ r_1, r_2 \in (N, +\infty)$;

3b) $r_1\theta_1 \leqslant 2\gamma$ since $r_1 \cdot \theta_1 = N(1+\delta)\cdot\theta_1 \leqslant 2 + 2\delta \leqslant 2\gamma$;

3c) $N(1+\delta)/r_2 < 2 + 2\theta_2 - 2\theta_2 N + \delta\cdot(1 - 2N\theta_2)$ since $2 + 2\theta_2 - 2\theta_2 N + \delta\cdot(1-2N\theta_2) = 2 + 2\theta_2 - 2\theta_2 N - \delta\cdot(N+1)/(N-1) + \delta(2N/(N-1) - 2N\cdot\theta_2) > 2 + 2\theta_2 - 2\theta_2\cdot N - \delta\cdot(N+1)/(N-1) > 1 + \theta_2 - N\cdot\theta_2 > (1+\delta)\cdot N/r_2$.

Now we conclude from the imbedding theorem and equation (15) the following estimates:

$$\begin{aligned} \|\varphi\|_{L^\infty(\Omega)} &\leqslant C\|\nabla\varphi\|_{L^{r_1}(\Omega)} \\ &\leqslant C(1 + a_2\|u^{(2)} - u^{(1)}\|^{2\theta_2+1}_{L^{(2\theta_2+1)r_1}(\Omega)} + a_0\|u^{(2)} - u^{(1)}\|_{L^{r_1}(\Omega)} \\ &\quad + a_1\|(\varrho^{(1)} + \varrho^{(2)})^{\theta_1}|u^{(2)} - u^{(1)}|\|_{L^{r_1}(\Omega)}). \end{aligned}$$

Furthermore, from (1) and the boundary condition (7) and by virtue of (3) we obtain the estimate

$$\begin{aligned} &\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(2)}\|_{L^{r_2}(\Omega)} \\ &\quad \leqslant C(1 + \|\varrho^{(1)}\|^\gamma_{L^{\gamma r_2}(\Omega)} + \|\varrho^{(2)}\|^\gamma_{L^{\gamma r_2}(\Omega)} + a_0\|u^{(2)} - u^{(1)}\|_{L^{Nr_2/(N+r_2)}(\Omega)} \\ &\qquad + a_1\|(\varrho^{(1)} + \varrho^{(2)})^{\theta_1}|u^{(2)} - u^{(1)}|\|_{L^{Nr_2/(N+r_2)}(\Omega)} + a_2\|u^{(2)} - u^{(1)}\|^{2\theta_2+1}_{L^{q_4}(\Omega)}), \end{aligned}$$

where $q_4 = (2\theta_2 + 1)Nr_2/(N + r_2)$. From the inequality $r_1 > Nr_2/(N + r_2)$ we obtain

$$\begin{aligned} &\|\varphi\|_{L^\infty(\Omega)} + \|\nabla u^{(1)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(2)}\|_{L^{r_2}(\Omega)} \\ &\quad \leqslant C(1 + \|\varrho^{(1)}\|^\gamma_{L^{\gamma r_2}(\Omega)} + \|\varrho^{(2)}\|^\gamma_{L^{\gamma r_2}(\Omega)} + a_0\|u^{(2)} - u^{(1)}\|_{L^{r_1}(\Omega)}) \\ &\qquad + a_1\|(\varrho^{(1)} + \varrho^{(2)})^{\theta_1}|u^{(2)} - u^{(1)}|\|_{L^{r_1}(\Omega)} \\ &\qquad + a_2\cdot\|u^{(2)} - u^{(1)}\|^{2\theta_2+1}_{L^{(2\theta_2+1)r_1}(\Omega)}. \end{aligned} \tag{43}$$

Finally, we estimate the last three expressions in the following way:

(A)
$$\begin{aligned}
&a_0\|u^{(2)}-u^{(1)}\|_{L^{r_1}(\Omega)}\\
&\quad\leqslant \varepsilon(\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)}+\|\nabla u^{(2)}\|_{L^{r_2}(\Omega)})+Ca_0\|u^{(2)}-u^{(1)}\|_{L^2(\Omega)}\\
&\quad\leqslant \varepsilon(\|\nabla u^{(2)}\|_{L^{r_2}(\Omega)}+\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)})+C(1+\|\varrho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\\
&\qquad\qquad+\|\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{1/2}.
\end{aligned}$$

Here

a) the number $\varepsilon\in(0,1)$ will be determined later;
b) the estimate (28) has been used.

