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HOW TO UNIFY THE TOTAL/LOCAL–LENGTH–CONSTRAINTS OF THE GRADIENT FLOW FOR THE BENDING ENERGY OF PLANE CURVES

Yuki Miyamoto, Takeyuki Nagasawa and Fumito Suto

The gradient flow of bending energy for plane curve is studied. The flow is considered under two kinds of constraints; one is under the area and total-length constraints; the other is under the area and local-length constraints. The fundamental results (the local existence and uniqueness) were obtained independently by Kurihara and the second author for the former one; by Okabe for the later one. For the former one the global existence was shown for any smooth initial curves, but the asymptotic behavior has not been studied. For the later one, the global existence was guaranteed for only curves with the rotation number one, and the behavior was well studied. It is desirable to compensate the results with each other. In this note, it is proposed how to unify the two flows.

Keywords: gradient flow, bending energy, total-length constraint, local-length constraint

AMS Subject Classification: 53C44, 53A04, 58J35, 35K30

1. INTRODUCTION

Let $\Gamma = \{\boldsymbol{f}(s) \,|\, 0 \leqq s \leqq L\}$ be a closed plane curve whose curvature vector, unit normal vector, and unit tangent vector are $\boldsymbol{\kappa}$, $\boldsymbol{\nu}$, and $\boldsymbol{\tau}$ respectively. Hereafter s is the arc-length parameter. Define the bending energy W and the area A by

$$W(\boldsymbol{f}) = \frac{1}{2}\int_0^L \|\boldsymbol{\kappa}\|^2 \,\mathrm{d}s, \quad A(\boldsymbol{f}) = \frac{1}{2}\int_0^L \langle \boldsymbol{f}, \boldsymbol{\nu}\rangle \,\mathrm{d}s.$$

Here $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^2$. Note that A is not always positive, however, it is an enclosed area when Γ is simple and positively oriented. The gradient flows of W under constraints of area and either total- or local-length have been investigated in [2] and [8] independently. In this note we would like to study the relation between two gradient flows and their variants.

The equation of gradient flow is

$$\partial_t \boldsymbol{f} = -\delta W(\boldsymbol{f}) + \boldsymbol{\alpha} + \boldsymbol{\beta},$$

where δ is the first variation. Two vector-valued functions $\boldsymbol{\alpha} = \alpha\boldsymbol{\tau}$ and $\boldsymbol{\beta} = \beta\boldsymbol{\nu}$ are unknown determined by constraints.

The second author of this note considered with Kurihara in [2] under the constraints of area $A \equiv \bar{A}$ and total-length $L \equiv \bar{L}$, where $\bar{A}$ and $\bar{L}$ are the area and length of initial curve respectively. It follows from the first variation formulas and the constraints that

$$0 = \frac{\mathrm{d}A}{\mathrm{d}t} = \int_0^L \langle \boldsymbol{\nu}, \partial_t \boldsymbol{f} \rangle \,\mathrm{d}s, \quad 0 = \frac{\mathrm{d}L}{\mathrm{d}t} = -\int_0^L \langle \boldsymbol{\kappa}, \partial_t \boldsymbol{f} \rangle \,\mathrm{d}s.$$

That is, $\partial_t \boldsymbol{f} \perp \operatorname{span}\{\boldsymbol{\nu}, \boldsymbol{\kappa}\}$. Here span is in the sense of $L^2(\mathbb{R}/L\mathbb{Z}\,\mathrm{d}s)$, not pointwise sense. We decompose $\boldsymbol{\alpha} + \boldsymbol{\beta}$ into $\operatorname{span}\{\boldsymbol{\nu}, \boldsymbol{\kappa}\}$-part and its complement:

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = \lambda_1 \boldsymbol{\kappa} + \lambda_2 \boldsymbol{\nu} + \boldsymbol{w}, \quad \boldsymbol{w} \in \operatorname{span}\{\boldsymbol{\nu}, \boldsymbol{\kappa}\}^{\perp}.$$

Lagrange multiplies λ_1 and λ_2 are functions of t but independent of s, which are determined by the orthogonality $\partial_t \boldsymbol{f} \perp \operatorname{span}\{\boldsymbol{\nu}, \boldsymbol{\kappa}\}$. We derive $\boldsymbol{w} \equiv \boldsymbol{o}$ as follows: If $\boldsymbol{f}(t)$ is a gradient flow of $W(\cdot)$, the time-derivative of $W(\boldsymbol{f})$ is equal to $-\|\partial_t \boldsymbol{f}\|_{L^2}^2$. This shows $\langle \boldsymbol{w}, \partial_t \boldsymbol{f} \rangle_{L^2} = 0$. Using this, we calculate the time-derivative again, and obtain

$$\frac{\mathrm{d}W(\boldsymbol{f})}{\mathrm{d}t} = -\|\delta W(\boldsymbol{f})\|_{L^2}^2 + \|\lambda_1 \boldsymbol{\kappa} + \lambda_2 \boldsymbol{\nu}\|_{L^2}^2 + \|\boldsymbol{w}\|_{L^2}^2.$$

