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# ABSORPTION IN STOCHASTIC EPIDEMICS

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A two dimensional stochastic differential equation is suggested as a stochastic model for the Kermack–McKendrick epidemics. Its strong (weak) existence and uniqueness and absorption properties are investigated. The examples presented in Section 5 are meant to illustrate possible different asymptotics of a solution to the equation.

*Keywords:* SIR epidemic models, stochastic epidemic models, stochastic differential equation, strong solution, weak solution, absorption, Kermack–McKendrick model

*AMS Subject Classification:* 92D25, 37N25

## 1. INTRODUCTION

Consider a constant $n_0 > 0$ and the 2-dimensional stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t &= -\,\varphi(X_t, Y_t)\,\mathrm{d}t + \psi(X_t, Y_t)\,\mathrm{d}W_t, && X_0 = x_0 \geq 0 \\ \mathrm{d}Y_t &= \varphi(X_t, Y_t)\,\mathrm{d}t - \psi(X_t, Y_t)\,\mathrm{d}W_t - \gamma Y_t\,\mathrm{d}t, && Y_0 = y_0 \geq 0, \end{aligned} \tag{1}$$

such that $x_0 + y_0 = n_0$.

We shall assume and denote

$$\begin{aligned} &\varphi, \psi : \mathbb{R}^2 \to \mathbb{R} \quad \text{Borel functions,} \\ &\varphi \geq 0 \quad \text{on} \quad [0, n_0]^2, \quad \gamma > 0. \end{aligned} \tag{2}$$

The SDE (1) is designed to make more realistic the classical Kermack–McKendrick epidemic model given by

$$\begin{aligned} \mathrm{d}X_t &= -\beta X_t Y_t, \quad X_0 = x_0 > 0 \\ \mathrm{d}Y_t &= \beta X_t Y_t - \gamma Y_t, \quad Y_0 = y_0 > 0 \\ \mathrm{d}Z_t &= \gamma Y_t, \quad Z_0 = 0, \end{aligned} \tag{3}$$

that assumes a fixed sized population of $n_0 = x_0 + y_0$ individuals, the population being divided into three subpopulations $X_t$, $Y_t$ and $Z_t = \gamma \int_0^t Y_s\,\mathrm{d}s$, that change their respective sizes $X_t$, $Y_t$ and $\gamma \int_0^t Y_s\,\mathrm{d}s$ in time by means of the differential equation (3). The individuals in $X_t$ (susceptibles) are those exposed to an infection,

$Y_t$ (infectives) denotes the size of individuals who are able to spread the infection and finally $Z_t$ (removals) collect all restored to health not being able to be infected again. Obviously, both in (1) and (3)

$$X_t + Y_t + \gamma \int_0^t Y_s \, \mathrm{d}s = x_0 + y_0 = n_0 \tag{4}$$

holds. Since $\varphi(X_t, Y_t)$ (or $\beta X_t Y_t$ in (3)) measures the speed of the transfer in the direction $X_t \to Y_t$, the constant $\beta > 0$ in (3) is called the intensity of infection. The constant $\gamma^{-1}$ is proportional to the average duration of the "state of being infected". The stochastic differentials $\pm\psi(X_t, Y_t)\mathrm{d}W_t$ are designed to model a random exchange between $X_t$ and $Y_t$ subpopulations in both directions, where a negative value of $\psi(X_t, Y_t)\mathrm{d}W_t$ suggests a possibility that some individuals might be infected again. The Figure illustrates a well behaved epidemics with $\varphi(x, y) = x^+y^+$ and $\psi(x, y) = \sqrt{x^+y^+}$ and a rather confusing model with $\varphi(x, y) = \psi(x, y) = xy$.

Our research is basically aimed at the problem of showing uniqueness and existence of nonnegative solutions $(X_t, Y_t)$ to (1).

A closely related problem is to determine whether the barriers $x = 0$ and $y = 0$ are absorbing or not.

The problem is how to recover the limit $X_\infty = n_0 - \gamma \int_0^\infty Y_s \, \mathrm{d}s$ (the equality follows by (4) since $Y_\infty = 0$ holds under fairly general conditions by Theorem 2.3).

Recall that all problems stated above are easily and completely answered for the equation (3) as follows ([9]):

Assuming that $x_0 > 0$, $y_0 > 0$, formula (3) has a unique solution $(X_t, Y_t)$ that is positive on $\mathbb{R}^+ = [0, \infty)$ with $Y_\infty = 0$. The limit $X_\infty = n_0 - \gamma \int_0^\infty Y_s \, \mathrm{d}s$ is received as a unique solution to

$$X_\infty = x_0 \exp\left\{-(n_0 - X_\infty)\frac{\beta}{\gamma}\right\} > 0. \tag{5}$$

The equation (3) is easily seen to be solved uniquely by

$$\dot{Z}_t = \gamma\left(n_0 - Z_t - x_0 \exp\left\{-\frac{\beta}{\gamma} Z_t\right\}\right), \ \dot{Z}_t = \gamma Y_t, \ Z_0 = 0, \tag{6}$$

however, the explicit form of $Z_t$ as a function of time is not available.

Consider any solution $(X_t, Y_t)$ to (1) and define

$$\tau_X := \inf\{t \geq 0 : X_t = 0\}, \quad \tau_Y := \inf\{t \geq 0 : Y_t = 0\}, \tag{7}$$

and $\tau := \tau_X \wedge \tau_Y$. In (3) we get $\tau_X = \tau_Y = \tau = \infty$, but Example 5.1 exhibits the equation with $\psi(x, y) = 0$ and $\varphi(x, y) = \gamma y^+ I_{(0,\infty)}(x)$ that is uniquely solved by $(X_t, Y_t)$ where $\tau_Y = \infty$ and $\tau_X < \infty$ which yields $Y_t > 0$ for all $t \geq 0$ and $X_\infty = X_{\tau_X} = 0$.

Example 5.2 with

$$\mathrm{d}X_t = Y_t^+ I_{[X_t > 0]} \, \mathrm{d}W_t, \quad \mathrm{d}Y_t = -\gamma Y_t \, \mathrm{d}t - Y_t^+ I_{[X_t > 0]} \, \mathrm{d}W_t$$

provides $\tau_Y = \infty$ a.s. and $0 < P[\tau_X = \infty] < 1$ if $\gamma > \frac{1}{2}$. By formula (14) below, we are able to prove that for any nonnegative solution $(X_t, Y_t)$ to the equation (1) the following implication holds almost surely:

$$\tau_X < \infty \Rightarrow X_t = 0, \quad Y_t = Y_{\tau_X} e^{-\gamma(t-\tau_X)}, \quad t \in [\tau_X, \infty) \tag{8}$$

The implication is illustrated by the examples. Our principal results that will be found in Section 3, are related to the unique strong and weak existence of the solution $(X_t, Y_t)$ to (1). For example, the Theorem 3.1 states that the equation has a unique solution $(X_t, Y_t) \in [0, n_0]^2$ that is absorbed by the barrier $B = \{x = 0\} \cup \{y = 0\}$ provided that $\psi$ and $\varphi$ are locally Lipschitz on $(0, n_0]^2$ and $\psi = \varphi = 0$ on $B$.