(B)
$$\begin{aligned}
&a_1\|(\varrho^{(1)}+\varrho^{(2)})^{\theta_1}\cdot|u^{(2)}-u^{(1)}|\|_{L^{r_1}(\Omega)}\\
&\quad\leqslant C\|u^{(2)}-u^{(1)}\|_{L^\infty(\Omega)}\left(\int_\Omega(\varrho^{(1)}+\varrho^{(2)})^{r_1\theta_1}\,\mathrm{d}x\right)^{1/r_1}\\
&\quad\leqslant C(1+\|\varrho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}+\|\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\theta_1/(2\gamma)}\\
&\qquad\times\|\nabla(u^{(2)}-u^{(1)})\|^{\alpha}_{L^{r_2}(\Omega)}\|u^{(2)}-u^{(1)}\|^{1-\alpha}_{L^{2N/(N-2)}(\Omega)}.
\end{aligned}$$

Here

a) $r_1\cdot\theta_1\leqslant 2\gamma$, since 3b) is satisfied;
b) $\alpha=(N-2)/2N\cdot((N-2)/2N-(N-r_2)/(Nr_2))^{-1}$
$=(\frac12-1/N)\cdot(\frac12-1/r_2)^{-1}\in(0,1)$ comes from Lemma 2.1.

Using (28) and the imbedding theorem we find

$$\begin{aligned}
&a_1\|(\varrho^{(1)}+\varrho^{(2)})^{\theta_1}|u^{(2)}-u^{(1)}|\|_{L^{r_1}(\Omega)}\\
&\quad\leqslant\varepsilon(\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)}+\|\nabla u^{(2)}\|_{L^{r_2}(\Omega)})\\
&\qquad+C(1+\|\varrho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}+\|\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{1/2+\theta_1/(2\gamma(1-\alpha))}.
\end{aligned}$$

It is important that we have $\frac12+\theta_1/(2\gamma(1-\alpha))<1$.

(C)
$$\begin{aligned}
&a_2\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{(2\theta_2+1)r_1(\Omega)}\\
&\quad\leqslant Ca_2\|\nabla(u^{(2)}-u^{(1)})\|^{\beta(2\theta_2+1)}_{L^{r_2}(\Omega)}\|u^{(2)}-u^{(1)}\|^{(1-\beta)(2\theta_2+1)}_{L^{2\theta_2+2}(\Omega)},
\end{aligned}$$

where

a) the inequality follows from Lemma 2.1,
b) $\beta=[1/(2\theta_2+2)-1/(2r_1\theta_2+r_1)](1/(2\theta_2+2)-(N-r_2)/(Nr_2))^{-1}\in(0,1)$.

Using (28) we find $\nu_3=((1-\beta)(2\theta_2+1))/(1-\beta(2\theta_2+1))\cdot1/(2\theta_2+2)$ such that

$$\begin{aligned}
&a_2\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{r_1(2\theta_2+1)}(\Omega)}\\
&\quad\leqslant\varepsilon(\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)}+\|\nabla u^{(2)}\|_{L^{r_2}(\Omega)})+C(1+\|\varrho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}+\|\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\nu_3}.
\end{aligned}$$

Here it is important that $\beta(2\theta_2+1)^2<1$. This inequality gives us the estimates

$$\beta(2\theta_2+1)<1,\quad (1-\beta)(2\theta_2+1)<(1-\beta(2\theta_2+1))(2\theta_2+2).$$

