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Prime Constellations in Triangles with Binomial Coefficient Congruences

Larry Ericksen

Abstract. The primality of numbers, or of a number constellation, will be determined from residue solutions in the simultaneous congruence equations for binomial coefficients found in Pascal's triangle. A prime constellation is a set of integers containing all prime numbers. By analyzing these congruences, we can verify the primality of any number. We present different arrangements of binomial coefficient elements for Pascal's triangle, such as by the row shift method of Mann and Shanks and especially by the diagonal representation of Ericksen. Primes of linear and polynomial forms are identified from congruences of their associated binomial coefficients.

This method of primality testing is extended to triangle elements created from q-binomial or Gaussian coefficients, using congruences with cyclotomic polynomials as a modulus. We apply Kummer's method of p-ary representation to binomial coefficient congruences to find prime constellations. Aside from their capacity to find prime numbers in binomial coefficient triangles, congruences are used to identify prime properties of composite numbers, represented as distinct prime factors or as prime pairs.

1 Introduction

1.1 Pascal's Triangle (n Primes)

The classical Pascal triangle from [16] is described in [1] as a number triangle, with entries as binomial coefficients $\binom{n}{h}$. Each element in row n and column h can be derived as a combinatorial ratio of factorials or generated recursively by

$$\binom{n}{h} = \frac{n!}{h!(nh)!} = \binom{n-1}{h} + \binom{n-1}{h-1}, \tag{1}$$

for $0 \leq h \leq n$ and $n \geq 0$ with initial conditions $\binom{0}{0} = \binom{1}{0} = \binom{1}{1} = 1$ and outer boundary conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all $n \geq 0$.

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The well-known connection between prime numbers and Pascal's triangle arises by examining the values of the binomial coefficients $\binom{n}{h}$ in each row n. We can state this relationship as a theorem: *In the n-th row of Pascal's triangle, the row number n is a prime number if and only if*

$$\binom{n}{h} \equiv 0 \pmod{n} \quad \textit{for all } h \textit{ over} \quad 1 \le h \le n-1\,. \tag{2}$$

So excluding the end points at $h = 0$ and $h = n$, all $n-1$ congruences modulo n for the binomial coefficients at each column h in any given row n will have zero residues if and only if n is a prime.

We can further reduce the number of simultaneous equations required to verify this primality relationship from (2), by restating the theorem: *In the n-th row of Pascal's triangle, the row number n is a prime number if and only if*

$$\binom{n}{h} \equiv 0 \pmod{n} \quad \textit{for all primes } h < n\,. \tag{3}$$

Thus, as proven in [5], we need only check the binomial coefficient congruences modulo n when the column number h is a prime number, in any given row n of Pascal's triangle. In this case, we need examine only $\pi(n-1)$ simultaneous equations, where $\pi(x)$ is the number of primes that are less than the number x. Every congruence will still have zero residues if and only if n is a prime.

1.2 Dilcher-Stolarsky Twin Primes ($2n \pm 1$ Primes)

In [5], Dilcher and Stolarsky came up with two sets of simultaneous equations, which together could identify twin primes. These two congruence sets can be summarized for modulus $2n-1$ and $2n+1$ both being prime. Their twin prime theorem can be described for the separate congruence equations by the following theorem: *Entries in the n-th rows of two related triangles, satisfying two sets of congruences with modulus $2n-1$ and $2n+1$, will identify numbers $p^* = 2n-1$ and $p = 2n+1$ as being prime if and only if terms $D^*_{n,h}$ and $D_{n,h}$ are given by*

$$\binom{n+h}{2h+1}\frac{(2n-1)(2n+1)}{2h+3} \equiv 0 \pmod{2n \pm 1} \quad \textit{for all } h \textit{ over } 0 \le h \le n-1\,, \tag{4}$$

with exactly one exception of a nonzero residue in the n-th row of each triangle.

1.3 Mann-Shanks Primality (h Primes)

1.3.1 Mann-Shanks Shift

As illustrated in [1] and proven in [15], Mann and Shanks used binomial coefficients as shifted elements in the rows of the Pascal triangle to obtain a criterion for determining the primality of a natural number, according to the following theorem: *A natural number $p = h$ is prime if and only if*

$$M_{n,h} = \binom{n}{h-2n} \equiv 0 \pmod{n} \quad \textit{for all } n \textit{ over} \left\lceil \frac{h}{3} \right\rceil \le n \le \left\lfloor \frac{h}{2} \right\rfloor. \tag{5}$$

Thus in the Mann-Shanks triangle, each row of the Pascal triangle is displaced successively two places to the right. Then at each candidate column h, the congruences modulo the set of row numbers n are examined for the requirement that all residues must be zero for h to be prime.