A generalization of the Kermack–McKendrick model (see (3)) is provided by Štěpán and Hlubinka in [11]. The intensity $\beta$ is assumed to be a function of $(X_t, Y_t, Z_t)$, or more simply a function of the removals subpopulation $Z_t$ while the population size $n_0$ is constructed to be time dependent and solves the Engelbert–Schmidt equation.

In [5], Greenwood, Gordillo and Kuske present a stochastic SIR model with infection rate $\beta$, removals rate $\gamma$ and birth and death rate $\mu$ and compare its behavior with the corresponding deterministic model. While the number of infectives converges as time tends to infinity to a steady equilibrium, for the deterministic model, the SIR model produces a permanent oscillation.

In [1], Allen and Kiruparaha offer both a deterministic and a stochastic epidemic model with multiple pathogens. The models are studied in detail in the case of two pathogens, the asymptotic stability of equilibrium is discussed, and the paper also presents a series of numerical examples.

For the sake of completeness we also recall some more recent references that relate to the deterministic dynamics of infections, further developing the research started by the classical Kermack–McKendrick equation (3) introduced in [9]. These are Bussenberg and Kenneth [3], Daley and Gani [4], Kalas and Pospíšil [7] and finally Wai-Yuan and Hulin [12] who offer a detailed review of the contemporary state of art in the field of mathematical epidemic models.

A lot of papers concerned to deterministic and stochastic epidemic models are collected also in [12].

## 2. PRELIMINARIES

The probabilistic framework for (1) is structured as $(\Omega, \mathcal{F}, P, W_t, \mathcal{F}_t)$, where $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(\mathcal{F}_t, t \geq 0)$ a $P$-complete right continuous filtration and finally $W_t$ is an $\mathcal{F}_t$-Wiener process ($W_0 = 0$). Our terminology and definitions coincide with those introduced by [8], e. g. the random variables $\tau_X$, $\tau_Y$ and $\tau$ defined by (7) are $\mathcal{F}_t$-stopping times. Throughout the present section, we shall assume (2) and consider a fixed solution $(X_t, Y_t)$ to the equation (1).

**Lemma 2.1.** If $0 < t < \tau$, then

$$\gamma \int_0^t Y_s \, \mathrm{d}s \in (0, n_0) \tag{9}$$

holds outside a $P$-null set. Especially, $X_t < n_0$ and $Y_t < n_0$ hold almost surely for $0 < t < \tau$. Moreover,

$$\tau^0 = \inf\left\{ t \geq 0 : \left( X_t, Y_t, \gamma \int_0^t Y_s \,\mathrm{d}s \right) \in \mathbb{R}^3 \setminus [0, n_0]^3 \right\} =: \lambda^0 \tag{10}$$

holds almost surely, where

$$\tau_X^0 := \inf\{t \geq 0 : X_t < 0\}, \quad \tau_Y^0 := \inf\{t \geq 0 : Y_t < 0\}, \quad \tau^0 := \tau_X^0 \wedge \tau_Y^0. \tag{11}$$

Proof. Consider $t \in (0, \tau)$, therefore $Y_s > 0$ and $X_s > 0$ for all $s \leq t$ which implies that $\gamma \int_0^t Y_s \,\mathrm{d}s > 0$. Together with (4) we get $X_t < n_0$, $Y_t < n_0$ and $\gamma \int_0^t Y_s \,\mathrm{d}s < n_0$, that proves (9). Obviously $\tau_X^0 \wedge \tau_Y^0 \geq \lambda^0$. If $(X_t, Y_t, Z_t) \notin [0, n_0]^3$ then $[X_t < 0] \vee [Y_t < 0] \vee [\exists\, s \leq t : Y_s < 0]$ which proves that $\tau_X^0 \wedge \tau_Y^0 \leq t$, hence $\tau_X^0 \wedge \tau_Y^0 \leq \lambda^0$. □

Assuming

$$\psi(x, y) = \varphi(x, y) = 0, \quad \forall\, (x, y) \notin (0, \infty)^2, \tag{12}$$

we may be more specific.

**Lemma 2.2.** $\tau_X^0 \wedge \tau_Y^0 = \lambda^0 = +\infty$ a.s. if $\psi$ and $\varphi$ satisfy (12).

In other words, (12) guarantees that $(X_t, Y_t, \gamma \int_0^t Y_s \,\mathrm{d}s)$ never exits the cube $[0, n_0]^3$.

Proof. Assume to the contrary that $\lambda^0 < \infty$. Then, there exists a $t_0 > 0$ such that either $X_{t_0} < 0$ or $Y_{t_0} < 0$. If $X_{t_0} < 0$, denote $s_0 := \sup\{0 \leq s \leq t_0 : X_s \geq 0\} < t_0$. Thus, $X_s \leq 0$ for all $s_0 \leq s \leq t_0$. Hence, according to (12),

$$X_{t_0} = X_{s_0} - \int_{s_0}^{t_0} \varphi(X_s, Y_s) \,\mathrm{d}s + \int_{s_0}^{t_0} \psi(X_s, Y_s) \,\mathrm{d}W_s = X_{s_0} \geq 0.$$

That is a contradiction.

If $Y_{t_0} < 0$, denote $s_0 = \sup\{0 \leq s \leq t_0 : Y_s \geq 0\} < t_0$, hence $Y_{s_0} = 0$ and $Y_s \leq 0$ on the interval $[s_0, t_0]$. For a $t \in [s_0, t_0]$, we write

$$Y_t = Y_{s_0} + \int_{s_0}^{t} \varphi(X_s, Y_s) \,\mathrm{d}s + \int_{s_0}^{t} \psi(X_s, Y_s) \,\mathrm{d}W_s - \gamma \int_{s_0}^{t} Y_s \,\mathrm{d}s$$

that together with (12) yields that $Y_t = -\gamma \int_{s_0}^t Y_s \,\mathrm{d}s$ holds on $[s_0, t_0]$. Hence, $Y_t = 0$ for arbitrary $t \in [s_0, t_0]$ that contradicts our assumption $Y_{t_0} < 0$. □

**Theorem 2.3.** Consider the SDE (1) and assume (12). Then $X_t$ is a nonnegative $\mathcal{F}_t$-supermartingal and almost surely the limits

$$X_\infty = \lim_{t\to\infty} X_t, \quad Y_\infty = \lim_{t\to\infty} Y_t = 0 \text{ a.s.}$$

exists. Moreover,

$$\tau_X < \infty \Rightarrow X_t = 0, \quad \text{for all} \quad t \geq \tau_X, \tag{13}$$

consequently

$$\tau_X < \infty \Rightarrow X_\infty = X_{\tau_X} = 0,$$

hold outside a $P$-null set.

P r o o f. It follows by (2) and Lemma 2.2 that

$$x_0 + \int_0^t \psi(X_s, Y_s)\,\mathrm{d}W_s = X_t + \int_0^t \varphi(X_s, Y_s)\,\mathrm{d}s$$

is a nonnegative martingale, hence $X_t$ is a nonnegative supermartingale which processes are known to have an integrable limits $X_\infty$ and to be absorbed by $x = 0$. Thus, it follows by (4) that the limit $Y_\infty$ exists almost surely, and because $\gamma \int_0^\infty Y_s\,\mathrm{d}s < n_0$, by Lemma 2.2 it follows that $Y_\infty = 0$ a.s. □

In particular, if $\varphi$ and $\psi$ satisfy (12) then

$$\tau_X < \infty \quad \Rightarrow \quad Y_t = Y_{\tau_X} e^{-\gamma(t-\tau_X)}, \quad t \geq \tau_X \tag{14}$$

holds almost surely.