Therefore, choosing $\varepsilon\in(0,1)$ appropriately we arrive at the inequality

$$\begin{aligned}\|\varphi\|_{L^\infty(\Omega)}&+\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)}+\|\nabla u^{(2)}\|_{L^{r_2}(\Omega)}\\&\leqslant C(1+\|\varrho^{(1)}\|^{\gamma r_2}_{L^{\gamma r_2}(\Omega)}+\|\varrho^{(2)}\|^{\gamma r_2}_{L^{\gamma r_2}(\Omega)})^{1/r_2}\\&\quad+C(1+\|\varrho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}+\|\varrho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\beta_2},\end{aligned}\tag{44}$$

where

a) $\beta_2=\max\{\frac12,\frac12+\theta_1/(2\gamma(1-\alpha)),$
$(1-\beta)(2\theta_2+1)/(1-\beta(2\theta_2+1))\cdot 1/(2\theta_2+2)\}\in(0,1).$

4) Here we look at the terms I_1, I_2, I_3 from (21) in the case $m>\gamma>1$, $N\geqslant 1$. By virtue of (21) and (39) we find, for all $t\in[0,T]$,

$$I_1\leqslant Cy(t)\int_\Omega(\varrho^{(1)}+\varrho^{(2)})^\gamma\,\mathrm{d}x\leqslant Cy(t),$$

where C is a positive constant not depending on m. Furthermore, the inequality

$$I_2\leqslant Cy(t)\|\varphi(t)\|_{L^\infty(\Omega)},\quad t\in[0,T],$$

holds and, again, the positive constant C does not depend on m.

Due to Lemma 2.3 the term I_3 can be estimated from below in the following way:

$$\begin{aligned}I_3&\geqslant\frac{M}{D^{(0)}}\int_\Omega\big(D^{(1)}k^{(1)}\nu_{22}(\varrho^{(1)})^{m+1}+D^{(2)}k^{(2)}\nu_{11}(\varrho^{(2)})^{m+1}\big)\\&\qquad\qquad\times(\varrho^{(1)}/\varrho^{(1)}_{\mathrm{ref}}+\varrho^{(2)}/\varrho^{(2)}_{\mathrm{ref}})^{\gamma-1}\,\mathrm{d}x\\&\geqslant C\int_\Omega\big(D^{(1)}(\varrho^{(1)})^{m+1}+D^{(2)}\cdot(\varrho^{(2)})^{m+1}\big)(\varrho^{(1)}+\varrho^{(2)})^{\gamma-1}\,\mathrm{d}x=CA(t),\end{aligned}$$

where $M=\mathrm{const}>0$ comes from Lemma 2.3 and C is a positive constant not depending on m. Hence we conclude from (20) the following inequality for all $t\in(0,T)$:

$$\frac{1}{m-1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}y(t)+C_1A(t)\leqslant C_2(y(t)+y(t)\cdot\|\varphi(t)\|_{L^\infty(\Omega)}),\tag{45}$$

where C_1, C_2 are positive constants independent of m.

4a) In the cases $N=1$, $N=2$ we obtain from (45) and (41), (42), for all $m>\gamma>1$, $t\in(0,T)$, the inequality

$$\frac{1}{m-1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}y(t)\leqslant G_1(t)\cdot y(t), \tag{46}$$

where $G_1(t)=C_2(1+\|\varphi(t)\|_{L^\infty(\Omega)})\in L^1(0,T)$ and the function $G_1(t)$ does not depend on m.

4b) In the case $N=3$ we obtain from (45) and (44) that, for all $m>\max(\gamma,\gamma(r_2-1))>1$, $t\in(0,T)$, the following inequality holds:

$$\begin{aligned}\frac{1}{m-1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}y(t)+C_3A(t)&\\ \leqslant C_4y(t)\cdot(1+(\|\varrho^{(1)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)}&+\|\varrho^{(2)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\beta_2}\\ +(\|\varrho^{(1)}(t)\|^{r_2\gamma}_{L^{r_2\gamma}(\Omega)}&+\|\varrho^{(2)}(t)\|^{r_2\gamma}_{L^{r_2\gamma}(\Omega)})^{1/r_2}).\end{aligned} \tag{47}$$

Due to the estimates for the densities (39) one easily checks the inequality

$$\begin{aligned}\left(\int_\Omega D^{(1)}(\varrho^{(1)})^m+D^{(2)}(\varrho^{(2)})^m\,\mathrm{d}x\right)\cdot(1+\|\varrho^{(1)}\|^{r_2\gamma}_{L^{r_2\gamma}(\Omega)}+\|\varrho^{(2)}\|^{r_2\gamma}_{L^{r_2\gamma}(\Omega)})^{1/r_2}&\\ \leqslant C\cdot\big((A(t))^{1-(\gamma/mr_2)}+y(t)\big),&\end{aligned}$$

where C is a positive constant not depending on m.