1.3.2 Mann-Shanks Criterion Extended

The Mann and Shanks primality criterion can be extended, as shown in [13], for certain other shift factor values by the following theorem: *For shift integer $\gamma \in \{1, 2, 3, 4\}$, the natural number $p = h$ is an odd prime if and only if*

$$M_{n,h} = \binom{n}{h - \gamma n} \equiv 0 \pmod{n} \quad \textit{for all } n \textit{ over } \left\lceil \frac{h}{1+\gamma} \right\rceil \leq n \leq \left\lfloor \frac{h-\delta}{\gamma} \right\rfloor, \quad (6)$$

where $\delta = 1$ if $\gamma = 1$, otherwise $\delta = 0$ for $2 \leq \gamma \leq 4$. In the case for $\gamma = 3$, only the odd values of h are evaluated, with the only exception occurring for odd $h = 25$ where all residues are zero. In [13], Hudson and Williams presented the binomial coefficient triangle for $\gamma = 1$ and proved the case for $\gamma = 3$. Harborth gave details for primality at $\gamma \leq 2$ in [10], at $\gamma = 3$ in [11] and at $\gamma = 4$ in [12].

1.4 Ericksen Diagonal Primality ($n + 2$ Primes)

In [7] we examined the binomial coefficients of the upward diagonals of the Pascal triangle and presented the following theorem: *In the n-th rows of the related diagonal triangle with binomial coefficient terms $\binom{n-h}{h}$ at columns h, the number $p = n + 2$ is prime if and only if*

$$E_{n,h} = \binom{n-h}{h} \equiv 0 \pmod{h+1} \quad \textit{for all } h \textit{ over } 0 \leq h \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad (7)$$

The terms of the binomial coefficient $\binom{n-h}{h}$ can be obtained variously by the combinatorial and recursive rules in (8), from Chebyshev polynomials of the second kind $U_n(z)$ in (9), and by the generating function in (10), displayed as

$$\binom{n-h}{h} = \frac{(n-h)!}{h!(n-2h)!} = \binom{n-h-1}{h} + \binom{n-h-2}{h-1}, \quad (8)$$

$$(-x)^{n/2} U_n\Big(\frac{1}{\sqrt{-4x}}\Big) = \sum_{h=0}^{\lfloor n/2 \rfloor} \binom{n-h}{h} x^h, \quad (9)$$

$$\frac{1}{1 - x - qx^2} = \sum_{n \geq 0} x^n \sum_{h=0}^{n/2} \binom{n-h}{h} q^h. \quad (10)$$

Incidentally it is also well known that the sum of these binomial coefficients gives the Fibonacci numbers F_m, according to $F_{n+1} = \sum_{h=0}^{n/2} \binom{n-h}{h}$.

In [9] we proved that it is not necessary to examine all columns h in the diagonal triangle to prove primality, only those where $h + 1$ is prime. Therefore we were able to restate (7) by the theorem: *In the n-th rows of the related diagonal triangle* *with binomial coefficient terms* $\binom{n-h}{h}$ *at select columns* h*, the number* $p = n+2$ *is prime if and only if*

$$E_{n,h} = \binom{n-h}{h} \equiv 0 \pmod{h+1} \quad \textit{for all } h \textit{ over } \begin{cases} 0 \le h \le \lfloor \frac{n}{2} \rfloor \,, \\ h+1 \textit{ is prime} \,. \end{cases} \tag{11}$$

2 Diagonal Primality Generalized

2.1 Diagonal Terms Extended

We generalize the diagonal triangle from (11) with the binomial coefficient terms $E_{n,h}$ by respectively weighting the row number n and column number h with integer factors of $\lambda > 0$ and any γ value. Thus we can extend (11) to a more general theorem: *In the* n*-th rows of a diagonal triangle with binomial coefficient terms* $\binom{\lambda n-\gamma h}{h}$ *at select columns* h*, the number* $t = \lambda n + \gamma + 1$ *is prime if and only if*

$$E_{n,h} = \binom{\lambda n - \gamma h}{h} \equiv 0 \pmod{h+1} \quad \textit{for each } h \textit{ over } \begin{cases} 0 \le h \le h_{\max} \,, \\ h+1 \textit{ is prime} \,, \end{cases} \tag{12}$$

where the largest value for h *is*

$$\begin{aligned} h_{\max} &= \lambda n + \gamma - 1 \quad \textit{when } \gamma \le 0 \textit{ for all } t > 2 \,, \\ h_{\max} &= \frac{\lambda n}{1+\gamma} \quad \textit{when } \gamma > 0 \textit{ for all } t > \gamma^2 \,. \end{aligned} \tag{13}$$