To verify this, combine (12) and (13) to get

$$Y_t = Y_{\tau_X} + \int_{\tau_X}^t \varphi(X_s, Y_s)\,\mathrm{d}s - \int_{\tau_X}^t \psi(X_s, Y_s)\,\mathrm{d}W_s - \gamma \int_{\tau_X}^t Y_s\,\mathrm{d}s = Y_{\tau_X} - \gamma \int_{\tau_X}^t Y_s\,\mathrm{d}s.$$

Hence, under (12),

$$\text{if } X_0 = 0, \text{ then } X = 0,\ Y_t = y_0 e^{-\gamma t} \text{ is a unique solution to (1).} \tag{15}$$

Observing (4) we get $X_t = n_0 - Y_t - \gamma \int_0^t Y_s\,\mathrm{d}s$. This formula may be inverted as follows:

$$Y_t = -X_t + e^{-\gamma t} n_0 + \gamma e^{-\gamma t} \int_0^t e^{\gamma s} X_s\,\mathrm{d}s,\ t \geq 0 \text{ a.s.} \tag{16}$$

To prove this, rewrite (4) as $Y_t = n_0 - X_t - \gamma \int_0^t Y_s\,\mathrm{d}s$ and solve the equation by means of proposition 21.2 in [8].

The implication (13) in Theorem 2.3 may be completed as follows:

**Lemma 2.4.** Assume (12) and let $n_0 > 0$. Outside a $P$-null set holds

$$\tau_Y < \infty \Rightarrow \tau_Y < \tau_X. \tag{17}$$

Moreover, if $(X, Y)$ is absorbed by $B = \{x = 0\} \cup \{y = 0\}$ then

$$\tau_Y < \infty \Rightarrow \tau_X = \infty \quad (\text{and } X_\infty = X_{\tau_Y} > 0) \quad \text{almost surely.} \tag{18}$$

P r o o f. If $\tau_X \leq \tau_Y < \infty$ then it follows by (13) that $X_{\tau_Y} = Y_{\tau_Y} = 0$ which contradicts (16) as $X \geq 0$ by Lemma 2.2.

Consider a solution $(X, Y)$ that is absorbed by the barrier $B$ and suppose that $\tau_Y < \infty$, $\tau_X < \infty$. Hence, $\tau_Y < \tau_X < \infty$ by (17) and $Y$ is absorbed by $y = 0$ at the time $\tau_Y$. It follows that $X_{\tau_X} = Y_{\tau_X} = 0$, a contradiction to (16), again. □

## 3. THE EXISTENCE AND UNIQUENESS

Throughout the rest of our presentation we shall consider the equation (1) with $x_0 > 0$, $y_0 > 0$ and fix

$$\begin{aligned} &\varphi, \psi : [0, n_0]^2 \to \mathbb{R},\ \varphi \geq 0,\ \varphi, \psi \text{ both locally Lipschitz on } (0, n_0]^2 \\ &\text{and bounded on } [0, n_0]^2 \end{aligned} \tag{19}$$

such that

$$\varphi(x, 0) = \psi(x, 0) = 0, \quad \varphi(0, y) = \psi(0, y) = 0,\ x, y \in [0, n_0]. \tag{20}$$

Our principal result is the following theorem:

**Theorem 3.1.** If $\varphi$ and $\psi$ satisfy (19) and (20), then there exists a unique process $(X, Y) \in [0, n_0]^2$ that is absorbed by the barrier $B = \{x = 0\} \cup \{y = 0\}$ and that solves the equation (1).

Especially, assuming that $\varphi$ and $\psi$ in (19) and (20) are Lipschitz on $[0, n_0]^2$, there exists a process $(X, Y) \in [0, n_0]^2$ absorbed by $B$ that is a unique nonnegative solution to (1).

P r o o f. Let $n_0 > a_1 > a_2 \ldots$ and $\lim_{n \to \infty} a_n = 0$, denote $D_n = [a_n, n_0]^2$ and assume that $(x_0, y_0) \in D_1$. Further, construct $\varphi_n, \psi_n : \mathbb{R}^2 \to \mathbb{R}$ Lipschitz and bounded such that

$$\varphi_n = \varphi \text{ and } \psi_n = \psi \text{ on } D_n, \quad \varphi_n \geq 0 \text{ and } \varphi_n = \psi_n = 0 \text{ on } \mathbb{R}^2 \setminus (0, \infty)^2.$$

The equation

$$\begin{aligned} \mathrm{d}X_t &= -\varphi_n(X_t, Y_t)\,\mathrm{d}t + \psi_n(X_t, Y_t)\,\mathrm{d}W_t, & X_0 = x_0 \\ \mathrm{d}Y_t &= \varphi_n(X_t, Y_t)\,\mathrm{d}t - \psi_n(X_t, Y_t)\,\mathrm{d}W_t - \gamma Y_t \mathrm{d}t, & Y_0 = y_0 \end{aligned} \tag{21}$$

has a unique strong solution $(X^n, Y^n)$ as the coefficients $\varphi_n(x,y), \psi_n(x,y)$ and $\gamma y$ are Lipschitz of a linear growth. It follows by Lemma 2.2 that $(X^n, Y^n)$ is a process that never leaves the cube $[0, n_0]^2$. Denote

$$\lambda_n := \inf\{t \geq 0 : (X^n, Y^n) \notin D_n\}.$$

Observe that $\lambda_n < \infty$ a.s. since $Y_\infty = 0$ a.s. by Theorem 2.3 and that the strong uniqueness property of equation (21) implies that

$$(X^{n+1}, Y^{n+1}) = (X^n, Y^n) \text{ on } [0, \lambda_n] \text{ and } \lambda_n < \lambda_{n+1},\ n \in \mathbb{N}$$

holds outside a $P$-null set $N$. Put $\lambda = \sup \lambda_n$ and for each $\omega \in \Omega$ define a continuous function

$$(X^0(\omega), Y^0(\omega)) : [0, \lambda(\omega)) \to [0, n_0]^2$$

by

$$(X^0(\omega), Y^0(\omega)) = (X^n(\omega), Y^n(\omega)) \text{ on } [0, \lambda_n(\omega)].$$

We shall prove that outside another $P$-null set

$$\begin{aligned} &\lambda < \infty \Rightarrow \text{ there exists the limit } (X^0_{\lambda-}, Y^0_{\lambda-}) \in [0, n_0]^2 \text{ such that} \\ &\text{either } X^0_{\lambda-} = 0 \text{ or } Y^0_{\lambda-} = 0 \text{ holds.} \end{aligned} \tag{22}$$

To verify this, note that both $\varphi(X^0, Y^0) I_{[0,\lambda)}$ and $\psi(X^0, Y^0) I_{[0,\lambda)}$ are $\mathcal{F}_t$-progressive bounded processes, hence

$$\begin{aligned} M_t =& x_0 - \int_0^t \varphi(X^0_s, Y^0_s) I_{[0,\lambda)}(s)\, \mathrm{d}s + \int_0^t \psi(X^0_s, X^0_s) I_{[0,\lambda)}(s)\, \mathrm{d}W_s \\ N_t =& y_0 + \int_0^t \varphi(X^0_s, Y^0_s) I_{[0,\lambda)}(s)\, \mathrm{d}s - \int_0^t \psi(X^0_s, X^0_s) I_{[0,\lambda)}(s)\, \mathrm{d}W_s \\ &- \gamma \int_0^t Y^0_s I_{[0,\lambda)}(s)\, \mathrm{d}s \end{aligned}$$

are well defined continuous $\mathcal{F}_t$-semimartingales on $\mathbb{R}^+$ such that

$$(M, N) = (X^0, Y^0) \text{ on } [0, \lambda) \tag{23}$$

holds almost surely.