Hence we obtain from (47) that

$$\frac{1}{m-1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}y(t)\leqslant C^m+G_2(t)\cdot y(t),\quad t\in(0,T), \tag{48}$$

with C being a positive constant not depending on m, and with the function G_2 defined by

$$G_2(t)=C(1+\|\varrho^{(1)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)}+\|\varrho^{(2)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)})^{\beta_2}.$$

We have $G_2(t)\in L^1(0,T)$.

Now we find from (46) and (48), for all $N\geqslant1$, $m>m_0$, $t\in[0,T]$, the estimate

$$\begin{aligned}y(t)&=\int_\Omega D^{(1)}\cdot(\varrho^{(1)})^m+D^{(2)}\cdot(\varrho^{(2)})^m\,\mathrm{d}x\\ &\leqslant(C(T))^m\cdot(1+\|\varrho_0^{(1)}\|_{L^m(\Omega)}+\|\varrho_0^{(2)}\|_{L^m(\Omega)})^m,\end{aligned}$$

where $C(T)$ is a positive constant not depending on m. Consequently,

$$\sup_{0<t<T}(\|\varrho^{(1)}(t)\|_{L^\infty(\Omega)}+\|\varrho^{(2)}(t)\|_{L^\infty(\Omega)})\leqslant C(1+\|\varrho_0^{(1)}\|_{L^\infty(\Omega)}+\|\varrho_0^{(2)}\|_{L^\infty(\Omega)}). \tag{49}$$

Now, we present an estimate *from below* for the densities of the mixture:

5) Let $n \in (0, +\infty)$ be a positive number. Then, analogously to equation (20), we find

$$-\frac{1}{n+1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\varrho^{(1)})^{-n}+(\varrho^{(2)})^{-n}\,\mathrm{d}x$$
$$+\frac{1}{D^{(0)}}\int_\Omega(\varrho^{(1)})^{-n}[\nu_{22}(p^{(1)}-p_1^{(1)})-\nu_{12}(p^{(2)}-p_1^{(2)})-(\nu_{22}+\nu_{12})\varphi]\,\mathrm{d}x$$
$$+\frac{1}{D^{(0)}}\int_\Omega(\varrho^{(2)})^{-n}[\nu_{11}(p^{(2)}-p_1^{(2)})-\nu_{21}(p^{(1)}-p_1^{(1)})+(\nu_{11}+\nu_{21})\varphi]\,\mathrm{d}x=0.$$

We define for all $t \in [0, T]$ a function $Z(t) = \int_\Omega (p^{(1)})^{-n} + (p^{(2)})^{-n}\,\mathrm{d}x$.

Then the estimate (49) implies the inequality

$$\frac{1}{n+1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}Z(t)\leqslant CZ(t)(1+\|\varphi(t)\|_{L^\infty(\Omega)}),$$

where C again is a positive constant not depending on n.

Taking into account (41), (42), (43) and (49) we derive from the differential inequality an estimate for all $n > 0$:

$$Z(t)=\int_\Omega(\varrho^{(1)})^{-n}+(\varrho^{(2)})^{-n}\,\mathrm{d}x\leqslant(C(T))^{n+1}(1+\|1/\varrho_0^{(1)}\|_{L^n(\Omega)}+\|1/\varrho_0^{(2)}\|_{L^n(\Omega)})^n,$$

where $C(T)$ is a positive constant not depending on n. Hence

$$\sup_{0<t<T}\Big(\Big\|\frac{1}{\varrho^{(1)}(t)}\Big\|_{L^\infty(\Omega)}+\Big\|\frac{1}{\varrho^{(2)}(t)}\Big\|_{L^\infty(\Omega)}\Big)\leqslant C\Big(1+\Big\|\frac{1}{\varrho_0^{(1)}}\Big\|_{L^\infty(\Omega)}+\Big\|\frac{1}{\varrho_0^{(2)}}\Big\|_{L^\infty(\Omega)}\Big).\tag{50}$$

5. Estimates for gradients of the velocities and densities

In this section we show that it is possible to estimate the first derivatives of the functions $u^{(1)}(x,t)$, $u^{(2)}(x,t)$, $\varrho^{(1)}(x,t)$, $\varrho^{(2)}(x,t)$. Let $s \in (N, +\infty)$ be any number.