Of course, along with the minimal conditions that $h_{\max} \ge 0$ and $\lambda n \ge 0$, we need enough simultaneous congruences to make a valid analysis of primality. In the cases $\gamma < 0$, we are looking at downward diagonals from Pascal's triangle, where we can get any necessary number of binomial coefficient entries for the congruences, as long as $\lambda n > 1 + |\gamma|$. But in the cases $\gamma > 0$, we are looking at upward diagonals in the Pascal triangle, which might yield an insufficient number of binomial coefficient entries at low values of λn. So for $\gamma > 0$, we require the number t to be sufficiently large ($t > \gamma^2$) to guarantee a minimum number of simultaneous equations for the general theorem to be effective.

2.2 Proof of the General Theorem

We define a natural number $t = \lambda n + \gamma + 1$, make the substitution for λn in the binomial coefficient in (14), and expand this into a ratio of factorials in (15).

$$\binom{\lambda n - \gamma h}{h} = \binom{t - (1+\gamma) - \gamma h}{h} \tag{14}$$

$$\begin{aligned} &= \binom{t - (1+\gamma) - (1+\gamma)h + h}{h} = \binom{t + h - (1+\gamma)(h+1)}{h} \\ &= \frac{(t+h-(1+\gamma)(h+1)) \ldots (t+1-(1+\gamma)(h+1))}{h!} \,. \end{aligned} \tag{15}$$

In (12), we are interested in the divisibility properties for $E_{n,h}$ when $h+1$ is prime. We know that $\pi = h+1$ does not divide the denominator $h!$ in (15) because of Wilson's Theorem, described in [2], which states

$$(\pi - 1)! \equiv -1 \pmod{\pi} \quad \text{if and only if } \pi \text{ is prime} \,.$$

Therefore the question of divisibility is the same for the binomial coefficient in (14) as for its numerator. So when we apply the congruence in (12) to the binomial coefficients, we can focus the congruence on the numerator of (15), given in reduced form as

$$\overbrace{(t+h)\dots(t+1)}^{h \text{ factors}} \pmod{h+1}\,. \tag{16}$$

Because the residue in (16) has h consecutive factors and the modulus is $h+1$ for all $1 \le h+1 \le h_{\max}+1 < t$, we can quickly evaluate the possible primality conditions. If t is composite, then there will be a factor $h+1$ that divides t and thus does not divide any of the h factors in (16); thus $E_{n,h} \not\equiv 0 \pmod{h+1}$. If t is prime, then no factor $h+1 \le h_{\max}+1 < t$ can divide that value t, so $h+1$ must divide one of the h factors in (16); therefore $E_{n,h} \equiv 0 \pmod{h+1}$. In the reverse direction, we evaluate the residues of $E_{n,h} \pmod{h+1}$ and obtain the same results corresponding to the primality of the t terms. The combination of these primality and congruence conditions completes the proof of the general theorem (12).

3 Prime Constellations

3.1 Definition of J-Tuples

As in [7] and [9], we define the J-tuple set $T = \{t_1, t_2, \dots, t_J\}$ of natural numbers t_j by a constellation of a form $\{p+k_0, p+k_1, \dots, p+k_{J-1}\}$ for $p = t_1$ and separation set $K = \{k_j\}$ where $k_j = t_{j+1} - p$ for $0 \le j \le J-1$. If all integers in the constellation are prime numbers, we call them generalized Prime Constellations [20] of the Prime J-tuples. We will further allow the outer difference $t_J - t_1$ between the first and last prime to be greater than the least possible separation.

Based on values of the separation set $K = \{k_j\}$, we add together rows of the individual binomial coefficient triangles to create constellation triangles. The congruences of the sum of these binomial coefficients yield the primality test for the whole number constellation.

3.2 Mann-Shanks Constellations

As an example, we state a constellation version of the Mann-Shanks triangle of (5) as a proposition: *The n-th row elements $M_{n,h}$ in a J-tuple triangle have congruences*

$$M_{n,h} \equiv 0 \pmod{n} \quad \textit{for all } n \textit{ over } \left\lceil \frac{h}{3} \right\rceil \le n \le \left\lfloor \frac{h+k_{J-1}}{2} \right\rfloor, \tag{17}$$
$$\textit{with } M_{n,h} = \sum_{j=0}^{J-1} \binom{n}{h+k_j-2n},$$

if and only if all terms t_j are prime in the J-tuple set, with $p = t_1 = h$ as the initial prime and the constellation values given by