It follows that $\lambda < \infty$ implies the existence of the left limit $(X_{\lambda^-}, Y_{\lambda^-}) = (M_\lambda, N_\lambda) \in [0, n_0]$ almost surely. Because either $(X^0_{\lambda_n}, Y^0_{\lambda_n}) = (a_n, Y^0_{\lambda_n})$ or $(X^0_{\lambda_n}, Y^0_{\lambda_n}) = (X^0_{\lambda_n}, a_n)$ and $\lambda_n \nearrow \lambda$, we conclude that either $X^0_{\lambda^-} = 0$ or $Y^0_{\lambda^-} = 0$. Finally, put

$$(X_t, Y_t) = (X^0_t, Y^0_t) I_{[0,\lambda)}(t) + (X^0_{\lambda\_}, Y^0_{\lambda\_}\, e^{-\gamma(t-\lambda)}) I_{[\lambda,\infty)}(t), \tag{24}$$

check that it is a continuous $\mathcal{F}_t$-adapted process that lives in $[0, n_0]^2$ and that is absorbed by the barrier $B$.

We shall prove the uniqueness of the absorbed process: Assume that

$$(X,Y) \text{ and } (X',Y') \text{ are both absorbed by } B \text{ and solve (1) for some } (\varphi,\psi). \tag{25}$$

It follows by Lemma 2.2 that processes $(X,Y)$ and $(X',Y')$ stay in $[0,n_0]^2$ forever. Denote

$$\lambda_n := \inf\{t \geq 0 : (X_t,Y_t) \notin D_n\} \wedge \inf\{t \geq 0 : (X'_t,Y'_t) \notin D_n\} < \infty \quad \text{a.s.}$$

and check that both $(X,Y)$ and $(X',Y')$ solves the (21) equation on the interval $[0,\lambda_n]$. Owing to the strong uniqueness property of the equation (21), we conclude that outside a $P$-null set

$$(X,Y) = (X',Y') \text{ holds on } [0,\lambda) \quad \text{where } \lambda = \sup \lambda_n. \tag{26}$$

It remains to prove that

$$(X,Y) = (X',Y') \text{ on } [\lambda,\infty) \text{ if } \lambda < \infty \tag{27}$$

is a statement to be true with probability one.

For these purposes note that (26) verifies that the implication

$$\lambda < \infty \Rightarrow (X_\lambda, Y_\lambda) = (X'_\lambda, Y'_\lambda) \in B$$

holds almost surely. Because both processes $(X,Y)$ and $(X',Y')$ are absorbed by $B$, we get that

$$\lambda < \infty \Rightarrow \varphi(X,Y) = \psi(X,Y) = 0 \text{ on } [\lambda,\infty) \text{ almost surely}$$

and therefore outside a $P$-null set

$$\lambda < \infty,\ t \geq \lambda \quad \Rightarrow \quad \begin{cases} (X_t,Y_t) = (X_\lambda, Y_\lambda - \gamma \int_\lambda^\infty Y_s \,\mathrm{d}s) \\ (X'_t,Y'_t) = (X'_\lambda, Y'_\lambda - \gamma \int_\lambda^\infty Y'_s \,\mathrm{d}s) \end{cases}$$

holds. Hence,

$$\lambda < \infty,\ t \geq \lambda \quad \Rightarrow \quad \begin{cases} X_t = X_\lambda = X'_\lambda = X'_t \\ Y_t = Y_\lambda e^{-\gamma(t-\lambda)} = Y'_\lambda e^{-\gamma(t-\lambda)} = Y'_t \end{cases}$$

which verifies (27).

The "especially part" now follows in a straightforward manner: We have already proved that there is an absorbed process $(X,Y) \in [0,n_0]^2$ that solve equation (1). Because $\varphi_+$ and $\psi_+$ are Lipschitz maps, the equation has a unique strong solution, hence the solution is absorbed by the barrier $B$. □

One can also apply the above reasoning to prove the following corollary:

**Corollary 3.2.** Assuming (19) and (20), the uniqueness in law holds for the equation (1) in the following sense:

If $(X_t, Y_t) \in (0, n_0)^2$ and $(X'_t, Y'_t) \in (0, n_0)^2$ are solutions to (1) (defined perhaps on different probability spaces) then $\mathcal{L}(X, Y) = \mathcal{L}(X', Y')$.

For the proof, just note that uniqueness in law holds for equation (21).

## 4. ABSORPTION

We are able to offer sufficient conditions for the equation (1) to produce solutions, which are absorbed by $\{y = 0\}$, hence absorbed by the barrier $B$.

**Theorem 4.1.** Let the uniqueness in law[1] holds for the equation (1). Then assuming (12), its arbitrary solution $(X, Y)$ is absorbed by the barrier $B$.

Proof. Just note that generally any solution $(X, Y)$ may be reorganized to a solution $(X^a, Y^a)$ that is absorbed by $\{y = 0\}$ as follows:

$$\begin{aligned} (X^a, Y^a) &= (X, Y) \quad \text{on} \quad [0, \tau_Y), \\ (X^a, Y^a) &= (X_{\tau_Y}, 0) \quad \text{on} \quad [\tau_Y, \infty). \end{aligned} \tag{28}$$

□

The uniqueness in law is not a property easy to recognize. Itô theorem (e.g. Theorem 21.3, p. 415, in [8]), that proves the property for $\varphi$ and $\psi$ Lipschitz on $[0, n_0]^2$, may not be always adequate in our context. A weaker sufficient condition is suggested by the following lemma.

**Theorem 4.3.** Let $\varphi$ and $\psi$ satisfy (12) and suppose that there exists an $\varepsilon > 0$ such that

$$0 \le y \le \varepsilon \quad \Rightarrow \quad \varphi(x, y) \le \gamma y \quad \text{for all} \quad x \in [0, n_0]. \tag{29}$$

Then arbitrary solution $(X, Y)$ to (1) is absorbed by the barrier $B$.