1) First we have by equation (1) and the boundary condition (7) the estimate

$$\|u^{(2)}\|_{W^{2,s}(\Omega)}+\|u^{(2)}\|_{W^{2,s}(\Omega)}\leqslant C(\|g\cdot(u^{(2)}-u^{(1)})\|_{L^s(\Omega)}+\|\nabla p^{(1)}\|_{L^s(\Omega)}+\|\nabla p^{(2)}\|_{L^s(\Omega)}).$$

Having completed (49) one proceeds with the inequality

$$(51)\qquad \|u^{(1)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)} \leqslant C(1 + \|(u^{(2)} - u^{(1)})\|^{2\theta_2+1}_{L^{s(2\theta_2+1)}(\Omega)} + \|\nabla\varrho^{(1)}\|_{L^s(\Omega)} + \|\nabla\varrho^{(2)}\|_{L^s(\Omega)}).$$

We take into account that (28) implies

$$(52)\qquad \|\nabla u^{(1)}\|_{L^2(\Omega)} + \|\nabla u^{(2)}\|_{L^2(\Omega)} \leqslant C.$$

1a) In the case $N = 1$, $N = 2$ we have the estimate

$$\|(u^{(2)} - u^{(1)})\|_{L^{s(2\theta_2+1)}(\Omega)} \leqslant C\|\nabla(u^{(2)} - u^{(1)})\|_{L^2(\Omega)},$$

and thus we find from (51) and (52) the estimate

$$(53)\quad \|u^{(1)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)} \leqslant C(1 + \|\nabla\varrho^{(1)}\|_{L^s(\Omega)} + \|\nabla\varrho^{(2)}\|_{L^s(\Omega)}).$$

1b) In the case $N \geqslant 3$ the inequality (Lemma 2.1)

$$\|u^{(2)} - u^{(1)}\|_{L^{s(2\theta_2+1)}(\Omega)} \leqslant C\|\nabla(u^{(2)} - u^{(1)})\|^{\alpha}_{L^\infty(\Omega)} \cdot \|u^{(2)} - u^{(1)}\|^{1-\alpha}_{L^{2N/(N-2)}(\Omega)}$$

holds with $\alpha \in (0,1)$, $\alpha = 1 - 2/N - 2/(2s\theta_2 + s)$ if $s(2\theta_2+1) > 2N/(N-2)$ and $\alpha = 0$ if $s(2\theta_2 + 1) \leqslant 2N/(N-2)$. Here we take into account that $\alpha \cdot (2\theta_2 + 1) < 1$, since $0 < \theta_2 < 1/(N\gamma - 1)$.

Therefore, by (52) and the estimate

$$\|u^{(2)} - u^{(1)}\|_{L^{2N/(N-2)}(\Omega)} \leqslant c\|\nabla(u^{(2)} - u^{(1)})\|_{L^2(\Omega)}$$

we conclude that the following inequality holds for all $\varepsilon > 0$:

$$(54)\qquad \|u^{(2)} - u^{(1)}\|^{2\theta_2+1}_{L^{s(2\theta_2+1)}(\Omega)} \leqslant \varepsilon\|\nabla(u^{(2)} - u^{(1)})\|_{L^\infty(\Omega)} + C(\varepsilon).$$

From this inequality we further obtain

$$(55)\qquad \|u^{(2)} - u^{(1)}\|^{2\theta_2+1}_{L^{s(2\theta_2+1)}(\Omega)} \leqslant \varepsilon[\|u^{(1)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)}] + C(\varepsilon).$$

Now, from (51) and (55) we find the estimate

$$(56)\qquad \|u^{(1)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)} \leqslant C(1 + \|\nabla\varrho^{(1)}\|_{L^s(\Omega)} + \|\nabla\varrho^{(2)}\|_{L^s(\Omega)}).$$

Hence we have proved the estimate (53), (56) in all cases $N \geqslant 1$.