$$\{t_1, t_2, \dots, t_J\} = \{h, h+k_1, \dots, h+k_{J-1}\}\,.$$

3.3 Diagonal Constellations

3.3.1 Constellations with Primes of Linear Form

By including the weight factors λ and γ as in (12), we extend the prime constellation theorem of [7] and [9] by stating a diagonal constellation theorem: *The n^{th} row elements $E_{n,h}$ in the J-tuple triangle have congruences*

$$E_{n,h} \equiv 0 \pmod{h+1} \quad \textit{for each } h \textit{ over} \begin{cases} 0 \le h \le h_{\max}\,, \\ h+1 \textit{ is prime}\,, \end{cases} \tag{18}$$

$$\textit{with } E_{n,h} = \sum_{j=0}^{J-1} \binom{\lambda n - \gamma h + k_j}{h}\,,$$

if and only if all terms t_j are prime in the J-tuple set, with $p = t_1 = \lambda n + \gamma + 1$ as the initial prime and the constellation values given by

$$\{t_1, t_2, \ldots, t_J\} = \{\lambda n + \gamma + 1, \lambda n + \gamma + 1 + k_1, \ldots, \lambda n + \gamma + 1 + k_{J-1}\}\,. \tag{19}$$

As in (12)*, the maximum allowed h value depends on the γ value, according to*

$$h_{\max} = \begin{cases} \lambda n + k_{J-1} + \gamma - 1 & \textit{for } \gamma \le 0\,, \\ \frac{\lambda n + k_{J-1}}{1+\gamma} & \textit{for } \gamma > 0\,. \end{cases} \tag{20}$$

The proof of the constellation theorem (18) follows that of the previous section by using a typical prime candidate $t = t_j = \lambda n + \gamma + 1 + k_{j-1}$, and then invoking the fact that the individual terms in the summation for $E_{n,h}$ are additive. In every term congruence with prime modulus, the residues must be zero for the constellation to be prime.

3.3.2 Constellations with Primes of Polynomial Form

We saw that the J-tuple terms t_j in (19) were linear in variable n. We can also construct a J-tuple where the integer terms t_j are given in polynomial form with integer value $P_j(n)$. We state our objective that the J-tuple is prime by the polynomial constellation theorem: *Given polynomials $f_j(x) = \sum_{i \ge 0} a_{i,j} x^i$, we can find a number n such that the J-tuple triangle in $E_{n,h}$ terms satisfy the congruences*

$$E_{n,h} \equiv 0 \pmod{h+1} \quad \textit{for each } h \textit{ over} \begin{cases} 0 \le h \le h_{\max}\,, \\ h+1 \textit{ is prime}\,, \end{cases} \tag{21}$$

$$\textit{with } E_{n,h} = \sum_{j=0}^{J-1} \binom{n f_j(n) + k_j - \gamma h}{h}\,,$$

if and only if all terms t_j are prime in the J-tuple set $T = \{t_1, \ldots, t_J\}$, with $p = t_1 = n f_0(n) + 1 + \gamma$ as the initial prime and the constellation values given by

$$T = \{n f_0(n) + 1 + \gamma, n f_1(n) + k_1 + 1 + \gamma, \ldots, n f_{J-1}(n) + k_{J-1} + 1 + \gamma\}\,. \tag{22}$$

In its most general polynomial form, we define the J-tuple set $\tau(x, y, z)$ of functions $F_j(x)$, $K_j(y)$ and $A_j(z)$ for $0 \le j \le J-1$. We take functional variables at integer values, such that $x = n$, $y = c$ and $z = b$, so that the number set $T = \{t_j\}$ can be evaluated for primality. We generalize the polynomial constellation theorem in (21) for the number constellation T to be prime if and only if the congruences in (21) are satisfied for the $E_{n,h}$ terms given by

$$E_{n,h} = \sum_{j=0}^{J-1} \binom{F_j(n) + K_j(c) - A_j(b)h}{h},$$

$$\text{with } h_{\max} = \begin{cases} F^*(n) + K^*(c) + A^*(b) - 1 & \text{for } A_j \le 0\,, \\ \frac{F^*(n)+K^*}{1+A^*} & \text{for } A_j > 0\,, \end{cases}$$

with the notation $G^*(m) = \max\{G_j(m)\}$ and where each constellation element $t_j = F_j(n) + K_j(c) + A_j(b) + 1$ with the requirement that $t_j > A_j^2(b)$ if $A_j(b) > 0$.

3.4 Examples for Primes of Special Form

Primes of a linear form, like the specific arithmetic progression $ax+b$, can be found as in (12) by using congruence terms

$$E_{n,h} = \binom{an - (b-1)h}{h}.$$

Dirichlet's Theorem from [6] on primes in arithmetic progression proves that there are an infinite number of these primes $t = an + b$ contained within the infinite set for $n \ge 0$, as long as a and b are coprime.