Proof. Note that

$$Z_t := -I_{[\tau_Y < \infty]} \int_{\tau_Y}^{t+\tau_Y} \psi(X_s, Y_s)\, \mathrm{d}W_s$$

is a continuous $\mathcal{F}_{\tau_Y + t}$-local martingale and

$$Y_{t+\tau_Y} = I_{[\tau_Y < \infty]} \int_{\tau_Y}^{t+\tau_Y} \varphi(X_s, Y_s) - \gamma Y_s \,\mathrm{d}s + Z_t, \quad t \ge 0$$

a continuous $\mathcal{F}_{\tau_Y + t}$-semimartingale. Denoting

$$\tau_\delta := \inf\{t \ge 0 : Y_{t+\tau_Y} \ge \delta\}, \quad \delta > 0$$

[1] If $(X, Y)$ and $(\acute{X}, \acute{Y})$ are solutions (perhaps on different probability spaces) then $(X, Y)$ and $(\acute{X}, \acute{Y})$ have the same probability distribution in $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^2)$.

we define an $\mathcal{F}_{\tau_Y+t}$-stopping time and by $Z_{t\wedge\tau_\delta}$ an $\mathcal{F}_{\tau_Y+t}$-local martingale. It follows that for arbitrary $0<\delta\le\varepsilon$

$$Z_{t\wedge\tau_\delta} = Y_{t\wedge\tau_\delta+\tau_Y} - I_{[\tau_Y<\infty]}\int_{\tau_Y}^{t\wedge\tau_\delta+\tau_Y}\varphi(X_s,Y_s)-\gamma Y_s\,\mathrm{d}s \ge 0$$

is a nonnegative $\mathcal{F}_{\tau_Y+t}$-local martingale, hence a nonnegative $\mathcal{F}_{\tau_Y+t}$-supermartingale. Therefore

$$Y_{t\wedge\tau_\delta+\tau_Y} = Y_{\tau_Y} = 0, \quad t\ge 0$$

holds almost surely for arbitrary $0<\delta\le\varepsilon$. Especially the implication

$$\tau_Y<\infty, \quad \tau_\delta<\infty \quad\Rightarrow\quad Y_{\tau_\delta+\tau_Y}=0$$

is true outside a $P$-null set. It follows that $P[\tau_Y<\infty,\tau_\delta<\infty]=0$ for all $0<\delta\le\varepsilon$, hence the process $Y$ is absorbed by $\{y=0\}$. □

## 5. EXAMPLES

**Example 5.1.** Consider the deterministic equation

$$\mathrm{d}X_t = -\gamma Y_t^+ I_{[X_t>0]}\,\mathrm{d}t, \quad \mathrm{d}Y_t = \gamma Y_t^+ I_{[X_t>0]}\,\mathrm{d}t - \gamma Y_t\,\mathrm{d}t \tag{30}$$

with $x_0=y_0=\gamma=1$. This is an equation (1) with $\varphi(x,y)=\gamma y^+ I_{(0,\infty)}(x)$ and $\psi(x,y)=0$, i. e. subject to (12). A solution is found easily as

$$X_t=(1-t)^+, \quad Y_t = e^{-(t-1)^+} \tag{31}$$

with $\tau_X=1$ and $\tau_Y=+\infty$. Because $\varphi(x,y)\le\gamma y$ for all $x\in[0,n_0]$ it follows by Lemma 2.2 and Theorem 4.3 that any solution to (30) is a nonnegative process absorbed by the barrier $B$. "The moreover part" of Theorem 3.1 further yields that (31) is a unique solution to (30) as $\varphi$ and $\psi$ are locally Lipschitz maps on $(0,n_0]^2$.

**Example 5.2.** The equation

$$\mathrm{d}X_t = Y_t^+ I_{[X_t>0]}\,\mathrm{d}W_t, \quad \mathrm{d}Y_t = -\gamma Y_t\,\mathrm{d}t - Y_t^+ I_{[X_t>0]}\,\mathrm{d}W_t, \quad x_0>0,\ y_0>0 \tag{32}$$

with $\varphi(x,y)=0$ and $\psi(x,y)=y^+ I_{(0,\infty)}(x)$ has a diffusion coordinate $X_t$. Theorem 3.1 later on with Theorem 4.3 states that (32) has a unique nonnegative solution $(X,Y)$. We shall verify that $\tau_Y=\infty$ a.s. It is possible to prove by using Itô formula (e. g. Theorem 17.18, p. 340, in [8]), that outside a $P$-null set, $Y_t = y_0\exp\left\{-\left(\gamma+\frac{1}{2}\right)t - W_t\right\}>0$, for all $t\in[0,\lambda]$, hence $\tau_Y=\infty$ almost surely.

We shall denote

$$Z_t=\exp\left\{-\left(\gamma+\frac{1}{2}\right)t-W_t\right\}, \quad \lambda=\inf\left\{t\ge 0: \int_0^t Z_s\,\mathrm{d}W_s = -\frac{x_0}{y_0}\right\}$$

and define

$$X_t = x_0 + y_0 \int_0^{t\wedge\lambda} Z_s \,\mathrm{d}W_s, \quad Y_t = \begin{cases} y_0 Z_t, & t < \lambda \\ y_0 Z_\lambda e^{-\gamma(t-\lambda)}, & t \geq \lambda. \end{cases} \tag{33}$$

Note that $\tau_X = \lambda$ and that $(X, Y)$ is the unique solution to (32) as $Z_t$ solves linear equation $\mathrm{d}Z_t = -\gamma Z_t \,\mathrm{d}t - Z_t \,\mathrm{d}W_t$ with $Z_0 = 1$.

First, observe that $P[\tau_X < \infty] > 0$. Obviously

$$\tau_X \leq \inf \left\{ t \geq 0 : Z_t = \frac{n_0}{y_0} \right\} := r$$

almost surely, because $r$ is time of the first entry of $Y_t$ into $\{n_0\}$. Thus,

$$\tau_X \leq r = \inf \left\{ t \geq 0 : \left(\gamma + \frac{1}{2}\right) t + W_t = \ln \frac{y_0}{n_0} \right\}.$$

It is a well known fact (see p. 18 in [10]) that

$$P[\tau_X < \infty] \geq P[r < \infty] = \exp\left\{ 2\left(\frac{1}{2} + \gamma\right) \ln \frac{y_0}{x_0} \right\} > 0$$

as $\ln \frac{y_0}{n_0} < 0$ and $\gamma + \frac{1}{2} > 0$.

On the other side, it may happen that $P[\tau_X = \infty] > 0$.

Denote $I_t = \int_0^t Z_s \,\mathrm{d}W_s$ and recall that $\tau_X = \lambda = \inf\{t \geq 0 : I_t = -\frac{x_0}{y_0} =: l_0\}$. Write $\beta(t) := \langle I \rangle(t) = \int_0^t Z_s^2 \,\mathrm{d}s$ and apply the DDS Theorem (e. g. Theorem 18.4, p. 352, in [8]) to exhibit a Wiener process $\mathcal{B}$ such that $\mathcal{B}_{\beta(t)} = I_t$ on $\mathbb{R}^+$ almost surely[2]. Denote by $\varepsilon$ the time of the first entry of $\mathcal{B}$ into $\{l_0\}$ and recall that $\varepsilon < \infty$ almost surely while $\mathrm{E}\varepsilon = \infty$. Obviously,

$$\tau_X = \lambda = \infty \iff \varepsilon \geq \beta(\infty) \quad \text{almost surely.} \tag{34}$$

We compute that

$$\mathrm{E}Z_t^2 = \mathrm{E}\exp\{(-2\gamma + 1)t - 2W_t - 2t\} = e^{(1-2\gamma)t}, \tag{35}$$

hence $\mathrm{E}\beta(\infty) < \infty$ if $\gamma > \frac{1}{2}$. It follows that $P[\varepsilon < \beta(\infty)] < 1$ and that $P[\tau_X = \infty] > 0$ by (34) for $\gamma > \frac{1}{2}$. Thus, $P[\tau_X < \infty] \in (0, 1)$ if $\gamma > \frac{1}{2}$.