Note that λ_1 and λ_2, are determined independently of $\boldsymbol{w}$. Hence the energy W decreases fastest when $\boldsymbol{w} \equiv \boldsymbol{o}$ under the constraints. Consequently we get

$$\boldsymbol{\alpha} = \boldsymbol{o}, \quad \boldsymbol{\beta} = \lambda_1 \boldsymbol{\kappa} + \lambda_2 \boldsymbol{\nu}.$$

On the other hand, Okabe [8] investigated the flow under the constraints of area $A \equiv \bar{A}$ and local-length $\gamma \equiv \bar{\gamma}$. The local-length is defined by $\gamma = \|\partial_\theta \boldsymbol{f}\|$, where θ is the parameter of reference curve. And $\bar{\gamma}$ is that of the initial curve. We may assume that $\bar{\gamma} \equiv 1$, because we choose the initial curve and its arch-length parameter as the reference curve and θ respectively. We can derive

$$\boldsymbol{\alpha} = \{\partial_s(\xi\gamma)\}\,\boldsymbol{\tau}, \quad \boldsymbol{\beta} = \xi\gamma\boldsymbol{\kappa} + \lambda\boldsymbol{\nu} \tag{1.1}$$

from constraints $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial \gamma}{\partial t} = 0$. Here $\xi = \xi(s,t)$ and $\lambda = \lambda(t)$, Lagrange's multipliers, are determined by

$$-\delta W(\boldsymbol{f}) + \partial_s(\xi\gamma\boldsymbol{\tau}) + \lambda\boldsymbol{\nu} \perp (Y + \operatorname{span}\{\boldsymbol{\nu}\}),$$

where

$$Y = \overline{\operatorname{span}\{\partial_s(\varphi\gamma\boldsymbol{\tau}) \mid \varphi \in C^\infty(\mathbb{R}/L\mathbb{R})\}}.$$

The closure is in the sense of $L^2(\mathbb{R}/L\mathbb{Z}\,\mathrm{d}s)$. The complete derivation of (1.1) is more or less complicated but basically in a manner similar to the case where $A \equiv \bar{A}$ and $L \equiv \bar{L}$. We can find it in papers [7]. See also [8], which is accessible easier than [7], but notation is different from ours.

In [2] the global existence of total-constraint-flow were shown for any smooth initial curve, however, its asymptotic behavior as $t \to \infty$ had not been investigated.

Okabe also showed the local existence of local-constraint-flow in [8]. He also showed the global existence and its behavior, however his argument is available only for curves *whose rotation number is one*. It is natural to clarify unknown parts. To this end, we would like to unify both arguments and compensate each other. In this note, we propose a family of gradient flow under the intermediate constraints which unifies two gradient flows in [2] and [8].

2. A RESULT AND ITS PROOF

In what follows, we denote the gradient flows considered in [2] and [8] by $\boldsymbol{f}_0$ and $\boldsymbol{f}_1$ respectively. We write geometrical quantities derived from $\boldsymbol{f}_j$ with subscript "j". For example, the curvature vector of $\boldsymbol{f}_j$ is $\boldsymbol{\kappa}_j$, and so on.

Let $\mu \in [0,1]$ be a constant. Consider the gradient flow of W under the constraints $A \equiv \bar{A}$ and $\gamma \equiv (1-\mu)\gamma_0 + \mu$. It follows from constraints

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0, \quad \frac{\partial \gamma}{\partial t} = (1-\mu)\frac{\partial \gamma_0}{\partial t}$$

that

$$\boldsymbol{\alpha} = \{\partial_s(\xi_\mu \gamma)\}\,\boldsymbol{\tau}, \quad \boldsymbol{\beta} = \xi_\mu \gamma \boldsymbol{\kappa} + \lambda_\mu \boldsymbol{\nu}.$$

Lagrange's multipliers $\xi_\mu = \xi_\mu(s,t)$ and $\lambda_\mu = \lambda_\mu(t)$ are determined by

$$-\delta W(\boldsymbol{f}) + \partial_s(\xi_\mu \gamma \boldsymbol{\tau}) + \lambda_\mu \boldsymbol{\nu} - (1-\mu)\int_0^s \gamma^{-1}\frac{\partial \gamma_0}{\partial t}\,\mathrm{d}s\,\boldsymbol{\tau} \perp (Y + \mathrm{span}\,\{\boldsymbol{\nu}\}).$$

For details, see [7].

We had better write down our equations explicitly. Note that notation is slightly different from those of [2, 8]. Since we shall compare one gradient flow with each other below, we had better use the coordinate θ, which is arc-length parameter of reference curve, than s. The range of θ is $[0, \bar{L}]$. In the following k is the rotation number of initial curve, and $\kappa_j = \langle \boldsymbol{\kappa}_j, \boldsymbol{\nu}_j \rangle$ and $\kappa = \langle \boldsymbol{\kappa}, \boldsymbol{\nu} \rangle$ are curvatures of $\boldsymbol{f}_j$ and $\boldsymbol{f}$ respectively.