2) In our considerations we use an important estimate for the velocities from [25], [26], [24], [16]. By virtue of the estimates for the densities (49), (50) we have $\varrho^{(i)} \in L^\infty(\Omega \times (0,T))$ for $i = 1,2$ and in view of the inequality from [25], [26], [24], [16] we conclude from (1), for all $s > N$, the estimate

$$\begin{aligned}
&\|\nabla u^{(1)}\|_{L^\infty(\Omega)} + \|\nabla u^{(2)}\|_{L^\infty(\Omega)} \\
&\qquad \leqslant C\big(1 + \ln(2 + \|\nabla \varrho^{(1)}\|_{L^s(\Omega)} + \|\nabla \varrho^{(2)}\|_{L^s(\Omega)}) + \|g \cdot (u^{(2)} - u^{(1)})\|_{L^s(\Omega)}\big).
\end{aligned}$$

Because of (49), (52) and (54) we have

$$\|\nabla u^{(1)}\|_{L^\infty(\Omega)} + \|\nabla u^{(2)}\|_{L^\infty(\Omega)} \leqslant C(1 + \ln(2 + \|\nabla \varrho^{(1)}\|_{L^s(\Omega)} + \|\nabla \varrho^{(2)}\|_{L^s(\Omega)})). \tag{57}$$

3) The estimates for the derivatives $\partial \varrho^{(i)}/\partial x_j(x,t)$, $i = 1,2$, $j = 1,\ldots,N$ are derived from the equation

$$\frac{\partial}{\partial t}(\nabla \varrho^{(i)}) + \nabla((u^{(i)} \cdot \nabla)\varrho^{(i)}) + \nabla(\varrho^{(i)} \cdot \operatorname{div} u^{(i)}) = 0,$$

which, in turn, follows from (2). Therefore, we obtain from (49) for $s \in (N,\infty)$ the estimate

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega |\nabla \varrho^{(1)}|^s + |\nabla \varrho^{(2)}|^s \,\mathrm{d}x\bigg) \\
&\qquad \leqslant C\bigg(\int_\Omega (|\nabla \varrho^{(1)}|^s + |\nabla \varrho^{(2)}|^s)(|\nabla u^{(1)}| + |\nabla u^{(2)}|) \,\mathrm{d}x\bigg) \\
&\qquad\quad + C\bigg(\int_\Omega (|\nabla \varrho^{(1)}|^s + |\nabla \varrho^{(2)}|^s)^{(s-1)/s}(|\nabla \operatorname{div} u^{(1)}| + |\nabla \operatorname{div} u^{(2)}|) \,\mathrm{d}x\bigg).
\end{aligned}$$

Using (56) and (57) we proceed to

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) \leqslant C(1 + L(t) + L(t)\ln(2 + L(t)))$$

with L defined by $L(t) = \int_\Omega |\nabla \varrho^{(1)}(t)|^s + |\nabla \varrho^{(2)}(t)|^s \,\mathrm{d}x$, $t \in [0,T]$.

From the last differential inequality we obtain, for $s \in (N,+\infty)$, the estimate

$$\sup_{0<t<T} (\|\nabla \varrho^{(1)}(t)\|_{L^s(\Omega)} + \|\nabla \varrho^{(2)}(t)\|_{L^s(\Omega)}) \leqslant C. \tag{58}$$

Furthermore, by (2) and (56) we have

$$\begin{aligned}
\sup_{0<t<T}\bigg(\Big\|\frac{\partial \varrho^{(1)}}{\partial t}(t)\Big\|_{L^s(\Omega)} + \Big\|\frac{\partial \varrho^{(2)}}{\partial t}(t)\Big\|_{L^s(\Omega)} + \|u^{(1)}(t)\|_{W^{2,s}(\Omega)} \\
+ \|u^{(2)}(t)\|_{W^{2,s}(\Omega)}\bigg) \leqslant C.
\end{aligned} \tag{59}$$