On the behavior of $\tau_X$ for more reasonable values $\gamma \leq \frac{1}{2}$ we may only remark that $\mathrm{E}\beta(\infty) = \infty$ (see (35)) and $\beta(\infty) < \infty$ almost surely (for all $\gamma > 0$). To verify the latter statement apply the SLLN for $W$ choosing $\delta < \frac{2\gamma+1}{2}$ and a $T_m > 0$ large enough that $|\frac{W_t(\omega)}{t}| \leq \delta$ for all $t > T_m$, hence

$$Z_t^2(\omega) = \exp\left\{ \left[ -(2\gamma + 1) - 2\frac{W_t(\omega)}{t} \right] t \right\} \leq \exp\{[-(2\gamma + 1) + 2\delta] t\}$$

for all $t \geq T_m$.

[2] To construct $\mathcal{B}$ we may need to extend the underlying probability space $(\Omega, \mathcal{F}, P)$ to its standard extension, see [8], p. 352.

**Example 5.3.** Consider $a \geq 0$, $c \in \mathbb{R}$ and the equation

$$\begin{aligned} \mathrm{d}X_t &= -cY_t^+ I_{[X_t>0]}\,\mathrm{d}t + \sqrt{2aY_t^+} I_{[X_t>0]}\,\mathrm{d}W_t, \\ \mathrm{d}Y_t &= (cY_t^+ I_{[X_t>0]} - \gamma Y_t)\,\mathrm{d}t - \sqrt{2aY_t^+} I_{[X_t>0]}\,\mathrm{d}W_t, \end{aligned} \tag{36}$$

with $X_0 = x_0 > 0$, $Y_0 = y_0 > 0$. The coefficients $\varphi(x,y) = -cy^+ I_{(0,\infty)}(x)$ and $\psi(x,y) = \sqrt{2ay^+ I_{(0,\infty)}(x)}$ are locally Lipschitz on $(0,n_0]^2$, hence there is a unique nonnegative by the barrier $B$ absorbed solution $(X,Y)$ to (36). Assuming $c \leq \gamma$ it follows by Theorem 4.3 that (36) has no other solution. Having $a \geq 0$ arbitrary, the solution is constructed as follows: the equation

$$Z_t = y_0 + (c-\gamma)\int_0^t Z_s\,\mathrm{d}s - \int_0^t \sqrt{2aZ_s^+}\,\mathrm{d}W_s \tag{37}$$

has a unique strong solution $Z_t \geq 0$, for a not completely trivial verification, see Example 8.2, p. 221 in [6]. Putting $\lambda_Z = \inf\{t \geq 0 : Z_t = 0\}$ one can verify that $P[\lambda_Z < \infty] > 0$ and that

$$c \leq \gamma \quad \Rightarrow \quad P[\lambda_Z < \infty] = 1 \quad \text{and} \quad Z_{\lambda_Z + t} = 0 \quad \forall\, t \geq 0 \text{ a.s.} \tag{38}$$

Also define

$$I_t = x_0 - c\int_0^t Z_s\,\mathrm{d}s + \int_0^t \sqrt{2aZ_s^+}\,\mathrm{d}W_s, \quad \lambda_I = \inf\{t \geq 0 : I_t = 0\},$$

and $\lambda = \lambda_Z \wedge \lambda_I$.

A straightforward calculation proves that

$$X_t = I_{t\wedge\lambda}, \quad Y_t = \begin{cases} Z_t, & t < \lambda \\ Z_\lambda e^{-\gamma(t-\lambda)}, & t \geq \lambda. \end{cases} \tag{39}$$

solve the equation (36), hence it is a unique nonnegative absorbed solution to (36) (a unique nonnegative solution to (36) if $c \leq \gamma$).

Note that

$$\begin{aligned} \lambda_I \leq \lambda_Z \quad &\Longrightarrow \quad \tau_X = \lambda_I \quad \text{and} \quad \tau_Y = \infty \\ \lambda_Z < \lambda_I \quad &\Longrightarrow \quad \tau_X = \infty \quad \text{and} \quad \tau_Y = \lambda_Z < \infty \end{aligned} \tag{40}$$

holds almost surely by Lemma 2.4 and therefore $\tau_X \wedge \tau_Y = \lambda$. Assume

$$y_0 \leq x_0, \quad 0 < c \leq \frac{\gamma}{2}, \quad a = \frac{1}{2} \quad (\Rightarrow c - \gamma \leq -c \text{ and } \lambda_Z < \infty \text{ a.s.}) \tag{41}$$

and prove that $P[\tau_Y < \infty] > 0$.

Put $\alpha(t) := \int_0^t Z_s\,\mathrm{d}s = \langle \int \sqrt{Z_s^+}\,\mathrm{d}W_s\rangle(t)$ and note that $\alpha$ is strictly increasing on $[0,\lambda_Z)$ and a constant on $[\lambda_Z,\infty)$. As in Example 5.2 we may exhibit a Wiener process $\mathcal{B}$ such that $\int_0^t \sqrt{Z_s^+}\,\mathrm{d}W_s = \mathcal{B}_{\alpha(t)}$ almost surely, hence assuming $a = \frac{1}{2}$, $Z_t = Z'_{\alpha(t)}$ and $I_t = I'_{\alpha(t)}$ hold almost surely where

$$Z'_t = y_0 + (c-\gamma)t + \mathcal{B}_t \quad \text{and} \quad I'_t = x_0 - ct - \mathcal{B}_t$$

is a pair of drifted Brownian motions. Now assume that $\tau_Y = \infty$ a.s. and observe that (40) yields that $\lambda_I \le \lambda_Z < \infty$ a.s. It follows that

$$\lambda_{I'} = \alpha(\lambda_I) \le \alpha(\lambda_Z) = \lambda_{Z'}. \tag{42}$$

On the other hand,

$$Z'_t = x_0 + (c-\gamma)t + \mathcal{B}_t \ge x_0 + ct + \mathcal{B}_t =: I''_t \quad \text{and} \quad \mathcal{L}(I'') = \mathcal{L}(I'). \tag{43}$$

Consequently,

$$\lambda_{Z'} \le \lambda_{I''} =: \inf\{t \le 0 : I''_t = 0\} \quad \text{and} \quad \mathcal{L}(\lambda_{I''}) = \mathcal{L}(\lambda_{I'}). \tag{44}$$

Combining (42) and (44) we get for any $t \le 0$

$$P[\lambda_{Z'} \ge t] \le P[\lambda_{I''} \ge t] = P[\lambda_{I'} \ge t] \le P[\lambda_{Z'} \ge t],$$

hence, $\mathcal{L}(\lambda_{Z'}) = \mathcal{L}(\lambda_{I'})$ which together with (42) yields $\lambda_{Z'} = \lambda_{I'}$ almost surely and a contradiction because $Z'$ and $I'$ are distinct processes almost surely, therefore $P[\tau_Y < \infty] > 0$.