The gradient flow with total-length-constraint ([2]):

$$\partial_t \boldsymbol{f}_0 = -\delta W(\boldsymbol{f}_0) + \lambda_1 \boldsymbol{\kappa}_0 + \lambda_2 \boldsymbol{\nu}_0, \tag{2.1}$$

$$\int_0^{\bar{L}} \left\{ (\gamma_0 \partial_\theta \kappa_0)^2 - \frac{1}{2}\kappa_0^4 \right\} \gamma_0\,\mathrm{d}\theta + 2W(\boldsymbol{f}_0)\lambda_1 - 2\pi k \lambda_2 = 0, \tag{2.2}$$

$$-\int_0^{\bar{L}} \frac{1}{2}\kappa_0^3 \gamma_0\,\mathrm{d}\theta - 2\pi k \lambda_1 + \lambda_2 L = 0. \tag{2.3}$$

The gradient flow with local-length-constraint ([8]):

$$\partial_t \boldsymbol{f}_1 = -\delta W(\boldsymbol{f}_1) + \gamma_1^{-1}\partial_\theta(\xi \partial_\theta \boldsymbol{f}_1) + \lambda \boldsymbol{\nu}_1 \tag{2.4}$$

$$\kappa_1 \left(\gamma_1^{-1}\partial_\theta\right)^2 \kappa_1 + \frac{1}{2}\kappa_1^4 + \left(\gamma_1^{-1}\partial_\theta\right)^2 (\xi\gamma_1) - \xi \kappa_1^2 \gamma_1 - \lambda \kappa_1 = 0, \tag{2.5}$$

$$\int_0^{\bar{L}} \left(-\frac{1}{2}\kappa_1^3 + \xi \kappa_1 \gamma_1 \right) \gamma_1\,\mathrm{d}\theta + \lambda L = 0. \tag{2.6}$$

The gradient flow in this note:

$$\partial_t \boldsymbol{f} = -\delta W(\boldsymbol{f}) + \gamma^{-1}\partial_\theta(\xi_\mu \partial_\theta \boldsymbol{f}) + \lambda_\mu \boldsymbol{\nu} \tag{2.7}$$

$$\kappa\left(\gamma^{-1}\partial_\theta\right)^2\kappa + \frac{1}{2}\kappa^4 + \left(\gamma^{-1}\partial_\theta\right)^2(\xi_\mu\gamma) - \xi_\mu\kappa^2\gamma - \lambda_\mu\kappa = (1-\mu)\gamma^{-1}\partial_t\gamma_0, \tag{2.8}$$

$$\int_0^{\bar{L}} \left(-\frac{1}{2}\kappa^3 + \xi_\mu\kappa\gamma\right)\gamma\,\mathrm{d}\theta + \lambda_\mu L = 0. \tag{2.9}$$

For the derivation of these, see [7].

Consequently (2.7)–(2.9) with $\mu = 1$ is the same as (2.4)–(2.5). It, however, is not clear that (2.7)–(2.9) with $\mu = 0$ coincides with (2.1)–(2.3).

In this note we shall show

Theorem. Let $\boldsymbol{f}_0(t)$ satisfy (2.1)–(2.3) on a time-interval $[0, \bar{T}_0)$, and let $\boldsymbol{f}(t)$ satisfy (2.7)–(2.9) with $\mu = 0$ on $[0, \bar{T})$. Assume that both intervals are maximal ones. If $\boldsymbol{f}_0(0) = \boldsymbol{f}(0)$, then $\bar{T}_0 = \bar{T}$ and $\boldsymbol{f}_0(t) = \boldsymbol{f}(t)$ for $t \in [0, \bar{T})$.

When $\mu = 0$, it follows from the constraint on length that $\partial_t(\gamma - \gamma_0) \equiv 0$. Since $\gamma \equiv \gamma_0$ holds for the gradient flow (2.1)–(2.3), it satisfies (2.7)–(2.9) with $\mu = 0$. We would like to show the converse. Hereafter $\boldsymbol{f}$ is a solution of (2.7)–(2.9) with $\mu = 0$. What we should show is $\boldsymbol{f}_0 \equiv \boldsymbol{f}$ provided $\boldsymbol{f}_0(0) = \boldsymbol{f}(0)$. The quantity μ seems to be related with "tangential redistribution" ([3]–[6]).

If the initial curve is a circle with rotation number k, then the curve at t coincides with the initial curve, so the gradient flows are completely static. Hence we have nothing to prove, and we may assume that the initial curve is not a circle. This is equivalent to $2LW(\boldsymbol{f}(0)) - (2\pi k)^2 > 0$. Because the rotation number is invariant under each gradient flow, there exist $\delta > 0$ and $T > 0$ such that

$$\inf_{t\in[0,T]}\left\{2LW(\boldsymbol{f}_0(t)) - (2\pi k)^2\right\} > \delta > 0, \quad \inf_{t\in[0,T]}\left\{2LW(\boldsymbol{f}(t)) - (2\pi k)^2\right\} > \delta > 0.$$