To obtain individual primes of the quadratic form $x^2 + 1 + \gamma$, the congruence in (21) might use

$$E_{n,h} = \binom{nf(n) - \gamma h}{h} = \binom{n^2 - \gamma h}{h}.$$

To find primes of a cyclotomic polynomial form $\Phi_i(x)$, we could examine congruences with

$$E_{n,h} = \binom{\Phi_i(n) + h}{h}.$$

The two term constellations for Sophie Germain primes of the form $\{x, 2x+1\}$ can use congruences with

$$E_{n,h} = \binom{n+h}{h} + \binom{2n+1+h}{h}.$$

The prime constellations of a four term J-tuple, containing a minimal total separation between the smallest and largest primes in the set, would be found using the congruence from (18) with congruence terms

$$E_{n,h} = \binom{n-h}{h} + \binom{n-h+2}{h} + \binom{n-h+6}{h} + \binom{n-h+8}{h}.$$

3.5 An Infinity of Prime Constellations

In the terminology of this paper, we mention two outstanding conjectures about the infinitude of prime constellations of special forms. In [4] Dickson posed a conjecture that a J-tuple set of numbers in linear form $\{a_j n + b_j\}$ as in (19) will constitute a prime constellation for infinitely many integers n. In [18] Schinzel and Sierpinski gave a conjecture called hypothesis H which generalized Dickson's Conjecture to cover a J-tuple set of functions $\{P_j(x)\}$ evaluated as irreducible polynomials $\{P_j(n)\}$ as in (22) which will create a prime constellation for infinitely many integers n.

4 Constellations With q-Binomial Coefficients

4.1 Definition of q-Binomial Coefficients

The q-binomial coefficients, also called Gaussian binomial coefficients, are defined in q-series notation by

$$\begin{bmatrix} m \\ h \end{bmatrix}_q = \frac{(q)_m}{(q)_{m-h}(q)_h} = \prod_{i=0}^{h-1} \frac{1-q^{m-i}}{1-q^{i+1}}, \tag{23}$$

where we define $\left[\begin{smallmatrix} m \\ h \end{smallmatrix}\right]_q = 0$ when $m < 0$. The q-binomial coefficient satisfies the recurrence equation given by

$$\begin{bmatrix} m+1 \\ h \end{bmatrix}_q = q^h \begin{bmatrix} m \\ h \end{bmatrix}_q + \begin{bmatrix} m \\ h-1 \end{bmatrix}_q .$$

In the limit $q \to 1^-$, the value of the q-binomial coefficient equals the value of classical binomial coefficient.

4.2 Mann-Shanks q-Constellation

We can extend the Mann-Shanks criterion of (5) to the functional representation for Gaussian binomial coefficients and select modulus polynomials. We extend proposition (17) to a q-version of the Mann-Shanks criterion by stating a constellation proposition: *The q-binomial coefficient terms $M_{n,h}$ in the J-tuple triangle have congruences with modulus polynomials $\phi_n(x)$ given by*

$$M_{n,h} \equiv 0 \pmod{\phi_n(q)} \quad \textit{at all } n \textit{ over } \left\lceil \frac{h}{3} \right\rceil \le n \le \left\lfloor \frac{h+k_{J-1}}{2} \right\rfloor,$$
$$\textit{with } M_{n,h} = \sum_{j=0}^{J-1} \begin{bmatrix} n \\ h+k_j-2n \end{bmatrix}_q \quad \textit{and} \quad \phi_n(q) = \frac{1-q^n}{1-q},$$

if and only if all terms t_j are prime in the J-tuple set, with the initial term $p = t_1 = h + k_0$ (where typically $k_0 = 0$) and the constellation values given by

$$\{t_1, t_2, \ldots, t_J\} = \{h+k_0, h+k_1, \ldots, h+k_{J-1}\}.$$

4.3 Diagonal q-Constellations

We can extend the congruence of (18) to the q-constellation with q-binomial coefficient terms in congruences with a cyclotomic polynomial modulus for any integer γ. For positive valued γ cases, we offer a q-constellation conjecture: *For any integer $\gamma > 0$ and $h_{\max} = \frac{\lambda n + k_{J-1}}{1+\gamma}$, the q-binomial coefficient term $E_{n,h}$ satisfies congruences* (24), *with modulus of irreducible cyclotomic polynomials $\Phi_{h+1} = \Phi_{h+1}(q)$, according to*