4) The estimates for the derivatives $\partial/\partial t(\partial u^{(i)}/\partial x_j(x,t))$, $i=1,2$, $j=1,\dots,N$, come from the following system which is, in turn, derived from (1):

$$\begin{aligned}(60)\quad & \frac{\partial}{\partial t}\Big(\sum_{j=1}^{2}\mu_{ij}\Delta u^{(j)}+(\mu_{ij}+\lambda_{ij})\nabla\operatorname{div}u^{(j)}\Big)\\ &\qquad+(-1)^{i+1}\Big(g\cdot\frac{\partial}{\partial t}(u^{(2)}-u^{(1)})+\frac{\partial g}{\partial t}\cdot(u^{(2)}-u^{(1)})\Big)-\nabla\Big(\frac{\partial p^{(i)}}{\partial t}\Big)=0.\end{aligned}$$

First, from (60), in view of the estimates (49), (50) and (59), we obtain the inequality

$$\Big\|\nabla\Big(\frac{\partial u^{(1)}}{\partial t}\Big)\Big\|^2_{L^2(\Omega)}+\Big\|\nabla\Big(\frac{\partial u^{(2)}}{\partial t}\Big)\Big\|^2_{L^2(\Omega)}\leqslant C, \tag{61}$$

since $g=g(x,t)\geqslant 0$.

Finally, by the properties of the system (60) and the imbedding theorem, we have for $s\in(N,+\infty)$ the estimate

$$\begin{aligned}&\Big\|\frac{\partial u^{(1)}}{\partial t}\Big\|_{W^{1,s}(\Omega)}+\Big\|\frac{\partial u^{(2)}}{\partial t}\Big\|_{W^{1,s}(\Omega)}\\ &\qquad\leqslant C\Big(\Big\|\frac{\partial p^{(1)}}{\partial t}\Big\|_{L^s(\Omega)}+\Big\|\frac{\partial p^{(2)}}{\partial t}\Big\|_{L^s(\Omega)}+\Big\|g\Big(\frac{\partial u^{(2)}}{\partial t}-\frac{\partial u^{(1)}}{\partial t}\Big)\Big\|_{L^s(\Omega)}\\ &\qquad\qquad+\Big\|\frac{\partial g}{\partial t}(u^{(2)}-u^{(1)})\Big\|_{L^s(\Omega)}\Big).\end{aligned}$$

Thus, applying (49), (50) and (59) we find the estimate

$$\Big\|\frac{\partial u^{(1)}}{\partial t}\Big\|_{W^{1,s}(\Omega)}+\Big\|\frac{\partial u^{(2)}}{\partial t}\Big\|_{W^{1,s}(\Omega)}\leqslant C\Big(1+\Big\|\frac{\partial u^{(2)}}{\partial t}-\frac{\partial u^{(1)}}{\partial t}\Big\|_{L^s(\Omega)}\Big).$$

Therefore, we conclude with Lemma 2.1 and (61):

$$\sup_{0<t<T}\Big(\Big\|\frac{\partial u^{(1)}}{\partial t}(t)\Big\|_{W^{1,s}(\Omega)}+\Big\|\frac{\partial u^{(2)}}{\partial t}(t)\Big\|_{W^{1,s}(\Omega)}\Big)\leqslant C. \tag{62}$$

5) In the case $\varrho_0^{(1)},\varrho^{(2)}\in W^{l,r}(\Omega)$, $r>1$, $l>1$, $r\cdot(l-1)>N$ it is easy to see, using (49), (50), (58), (59) and (62), that for all $k=1,2,\dots,l$ and $i=1,2$ the following inclusions hold:

$$\frac{\partial^k u^{(i)}}{\partial t^k}\in L^\infty(0,T;W^{l+1-k,r}(\Omega)),\qquad \frac{\partial^k\varrho^{(i)}}{\partial t^k}\in L^\infty(0,T;W^{l-k,r}(\Omega)). \tag{63}$$

Thus we have proved all a priori estimates stated in the theorem.

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