Under this situation, λ_1 and λ_2 in (2.2)–(2.3) are determined by

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2W(\boldsymbol{f}_0) & -2\pi k \\ -2\pi k & L \end{pmatrix}^{-1} \begin{pmatrix} \displaystyle\int_0^{\bar{L}} \left\{-\left(\gamma_0^{-1}\partial_\theta\kappa_0\right)^2 + \frac{1}{2}\kappa_0^4\right\}\gamma_0\,\mathrm{d}\theta \\ \displaystyle\int_0^{\bar{L}} \frac{1}{2}\kappa_0^3\gamma_0\,\mathrm{d}\theta \end{pmatrix}.$$

We denote these quantities $\lambda_1(\boldsymbol{f}_0)$ and $\lambda_2(\boldsymbol{f}_0)$. If $\xi_\mu\gamma|_{\mu=0}$ is independent of θ, then from (2.8)–(2.9) we have

$$\xi_\mu\gamma|_{\mu=0} = \lambda_1(\boldsymbol{f}), \quad \lambda_\mu|_{\mu=0} = \lambda_2(\boldsymbol{f}),$$

where

$$\begin{pmatrix} \lambda_1(\boldsymbol{f}) \\ \lambda_2(\boldsymbol{f}) \end{pmatrix} = \begin{pmatrix} 2W(\boldsymbol{f}) & -2\pi k \\ -2\pi k & L \end{pmatrix}^{-1} \begin{pmatrix} \displaystyle\int_0^{\bar{L}} \left\{-\left(\gamma^{-1}\partial_\theta\kappa\right)^2 + \frac{1}{2}\kappa^4\right\}\gamma\,\mathrm{d}\theta \\ \displaystyle\int_0^{\bar{L}} \frac{1}{2}\kappa^3\gamma\,\mathrm{d}\theta \end{pmatrix}.$$

And therefore (2.7) coincides with (2.1). This is our strategy to prove Theorem.

Now we put

$$\rho_1 = \xi_\mu \gamma\big|_{\mu=0} - \lambda_1(\boldsymbol{f}_0), \quad \rho_2 = \lambda_\mu\big|_{\mu=0} - \lambda_2(\boldsymbol{f}_0),$$

and shall show $\rho_1 = 0$, which implies that $\xi_\mu \gamma\big|_{\mu=0}$ is independent of θ.

It is not difficult to show that γ_0 satisfies

$$\gamma_0^{-1}\kappa\left(\gamma_0^{-1}\partial_\theta\right)^2\kappa_0 + \frac{1}{2}\kappa_0^4 - \kappa_0\left(\lambda_1(\boldsymbol{f}_0)\kappa_0 + \lambda_2(\boldsymbol{f}_0)\right) = 0.$$

Combining this with (2.8), $\mu = 0$, and $\gamma = \gamma_0$, we get

$$\begin{aligned}
&\kappa\left(\gamma^{-1}\partial_\theta\right)^2\kappa + \frac{1}{2}\kappa^4 + \left(\gamma^{-1}\partial_\theta\right)^2\left(\xi_\mu\lambda\big|_{\mu=0}\right) - \xi_\mu\kappa^2\gamma\big|_{\mu=0} - \lambda_\mu\kappa\big|_{\mu=0}\\
&= \gamma^{-1}\partial_t\gamma_0 = \gamma_0^{-1}\partial_t\gamma_0\\
&= \kappa_0\left(\gamma^{-1}\partial_\theta\right)^2\kappa_0 + \frac{1}{2}\kappa_0^4 - \kappa_0\left(\lambda_1(\boldsymbol{f}_0)\kappa_0 + \lambda_2(\boldsymbol{f}_0)\right).
\end{aligned}$$

Therefore we have

$$\left(\gamma^{-1}\partial_\theta\right)^2\rho_1 - \kappa^2\rho_1 - \kappa\rho_2 = \Delta_1, \tag{2.10}$$

where

$$\begin{aligned}
\Delta_1 = {}& \kappa_0\left(\gamma^{-1}\partial_\theta\right)^2\kappa_0 - \kappa\left(\gamma^{-1}\partial_\theta\right)^2\kappa\\
&+\frac{1}{2}\kappa_0^4 - \frac{1}{2}\kappa^4 - \left(\kappa_0^2 - \kappa^2\right)\lambda_1(\boldsymbol{f}_0) - (\kappa_0 - \kappa)\lambda_2(\boldsymbol{f}_0).
\end{aligned} \tag{2.11}$$

Since $\displaystyle\int_0^{\bar{L}} \gamma_0\,\mathrm{d}\theta \equiv \bar{L}$, we have

$$L = \int_0^{\bar{L}} \gamma\,\mathrm{d}\theta = (1-\mu)\int_0^{\bar{L}} \gamma_0\,\mathrm{d}\theta + \mu\int_0^{\bar{L}} \mathrm{d}\theta = (1-\mu)\bar{L} + \mu\bar{L} = \bar{L}.$$

From this relation, $-2\pi k = \displaystyle\int_0^{\bar{L}} \kappa\gamma\,\mathrm{d}\theta$, and $\gamma_0 = \gamma$, we obtain

$$\begin{aligned}
0 &= -\int_0^{\bar{L}} \frac{1}{2}\kappa_0^3\gamma_0\mathrm{d}\theta - 2\pi k\lambda_1(\boldsymbol{f}_0) + \lambda_2(\boldsymbol{f}_0)\bar{L}\\
&= -\int_0^{\bar{L}} \frac{1}{2}\kappa_0^3\gamma\,\mathrm{d}\theta + \int_0^{\bar{L}} \kappa\lambda_1(\boldsymbol{f}_0)\gamma\,\mathrm{d}\theta + \lambda_2(\boldsymbol{f}_0)L.
\end{aligned}$$