$$E_{n,h} \equiv 0 \pmod{\Phi_{h+1}} \quad \textit{for every } h \textit{ where } \begin{cases} 0 \le h \le h_{\max}\,, \\ h+1 \textit{ is prime}\,, \end{cases} \tag{24}$$
$$\textit{with } E_{n,h} = \sum_{j=0}^{J-1} \begin{bmatrix} \lambda n - \gamma h + k_j \\ h \end{bmatrix}_q \quad \textit{and} \quad \Phi_{h+1}(q) = \frac{1-q^{h+1}}{1-q}\,,$$

if and only if the J-tuple terms of all $t_j > \gamma^2$ are prime, with all corresponding cyclotomic integers Φ_{t_j} likewise being prime and irreducible, given by

$$\{t_1, t_2, \ldots, t_J\} = \{\lambda n + \gamma + 1, \lambda n + \gamma + 1 + k_1, \ldots, \lambda n + \gamma + 1 + k_{J-1}\}\,,$$
$$\{\Phi_{t_1}, \Phi_{t_2}, \ldots, \Phi_{t_J}\} = \{\Phi_{\lambda n+\gamma+1}, \Phi_{\lambda n+\gamma+1+k_1}, \ldots, \Phi_{\lambda n+\gamma+1+k_{J-1}}\}\,.$$

The above correspondence uses the fact that cyclotomic polynomials $\Phi_p = \Phi_p(q)$ of prime index p are irreducible. Cyclotomic polynomials with prime index p can be written as

$$\Phi_p(q) = \frac{1-q^p}{1-q} = \sum_{i=0}^{p-1} q^i\,,$$

which reduces in the limit $q \to 1^-$ to the index value p. Combined with the fact that the q-binomial coefficient equals the classical binomial coefficient in the limit $q \to 1^-$, the congruence of (24) then reduces to the congruence of (18).

Next for non-positive valued γ cases, we have the companion conjecture: *For any integer $\gamma \le 0$ and $h_{\max} = \lambda n + k_{J-1} + \gamma - 1$, the q-binomial coefficient $E_{n,h}$ satisfies the congruences* (25), *with the irreducible cyclotomic polynomials Φ_{h+1} as modulus, similarly given by*

$$E_{n,h} \equiv 0 \pmod{\Phi_{h+1}} \quad \textit{for every } h \textit{ where } \begin{cases} 0 \le h \le h_{\max}\,, \\ h+1 \textit{ is prime}\,, \end{cases} \tag{25}$$
$$\textit{with } E_{n,h} = \sum_{j=0}^{J-1} \begin{bmatrix} \lambda n - \gamma h + k_j \\ h \end{bmatrix}_q \quad \textit{and} \quad \Phi_{h+1}(q) = \frac{1-q^{h+1}}{1-q}\,,$$

if and only if all J-tuple terms t_j are prime, with all cyclotomic integers Φ_{t_j} likewise being prime and irreducible, given by

$$\{t_1, t_2, \ldots, t_J\} = \{\lambda n + \gamma + 1, \lambda n + \gamma + 1 + k_1, \ldots, \lambda n + \gamma + 1 + k_{J-1}\}\,,$$
$$\{\Phi_{t_1}, \Phi_{t_2}, \ldots, \Phi_{t_J}\} = \{\Phi_{\lambda n+\gamma+1}, \Phi_{\lambda n+\gamma+1+k_1}, \ldots, \Phi_{\lambda n+\gamma+1+k_{J-1}}\}\,.$$

5 Reduced Diagonal Triangles

5.1 Residues in Reduced Triangles

The full diagonal triangles were constructed from the Pascal triangle having each row n with column values h consecutively covering all the integers from zero to some $h_{\max}$. However in these diagonal triangles $E_{n,h}$, we concentrated our attention on congruences with prime modulus $h+1$. Because of that selection, we could identify numbers t to be primes of some form when all the residues were zero. When the number t is composite, we found that the only residues possible were zero or one, when the congruences had prime modulus $h+1$.

Now we will construct a reduced triangle with only residue elements $r_{n,h}$ from the full diagonal triangle whose columns h where $h+1$ is prime. For this reduced residue triangle, we state the following conjectures, dependent on the value of the variable γ.