Finally, the procedure introduced above may serve to exhibit the probability distribution of the random variable $n_0 - (X_\lambda + Y_\lambda) = \gamma \int_0^\lambda Y_s\,\mathrm{d}s$ that defines the size of "Removals subpopulation" at the time $\lambda = \tau_Y \wedge \tau_X = \lambda_Z \wedge \lambda_I$.

We assume (38) again to get $\lambda_Z < \infty$ a.s. This verifies that $\lambda_I \le \lambda_Z$ iff $\lambda_{I'} \le \lambda_{Z'}$, hence $\alpha(\lambda) = \lambda_{I'} \wedge \lambda_{Z'}$ a.s. and

$$n_0 - (X_\lambda + Y_\lambda) = \gamma \int_0^\lambda Y_s\,\mathrm{d}s = \gamma \int_0^\lambda Z_s\,\mathrm{d}s = \gamma\alpha(\lambda) = \gamma(\lambda_{I'} \wedge \lambda_{Z'}) \quad \text{a.s.}$$

**Example 5.4.** Consider the equation

$$\mathrm{d}X_t = I_{[X_t>0,Y_t>0]}\,\mathrm{d}W_t, \quad \mathrm{d}Y_t = -\gamma Y\,\mathrm{d}t - I_{[X_t>0,Y_t>0]}\,\mathrm{d}W_t \tag{45}$$

with $x_0 > y_0 > 0$, i.e. $\varphi(x,y) = 0$ and $\psi(x,y) = I_{(0,\infty)^2}(x,y)$. Using Theorem 3.1 together with Theorem 4.3 we get that equation (45) has a unique solution which is absorbed by the barrier $B$.

Because $X_t = x_0 + W_{t\wedge\tau}$ and $Y_t = y_0 - \gamma\int_0^t Y_s\,\mathrm{d}s - W_{t\wedge\tau} \le y_0 - W_{t\wedge\tau}$ almost surely then $\tau_X = \inf\{t \ge 0 : W_t = -x_0\}$ and $\tau \le \tau_{(-x_0,y_0)}$ where $\tau_{(-x_0,y_0)} := \inf\{t \ge 0 : W_t \notin (-x_0, y_0)\}$, hence $\tau < \infty$ almost surely (see Proposition 7.3, p. 14, in [10]).

It remains to prove that $P[\tau_X < \infty] > 0$ and $P[\tau_Y < \infty] > 0$. First define $\tau_{\breve{y}} := \inf\{t \geq 0 : W_t = y_0 - \gamma n_0 t\}$ and note that $\tau_Y \geq \tau_{\breve{y}}$. Obviously $\tau_X \wedge \tau_{\breve{y}} \leq \frac{1}{\gamma}$, hence (see p. 295 in [2])

$$P[\tau_X < \infty] \geq P\left[\tau_X \leq \frac{1}{\gamma}\right] \geq P\left[\tau_{\breve{y}} > \frac{1}{\gamma}\right]$$
$$= \int_{\frac{1}{\gamma}}^{\infty} \frac{y_0}{\sqrt{2\pi} t^{\frac{3}{2}}} \exp\left\{-\frac{(y_0 - \gamma n_0 t)^2}{2t}\right\} \mathrm{d}t > 0$$

holds. On the other hand, if we denote $\tau_{y_0} = \inf\{t \geq 0 : W_t = y_0\}$, then $P[\tau_Y < \tau_X] \geq P[\tau_{y_0} \leq \tau_X] = \frac{x_0}{n_0} > 0$ (see Proposition 7.3, p. 14, in [10] again), therefore $P[\tau_Y < \infty] \geq \frac{x_0}{n_0} > 0$.

**Example 5.5.** Another equation that provides a unique solution $(X, Y)$ with $P[\tau_Y < \infty] > 0$ is

$$\begin{aligned} \mathrm{d}X_t &= \sqrt{X_t^+ \wedge Y_t^+}\, \mathrm{d}W_t, & X_0 &= x_0 > 0 \\ \mathrm{d}Y_t &= -\sqrt{X_t^+ \wedge Y_t^+}\, \mathrm{d}W_t - \gamma Y_t\, \mathrm{d}t, & Y_0 &= y_0 > 0. \end{aligned} \tag{46}$$

The coefficients $\varphi(x, y) = 0$ and $\psi(x, y) = \sqrt{x^+ \wedge y^+}$ are chosen to be locally Lipschitz on $(0, n_0]^2$ such that $\varphi = \psi = 0$ on $\mathbb{R}^2 \setminus (0, \infty)^2$ and $\varphi(x, y) \leq \gamma y$ when $y \geq 0$. It follows from Theorem 4.3 and Theorem 3.1 that (46) has a unique solution $(X, Y)$ which is nonnegative and absorbed by the barrier $B = [x = 0] \cup [y = 0]$. Moreover, the process $X_t$ is a bounded martingale, hence $\mathrm{E}X_\infty = x_0 > 0$, and therefore $P[X_\infty > 0] = p > 0$. Because $Y_\infty = 0$ almost surely, we may construct a $T > 0$ such that $P(A) \geq p/2$, where $A = [X_t > Y_t, t \geq T]$. The equation

$$Z_t = Y_T - \int_T^t \sqrt{Z_s^+}\, \mathrm{d}W_s - \gamma \int_T^t Z_s\, \mathrm{d}s, \quad t \geq T$$

has a unique strong solution $Z \geq 0$ with $\lambda_Z = \inf\{t \geq T : Z_t = 0\} < \infty$ a.s. by Example 8.2 in [6] again. It follows that there is a $P$-null set $N$ such that

$$\omega \in A \setminus N \quad \Rightarrow \quad Y_t(\omega) = Z_t(\omega) \quad \text{for all} \quad t \geq T \quad \text{and} \quad \tau_Y = \lambda_Z < \infty$$

hence, $P[\tau_Y < \infty] \geq P(A) \geq p/2 > 0$. Using Theorem 2.3, we get $P[\tau_X = \infty] \geq P[X_\infty > 0] = p > 0$.

**Example 5.6.** Consider a bounded $\varphi$, locally Lipschitz and positive on $(0, n_0]^2$ such that $\varphi(x, y) = 0$ on $\mathbb{R}^2 \setminus (0, \infty)^2$. Then $\psi(x, y) = \sqrt{\varphi(x, y)}$ is a locally Lipschitz function on $(0, n_0]^2$ and we shall scrutinize the following equation:

$$\begin{aligned} \mathrm{d}X_t &= -c\varphi(X_t, Y_t)\, \mathrm{d}t + \sqrt{\varphi(X_t, Y_t)}\, \mathrm{d}W_t, & X_0 &= x_0 > 0 \\ \mathrm{d}Y_t &= c\varphi(X_t, Y_t)\, \mathrm{d}t - \gamma Y_t\, \mathrm{d}t - \sqrt{\varphi(X_t, Y_t)}\, \mathrm{d}W_t, & Y_0 &= y_0 > 0 \end{aligned} \tag{47}$$

where $c > 0$, a more general version of the equation (36). It follows by the Theorem 3.1 that (47) has a unique nonnegative solution $(X, Y)$ that is absorbed by the barrier $B = [x = 0] \cup [y = 0]$. Note that

$$\tau_X \wedge \tau_Y = \tau = \inf\{t \geq 0 : (X_t, Y_t) \in B\} = \inf\{t \geq 0 : \varphi(X_t, Y_t) = 0\}$$