Combining this with (2.9), we get

$$\int_0^{\bar{L}} \rho_1\kappa\gamma\,\mathrm{d}\theta + \rho_2 L = \int_0^{\bar{L}} \Delta_2\gamma\,\mathrm{d}\theta, \tag{2.12}$$

where

$$\Delta_2 = \frac{1}{2}\left(\kappa^3 - \kappa_0^3\right). \tag{2.13}$$

We multiply both sides of (2.10) by $-\rho_1\gamma$, and integrate. Using (2.12) and Schwarz' inequality, we have

$$\begin{aligned}
\int_0^{\bar{L}} \left\{ \left(\gamma^{-1}\partial_\theta \rho_1\right)^2 + \rho_1^2\kappa^2 \right\} \gamma \, \mathrm{d}\theta &= -\int_0^{\bar{L}} \left(\rho_2\kappa + \Delta_1\right) \rho_1 \gamma \, \mathrm{d}\theta \\
&= \frac{1}{L}\int_0^{\bar{L}} \left(\rho_1\kappa - \Delta_2\right) \gamma \, \mathrm{d}\theta \int_0^{\bar{L}} \rho_1\kappa\gamma \, \mathrm{d}\theta - \int_0^{\bar{L}} \Delta_1\rho_1\gamma \, \mathrm{d}\theta \\
&\leqq \int_0^{\bar{L}} \rho_1^2\kappa^2\gamma \, \mathrm{d}\theta - \frac{1}{L}\int_0^{\bar{L}} \Delta_2\gamma \, \mathrm{d}\theta \int_0^{\bar{L}} \rho_1\kappa\gamma \, \mathrm{d}\theta - \int_0^{\bar{L}} \Delta_1\rho_1\gamma \, \mathrm{d}\theta.
\end{aligned}$$

This shows

$$\int_0^{\bar{L}} \left(\gamma^{-1}\partial_\theta \rho_1\right)^2 \gamma \, \mathrm{d}\theta \leqq -\frac{1}{L}\int_0^{\bar{L}} \Delta_2\gamma \, \mathrm{d}\theta \int_0^{\bar{L}} \rho_1\kappa\gamma \, \mathrm{d}\theta - \int_0^{\bar{L}} \Delta_1\rho_1\gamma \, \mathrm{d}\theta. \tag{2.14}$$

Put $Q_T = \mathbb{R}/\bar{L}\mathbb{Z} \times [0,T]$. In what follows C is a generic constant depending possibly on

$$\sup_{Q_T} \left(|\kappa| + |\gamma^{-1}\partial_\theta\kappa| + |\kappa_0| + |\gamma^{-1}\partial_\theta\kappa_0| + \gamma \right).$$

$C(\delta)$ is that depending on the above quantity and δ. Another notation $C(\cdot,\delta)$ has a similar meaning.

Lemma. Assume that $\inf_{t\in[0,T]} \left\{ 2LW(\boldsymbol{f}(t)) - (2\pi k)^2 \right\} > \delta > 0$. Then it holds for each $t \in [0,T]$ that

$$\|\rho_1\|_{L^\infty} + \|\gamma^{-1}\partial_\theta\rho_1\|_{L^2_\gamma} \leqq C(\delta) \left\{ \|\kappa - \kappa_0\|_{L^2_\gamma} + \|\gamma^{-1}\partial(\kappa - \kappa_0)\|_{L^2_\gamma} \right\}.$$

Here $L^\infty = L^\infty(\mathbb{R}/\bar{L}\mathbb{Z})$, $L^2_\gamma = L^2(\mathbb{R}/\bar{L}\mathbb{Z}, \gamma\, d\theta)$.

Proof. It is enough to show

$$\|\rho_1\|_{L^\infty}^2 \leqq C(\delta) \left\{ \|\gamma^{-1}\partial_\theta\rho_1\|_{L^2_\gamma}^2 + \|\kappa - \kappa_0\|_{L^2_\gamma}^2 + \|\gamma^{-1}\partial_\theta(\kappa - \kappa_0)\|_{L^2_\gamma}^2 \right\}, \tag{2.15}$$

and

$$\begin{aligned}
\|\gamma^{-1}\partial_\theta\rho_1\|_{L^2_\gamma}^2 \leqq C(\delta) \Big[&\|\rho_1\|_{L^\infty} \left\{ \|\kappa - \kappa_0\|_{L^2_\gamma} + \|\gamma^{-1}\partial_\theta(\kappa - \kappa_0)\|_{L^2_\gamma} \right\} \\
&+ \|\kappa - \kappa_0\|_{L^2_\gamma}^2 + \|\gamma^{-1}\partial_\theta(\kappa - \kappa_0)\|_{L^2_\gamma}^2 \Big].
\end{aligned} \tag{2.16}$$