Residue Conjecture 1: *For any integer $\gamma \le 0$ and $h_{\max} = \lambda n + \gamma - 1$, the only possible residues are $r_{n,h} \in \{0,1\}$ in the congruence:*

$$E_{n,h} = \binom{\lambda n - \gamma h}{h} \equiv r_{n,h} \pmod{h+1} \quad \textit{for any } h \textit{ such that } \begin{cases} 0 \le h \le h_{\max}\,, \\ h+1 \textit{ is prime}\,. \end{cases}$$

Residue Conjecture 2: *For any integer $\gamma > 0$ and $h_{\max} = \frac{\lambda n}{1+\gamma}$, the only possible residues are $r_{n,h} \in \{0,1\}$ in the congruence:*

$$E_{n,h} = \binom{\lambda n - \gamma h}{h} \equiv r_{n,h} \pmod{h+1} \quad \textit{for any } h \textit{ such that } \begin{cases} 0 \le h \le h_{\max}\,, \\ h+1 \textit{ is prime}\,. \end{cases}$$

So in a reduced diagonal triangle of residue $r_{n,h}$ values, we get a primality rule: *From the above congruences, a number $t = \lambda n + \gamma + 1$ is prime if all $r_{n,h} = 0$ in row n, and the number t is composite if any $r_{n,h} = 1$ in row n.*

5.2 Distinct Prime Factors from Residue Triangles

Using congruence terms $E_{n,h}$ with weight factors $\lambda = 1$ and $\gamma = 0$, we build reduced residue triangles $r_{n,h}$ from each congruence given by

$$E_{n,h} = \binom{n}{h} \equiv r_{n,h} \pmod{h+1} \quad \text{for any } h \text{ such that } \begin{cases} 0 \le h \le h_{\max}\,, \\ h+1 \text{ is prime}\,. \end{cases}$$

Next we observe that the set H of all primes $h+1$ for which the residues $r_{n,h} = 1$ constitutes a survey of all the distinct prime factors of a composite number t. By the fundamental theorem of arithmetic, we write the factorization of the natural number t as

$$t = \prod_{i \ge 1} p_i^{e_i} \quad \text{for } p_i \in H\,.$$

We define a variable $\omega(n)$ to be number of distinct prime factors of the number t. Consequently the $\omega(n)$ value will equal the cardinality of the set H. We let a variable G_n count of the total number of residues $r_{n,h} = 1$ in the reduced triangle, containing all the rows up to row number n, so that $G_n = \sum_2^n \omega(n)$. We have determined experimentally that

$$n(\log\log n - cB_1) < G_n < n(\log\log n + cB_1)\,, \tag{26}$$

where $B_1 = 0.261497\ldots$ is the Mertens constant and c has a value is $0 \leq c < 1$. In the limit as $n \to \infty$, we get $G_n \to n \log\log n$. This result (26) compares favorably with bracketing inequalities for $\sum_2^n \omega(n)$ from [19], including $\omega(p) = 1$ for primes p, which thus has another term in the upper limit given as $O(\frac{n}{\log n})$.

6 Diagonal Triangles in p-Ary Representation

6.1 Kummer's Theorem for Binomial Coefficients

A divisibility property for binomial coefficients from [14] can be stated by Kummer's Theorem: *The power to which prime p divides the binomial coefficient $\binom{n}{m}$ is given by the number of 'carries' when we add m and $n-m$ in the base p.* As mentioned in [1], Kummer gave a formula for determining the highest power s of the prime p for which $\binom{n}{m}$ is exactly divisible by p^s, by using the summation formula with $s = s_{n,m}$ as:

$$s = \frac{1}{p-1}\sum_{k=0}^{r}(c_k + b_k - a_k)\,, \tag{27}$$

where the p-ary representations of integers n, m and $n-m$ are given by

$$n = (a_r a_{r-1} \ldots a_0)\,, \quad m = (b_r b_{r-1} \ldots b_0)\,, \quad n - m = (c_r c_{r-1} \ldots c_0)\,.$$

6.2 Kummer's Theorem for Diagonal Triangles

In diagonal triangle congruences like (12), we can see that the methodology for the primality testing relies on binomial coefficients being divisible by all primes $h+1$. In the terminology of the Kummer formula (27), that would imply that $s > 0$ for each prime p. Given Kummer's theorem for primes $p = h+1$, we can interpret the diagonal congruence for $E_{n,h}$ from (18) and an expanded Kummer's formula $S_{n,h}$ as

$$E_{n,h} = \sum_{j=0}^{J-1}\binom{n+k_j-\gamma h}{h} \quad \text{and} \quad S_{n,h} = \frac{1}{h}\prod_{j=0}^{J-1}\sum_{k=0}^{r}(c_{j,k} + b_{j,k} - a_{j,k})\,. \tag{28}$$

Thus for the binomial coefficient sum $E_{n,h}$ and its p-ary representation $S_{n,h}$ in (28), as defined by Kummer's Theorem in (27), we state the related theorem: *The J-tuple number set $\{t_j\} = \{n + k_j + \gamma + 1\}$ for $1 \leq j \leq J$ is a prime constellation if and only if condition* (29) *is satisfied, according to*

$$S_{n-\gamma h,h} > 0 \quad \textit{for every } h \textit{ such that } \begin{cases} 0 \leq h \leq h_{\max}\,, \\ h+1 \textit{ is prime}\,, \end{cases} \tag{29}$$

where $h_{\max}$ is the same as defined previously in (20).