and observe that $\alpha(t) = \int_0^t \varphi(X_s, Y_s)\,\mathrm{d}s = \langle \int \sqrt{\psi(X_s, Y_s)}\,\mathrm{d}W_s \rangle(t)$ is a process strictly increasing on $[0, \tau)$ and constant on $[\tau, \infty)$. Apply the DDS theorem again to write $\int_0^t \sqrt{\varphi(X_s, Y_s)}\,\mathrm{d}W_s = \mathcal{B}_{\alpha(t)}$, where $\mathcal{B}$ is a Wiener process. Thus, the coordinate $X$ may be represented as $X_t = X'_{\alpha(t)}$, where

$$X'_t = x_0 - ct + \mathcal{B}(t)$$

is a drifted Brownian motion. Consider $\varphi(x, y)$ a Lipschitz on $[0, n_0]^2$ with Lipschitz constant $C_\varphi$, then

$$\alpha(\infty) = \int_0^\infty \varphi(X_s, Y_s)\,\mathrm{d}s \leq \int_0^\infty C_\varphi Y_s\,\mathrm{d}s = \frac{C_\varphi}{\gamma}(n_0 - X_\infty) \leq \frac{C_\varphi n_0}{\gamma}.$$

Denote $\lambda_{X'} := \inf\{t \geq 0 : X'_t = 0\}$, then

$$P[\tau_X = \infty] = P[\lambda_{X'} \geq \alpha(\infty)] \geq P\left[\lambda_{X'} \geq \frac{C_\varphi n_0}{\gamma}\right] > 0 \tag{48}$$

(see p. 295 in [2] again).

An interesting specification of the equation (47) is

$$\begin{aligned} \mathrm{d}X_t &= -\beta X_t^+ Y_t^+ \mathrm{d}t + \sqrt{\beta X_t^+ Y_t^+}\,\mathrm{d}W_t \\ \mathrm{d}Y_t &= \beta X_t^+ Y_t^+ \mathrm{d}t - \gamma Y_t\,\mathrm{d}t - \sqrt{\beta X_t^+ Y_t^+}\,\mathrm{d}W_t, \end{aligned} \tag{49}$$

where $\beta > 0$. Note that the solution $(X, Y)$ to (49) terminates as in (48), i. e. we know only that $P[\tau_X = \infty]$ is positive.

On the other hand, having a solution to

$$\begin{aligned} \mathrm{d}X_t &= -\beta X_t Y_t\,\mathrm{d}t + \beta X_t Y_t\,\mathrm{d}W_t \\ \mathrm{d}Y_t &= \beta X_t Y_t\,\mathrm{d}t - \gamma Y_t\,\mathrm{d}t - \beta X_t Y_t\,\mathrm{d}W_t \end{aligned} \tag{50}$$

then

$$X_t = x_0 \exp\left\{-\beta \int_0^t Y_s\,\mathrm{d}s + \beta \int_0^t Y_s\,\mathrm{d}W_s - \frac{1}{2}\beta^2 \int_0^t Y_s^2\,\mathrm{d}s\right\} > 0$$
$$Y_t = y_0 \exp\left\{\beta \int_0^t X_s\,\mathrm{d}s - \beta \int_0^t X_s d\,\mathrm{d}W_s - \frac{1}{2}\beta^2 \int_0^t X_s^2\,\mathrm{d}s - \gamma t\right\} > 0$$

hence, $\tau_X = \tau_Y = \infty$ almost surely. The choice of the diffusion coefficient as a square root of the trend coefficient (see (49)) is frequently used, see [1] and [5]. Adding this diffusion coefficient to the Kermack–McKendrick model, the behavior of the model does not change dramatically, while the choice in the equation (50) provide more rugged paths, see Figure.

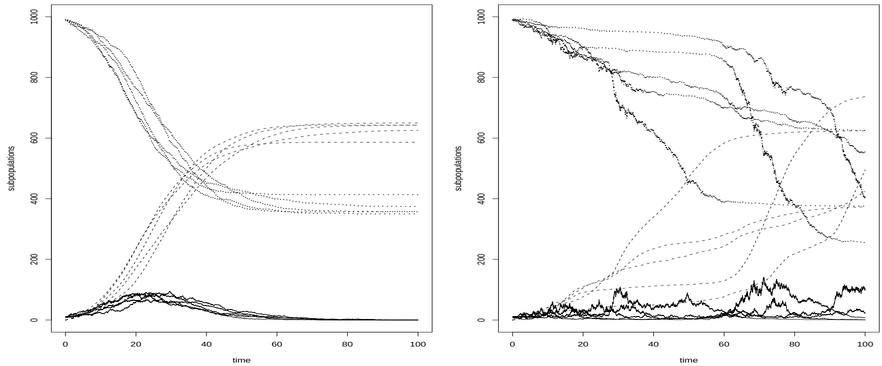


**Fig.** Five simulations of $(X_t, Y_t, \gamma \int_0^t Y_s \, \mathrm{d}s)$, where $(X, Y)$ is a solution to (49) (left) and (50) (right). Dotted line is used for $X_t$, solid line for $Y_t$ and dashed line for $\gamma \int_0^t Y_s \, \mathrm{d}s$. The parameters are $\beta = 0.38$ and $\gamma = 0.25$.

**Example 5.7.** Consider a Langevin type of the equation (1):

$$\begin{aligned} \mathrm{d}X_t &= -\beta I_{[X_t>0]} Y_t^+ \mathrm{d}t + I_{[X_t>0, Y_t>0]} \, \mathrm{d}W_t, && X_0 = x_0 > 0 \\ \mathrm{d}Y_t &= \beta I_{[X_t>0]} Y_t^+ \mathrm{d}t - \gamma Y_t \, \mathrm{d}t - I_{[X_t>0, Y_t>0]} \, \mathrm{d}W_t, && Y_0 = y_0 > 0. \end{aligned} \tag{51}$$

According to Theorem 3.1 the equation has a unique nonnegative absorbed solution $(X, Y)$ and has unique nonnegative solution $(X, Y)$ assuming that $\beta \leq \gamma$ according to Theorem 4.3. The solution is constructed as follows: Solve first the Langevin equation

$$\mathrm{d}Z_t = (\beta - \gamma) Z_t \, \mathrm{d}t - \mathrm{d}W_t, \quad Z_0 = y_0 \tag{52}$$

to get a unique strong solution

$$Z_t = e^{-(\beta-\gamma)t} \left( y_0 - \int_0^t e^{(\gamma-\beta)s} \, \mathrm{d}W_s \right).$$

Define

$$I_t = x_0 - \beta \int_0^t Z_s \, \mathrm{d}s + W_t, \quad \lambda_Z = \inf\{t \geq 0 : Z_t = 0\}, \quad \lambda_I = \inf\{t \geq 0 : I_t = 0\}$$

and $\lambda = \lambda_Z \wedge \lambda_I$. Then the absorbed solution $(X, Y)$ to (51) reads as follows:

$$X_t = I_{t \wedge \lambda}, \quad Y_t = \begin{cases} Z_t, & t < \lambda \\ Y_\lambda e^{-\gamma(t-\lambda)}, & t \geq \lambda. \end{cases}$$

One can easily verify the implications (40), hence $\tau_X \wedge \tau_Y = \lambda$ a.s.

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