Firstly we shall show (2.15). By the mean-value theorem of integration, there exists $\theta_0 = \theta_0(t)$ such that

$$2\, \rho_1\big|_{\theta=\theta_0} W = \int_0^{\bar{L}} \rho_1\kappa^2\gamma \, \mathrm{d}\theta.$$

Multiplying both sides of (2.10) by γ, and integrating, we get

$$-\int_0^{\bar{L}} \rho_1 \kappa^2 \gamma \,\mathrm{d}\theta + 2\pi k \rho_2 = \int_0^{\bar{L}} \Delta_1 \gamma \,\mathrm{d}\theta.$$

It follows from these and (2.12) that

$$\begin{aligned}
2\, \rho_1|_{\theta=\theta_0} W &= 2\pi k \rho_2 - \int_0^{\bar{L}} \Delta_1 \gamma \,\mathrm{d}\theta \\
&= \frac{2\pi k}{L} \left(\int_0^{\bar{L}} \Delta_2 \gamma \,\mathrm{d}\theta - \int_0^{\bar{L}} \rho_1 \kappa \gamma \,\mathrm{d}\theta \right) - \int_0^{\bar{L}} \Delta_1 \gamma \,\mathrm{d}\theta \\
&= \frac{(2\pi k)^2}{L} \rho_1|_{\theta=\theta_0} - \frac{2\pi k}{L} \int_0^{\bar{L}} \left(\int_{\theta_0}^{\theta} \partial_\theta \rho_1 \mathrm{d}\theta \right) \kappa \gamma \,\mathrm{d}\theta \\
&\qquad + \int_0^{\bar{L}} \left(-\Delta_1 + \frac{2\pi k}{L} \Delta_2 \right) \gamma \,\mathrm{d}\theta.
\end{aligned}$$

Here we have used

$$\rho_1 = \rho_1|_{\theta=\theta_0} + \int_{\theta_0}^{\theta} \partial_\theta \rho_1 \,\mathrm{d}\theta. \tag{2.17}$$

Consequently

$$\begin{aligned}
\rho_1|_{\theta=\theta_0} = &\frac{1}{2LW - (2\pi k)^2} \\
&\times \left\{ -2\pi k \int_0^{\bar{L}} \left(\int_{\theta_0}^{\theta} \partial_\theta \rho_1 \mathrm{d}\theta \right) \kappa \gamma \,\mathrm{d}\theta + \int_0^{\bar{L}} (-L\Delta_1 + 2\pi k \Delta_2) \gamma \,\mathrm{d}\theta \right\}.
\end{aligned}$$

Furthermore we have easily

$$\left| \int_{\theta_0}^{\theta} \partial_\theta \rho_1 \mathrm{d}\theta \right| \leqq C \|\gamma^{-1} \partial_\theta \rho_1\|_{L^2_\gamma}, \tag{2.18}$$

$$\int_0^{\bar{L}} |\kappa| \gamma \,\mathrm{d}\theta \leqq C,$$

$$\begin{aligned}
\left| \int_0^{\bar{L}} \Delta_1 \gamma \,\mathrm{d}\theta \right| = &\left| \int_0^{\bar{L}} \left[\left\{ \gamma^{-1} \partial_\theta (\kappa - \kappa_0) \right\} \left\{ \gamma^{-1} \partial_\theta (\kappa + \kappa_0) \right\} \right.\right. \\
&\qquad - \frac{1}{2} (\kappa - \kappa_0) \left(\kappa^3 + \kappa^2 \kappa_0 + \kappa \kappa_0^2 + \kappa_0^3 \right) \\
&\qquad \left.\left. - (\kappa - \kappa_0)(\kappa + \kappa_0) \lambda_1(\boldsymbol{f}_0) \right] \gamma \,\mathrm{d}\theta \right| \\
\leqq\ & C(\delta) \left\{ \|\gamma^{-1} \partial_\theta (\kappa - \kappa_0)\|_{L^2_\gamma} + \|\kappa - \kappa_0\|_{L^2_\gamma} \right\},
\end{aligned}$$

$$\left| \int_0^{\bar{L}} \Delta_2 \gamma \,\mathrm{d}\theta \right| = \left| \frac{1}{2} \int_0^{\bar{L}} (\kappa - \kappa_0) \left(\kappa^2 + \kappa \kappa_0 + \kappa_0^2 \right) \gamma \,\mathrm{d}\theta \right| \leqq C \|\kappa - \kappa_0\|_{L^2_\gamma}. \tag{2.19}$$

Hence it holds that

$$\left|\rho_1|_{\theta=\theta_0}\right| \leqq C(\delta)\left\{\|\gamma^{-1}\partial_\theta\rho_1\|_{L^2_\gamma} + \|\kappa-\kappa_0\|_{L^2_\gamma} + \|\gamma^{-1}\partial_\theta(\kappa-\kappa_0)\|_{L^2_\gamma}\right\}.$$

From (2.17) and (2.18) we obtain

$$|\rho_1| \leqq \left|\rho_1|_{\theta=\theta_0}\right| + \left|\int_{\theta_0}^{\theta}\partial_\theta\rho_1 \mathrm{d}\theta\right| \leqq \left|\rho_1|_{\theta=\theta_0}\right| + C\|\gamma^{-1}\partial_\theta\rho_1\|_{L^2_\gamma}.$$

Consequently (2.15) has been proven.

Next we shall show (2.16). It follows from (2.14), (2.19) and

$$\left|\int_0^{\bar{L}}\rho_1\kappa\gamma\,\mathrm{d}\theta\right| \leqq C\|\rho_1\|_{L^\infty}$$

that

$$\|\gamma^{-1}\partial_\theta\rho_1\|^2_{L^2_\gamma} \leqq C\|\rho_1\|_{L^\infty}\|\kappa-\kappa_0\|_{L^2_\gamma} + \left|\int_0^{\bar{L}}\Delta_1\rho_1\gamma\,\mathrm{d}\theta\right|.$$

Hence

$$\begin{aligned}\left|\int_0^{\bar{L}}\Delta_1\rho_1\gamma\,\mathrm{d}\theta\right| \leqq &\left|\int_0^{\bar{L}}\left\{\kappa\left(\gamma^{-1}\partial_\theta\right)^2\kappa - \kappa_0\left(\gamma^{-1}\partial_\theta\right)^2\kappa_0\right\}\rho_1\gamma\,\mathrm{d}\theta\right| \\ &+ \left|\int_0^{\bar{L}}\left\{\frac{1}{2}\left(\kappa^4-\kappa_0^4\right) + \left(\kappa^2-\kappa_0^2\right)\lambda_1(\boldsymbol{f}_0) + (\kappa-\kappa_0)\lambda_2(\boldsymbol{f}_0)\right\}\rho_1\gamma\,\mathrm{d}\theta\right|.\end{aligned}$$

The second term in the right-hand side can be dominated by $C(\delta)\|\rho_1\|_{L^\infty}\|\kappa-\kappa_0\|_{L^2_\gamma}$. By the integration by parts, the first term is estimated by

$$\begin{aligned}&\left|\int_0^{\bar{L}}\left[-\left\{\gamma^{-1}\partial_\theta(\kappa-\kappa_0)\right\}\left\{\gamma^{-1}\partial_\theta(\kappa+\kappa_0)\right\}\rho_1\right.\right. \\ &\qquad \left.\left. - \left(\kappa\gamma^{-1}\partial_\theta\kappa - \kappa_0\gamma^{-1}\partial_\theta\kappa_0\right)\gamma^{-1}\partial_\theta\rho_1\right]\gamma\,\mathrm{d}\theta\right| \\ &\qquad \leqq C\|\rho_1\|_{L^\infty}\|\gamma^{-1}\partial_\theta(\kappa-\kappa_0)\|_{L^2_\gamma} + \frac{1}{2}\|\gamma^{-1}\partial_\theta\rho_1\|^2_{L^2_\gamma} \\ &\qquad\quad + C\left\{\|\kappa-\kappa_0\|^2_{L^2_\gamma} + \|\gamma^{-1}\partial_\theta(\kappa-\kappa_0)\|^2_{L^2_\gamma}\right\}.\end{aligned}$$

Thus (2.16) has been proven. □

Corollary. Assume that $\inf_{t\in[0,T]}\left\{2LW(\boldsymbol{f}(t)) - (2\pi k)^2\right\} > \delta > 0$. Then it holds for each $t \in [0,T]$ that

$$|\rho_2| \leqq C(\delta)\left\{\|\kappa-\kappa_0\|_{L^2_\gamma} + \|\gamma^{-1}\partial(\kappa-\kappa_0)\|_{L^2_\gamma}\right\}.$$

Proof. The assertion is derived from (2.12) and Lemma. □

Proof of Theorem. As stated before, we may assume

$$\inf_{t\in[0,T]}\left\{2LW(\boldsymbol{f}(t))-(2\pi k)^2\right\}>\delta>0.$$

We would like to show $\rho_1=\rho_2=0$, $\kappa=\kappa_0$. It is not difficult to see

$$\begin{aligned}\partial_t\kappa = & -\left(\gamma^{-1}\partial_\theta\right)^4\kappa-\frac{5}{2}\kappa^2\left(\gamma^{-1}\partial_\theta\right)^2\kappa-3\kappa\left(\gamma^{-1}\partial_\theta\kappa\right)^2-\frac{1}{2}\kappa^5\\ &+\left(\gamma^{-1}\partial_\theta\right)^2(\xi_\mu\kappa\gamma+\lambda_\mu)|_{\mu=0}+\kappa^2\,(\xi_\mu\kappa\gamma+\lambda_\mu)|_{\mu=0}\\ &+\left\{\gamma^{-1}\partial_\theta(\xi_\mu\gamma|_{\mu=0})\right\}\gamma^{-1}\partial_\theta\kappa,\\ \partial_t\kappa_0 = & -\left(\gamma^{-1}\partial_\theta\right)^4\kappa_0-\frac{5}{2}\kappa_0^2\left(\gamma^{-1}\partial_\theta\right)^2\kappa_0-3\kappa_0\left(\gamma^{-1}\partial_\theta\kappa_0\right)^2-\frac{1}{2}\kappa_0^5\\ &+\lambda_1(\boldsymbol{f}_0)\left(\gamma^{-1}\partial_\theta\right)^2\kappa_0+\kappa_0^2\,(\lambda_1(\boldsymbol{f}_0)\kappa_0+\lambda_2(\boldsymbol{f}_0)).\end{aligned}$$

We have used $\gamma_0=\gamma$ when deriving the equation for κ_0. We subtract the second equation from the first one, multiply both sides by $(\kappa-\kappa_0)\gamma$, and integrate. Using Lemma and Corollary, we get the estimate

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\kappa-\kappa_0\|_{L^2_\gamma}^2-\frac{1}{2}\int_0^{\bar{L}}(\kappa-\kappa_0)^2(\partial_t\gamma)\,\mathrm{d}\theta+\left\|\left(\gamma^{-1}\partial_\theta\right)^2(\kappa-\kappa_0)\right\|_{L^2_\gamma}^2\\ &\quad\leqq C\left\{\|\kappa-\kappa_0\|_{L^2_\gamma}^2+\|\gamma^{-1}\partial_\theta(\kappa-\kappa_0)\|_{L^2_\gamma}^2+\|\rho_1\|_{L^\infty}^2+\|\gamma^{-1}\partial_\theta\rho_1\|_{L^2_\gamma}^2+|\rho_2|^2\right\}\\ &\quad\leqq C(\delta)\left\{\|\kappa-\kappa_0\|_{L^2_\gamma}^2+\|\gamma^{-1}\partial_\theta(\kappa-\kappa_0)\|_{L^2_\gamma}^2\right\}.\end{aligned}$$

Since $\gamma\equiv\gamma_0$, it holds that

$$\frac{1}{2}\int_0^{\bar{L}}(\kappa-\kappa_0)^2(\partial_t\gamma)\,\mathrm{d}\theta\leqq C\left(\sup_{Q_T}|\partial_t\log\gamma_0|\right)\|\kappa-\kappa_0\|_{L^2_\gamma}^2.$$

By the integration by parts and Schwarz' inequality, we obtain

$$\begin{aligned}\|\gamma^{-1}\partial_\theta(\kappa-\kappa_0)\|_{L^2_\gamma}^2 &= -\int_0^{\bar{L}}(\kappa-\kappa_0)\left\{\left(\gamma^{-1}\partial_\theta\right)^2(\kappa-\kappa_0)\right\}\gamma\,\mathrm{d}\theta\\ &\leqq \frac{1}{4\varepsilon}\|\kappa-\kappa_0\|_{L^2_\gamma}^2+\varepsilon\left\|\left(\gamma^{-1}\partial_\theta\right)^2(\kappa-\kappa_0)\right\|_{L^2_\gamma}^2.\end{aligned}$$

Here we can take $\varepsilon>0$ as small as we like. Therefore we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\kappa-\kappa_0\|_{L^2_\gamma}^2\leqq C\left(\sup_{Q_T}|\partial_t\log\gamma_0|,\delta\right)\|\kappa-\kappa_0\|_{L^2_\gamma}^2.$$

We integrate with respect to t, and apply Gronwall's lemma. Since $\kappa|_{t=0}-\kappa_0|_{t=0}=0$, it holds that $\|\kappa-\kappa_0\|_{L^2_\gamma}\equiv 0$. It follows from Lemma and Corollary that $\rho_1=\rho_2=0$. Thus we have shown that $\boldsymbol{f}_0(t)\equiv\boldsymbol{f}(t)$ as long as they are not circles.

Now we assume that one of $\boldsymbol{f}_0(t)$ or $\boldsymbol{f}(t)$ becomes a circle at $t=T_*<\infty$. Then both must become circles simultaneously. Clearly both have the same length and the same rotation number. Since we can show the centers of gravity $\frac{1}{L}\int_0^{\bar{L}}\boldsymbol{f}_0(t)\gamma_0\,\mathrm{d}\theta$ and $\frac{1}{L}\int_0^{\bar{L}}\boldsymbol{f}(t)\gamma\,\mathrm{d}\theta$ are independent of t, we can conclude that $\boldsymbol{f}_0(T_*)$ and $\boldsymbol{f}(T_*)$ are congruent and concentric circles. Hence $\boldsymbol{f}_0(t)\equiv\boldsymbol{f}(t)$ for $t\geqq T_*$. Thus the proof is complete. $\square$

Remark. If the rotation number is one (or minus one), and if the initial curve is not a circle, then curve at $t > 0$ is also not a circle. This is derived from the constraints $A \equiv \bar{A}$, $L \equiv \bar{L}$, and the isoperimetric inequality. This is the reason why Okabe [8] assumed that the initial curve has the rotation number one. It was shown in [2, Proposition 4.1] without assumption of rotation number that for the gradient flow constructed by a singular limit there, $\boldsymbol{f}_0(t)$ is not a circle for every $t > 0$ provided $\boldsymbol{f}_0(0)$ is not a circle. It has not been shown that the same property holds or not for every solution to (2.1)–(2.3). However, it still holds under some additional conditions, see [9].

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