6.3 A Quick p-Ary Primality Test

Because of the special parameters connecting the congruence modulus with the diagonal binomial coefficients, we can simplify the requirements of theorem (29) and offer the theorem: *The J-tuple number set $T = \{t_j\} = \{n + k_j + \gamma + 1\}$ is a prime constellation if and only if condition* (30) *is satisfied, according to*

$$C_{n,h} > 0 \quad \textit{for every } h \textit{ such that } \begin{cases} 0 \le h \le h_{\max}\,, \\ h+1 \textit{ is prime}\,, \end{cases} \tag{30}$$

where $C_{n,h} = \prod_{j=0}^{J-1} c_{j,0}$ is the product of the initial terms from the $(h+1)$-ary representations of each integer η_j for $1 \le j \le J$ given by

$$\eta_j = n + k_j - (\gamma+1)h = (c_{j,r}c_{j,r-1}\dots c_{j,0})\,. \tag{31}$$

The proof of (30) relies on the binomial coefficients in $E_{n,h}$ being a special case of the Kummer formula, due to the following conditions:

1. h in base $p = h+1$ is h, which leads to the ratio $\frac{h}{p-1} = 1$ for the $b_{j,k}$ term in the Kummer sum.

2. $n - m$ in base $h+1$ is the top factorial term minus h. The only way to avoid a carry is to have $c_{j,0} = 0$, giving the difference $c_{j,0} - a_{j,0} = -h$. The resulting contribution $\frac{-h}{p-1} = -1$ cancels the $b_{j,k}$ value, so the Kummer sum is equal to zero as the power of $h+1$ that divides the binomial coefficient. Thus a constellation term t_j is a composite number if $c_{j,0} = 0$, otherwise it is prime with $c_{j,0} > 0$.

We mention the special case for a single binomial coefficient $E_{n,h} = \binom{n+1}{h}$ in (28) for $\gamma = -1$ with $k_0 = 0$ at $J = 1$. In that case, we get the obvious condition that n is prime if and only if all primes $h+1 < n$ do not divide the prime $t_1 = n$. In this situation, the primality test (30) means that, for the integer $\eta_1 = n$ to be prime, we must have $\prod_{j=0}^{J-1} c_{j,0} > 0$. For example with the prime $t_1 = n = 7$, we get the p-ary representations of $\eta_1 = n$ for primes $p = h+1 < t_1 = n$ as $\{1,1,1\}, \{2,1\}, \{1,2\}$ in respective bases $p = \{2,3,5\}$, with each initial term $c_{j,0} > 0$ as required.

7 Composites as Prime Pairs

7.1 Prime Pairs in Diagonal Triangles

The primality tests in (12) and (21) for diagonal triangles evaluate the congruences taken over the triangle columns $h < h_{\max}$. For individual congruences in (21) with weight factor $\gamma < 0$, we use each prime modulus $h+1$ up to $h_{\max} = nf(n) + \gamma - 1$. If any one of the congruences did not have a residue of value zero, the number $t = nf(n) + \gamma + 1$ was determined to be a composite number.

We want to examine a particular composite number $t = (n-k)(n+k)$, which would be the product of two primes $(n-k)$ and $(n+k)$. Expanding the product, we get $t = n^2 - k^2$. From this, we can restate the formula for the t value with the substitutions $n^2 = nf(n)$ and $k^2 = \gamma + 1$.

Now we examine the binomial coefficient congruences taken over a maximum number of columns h chosen to be $h_{\max} = \sqrt{n}$, a smaller number of terms than was used to identify individual primes. From experimental evidence, we can identify these numbers t with prime pair factors as satisfying the prime pair conjecture:

For $\gamma = -k^2 - 1 < 0$ and $n > |\gamma|$, then a number $t = n^2 - k^2$ has two prime factors $(n - k)$ and $(n + k)$ if and only if the congruence (32) is satisfied and is given by

$$E_{n,h} \equiv 0 \pmod{h+1} \quad \text{for each } h \text{ over } \begin{cases} 0 \le h \le \sqrt{n}\,, \\ h+1 \text{ is prime}\,, \end{cases} \tag{32}$$

$$\text{with } E_{n,h} = \binom{n^2 + (k^2+1)h}{h} \quad \text{for } k \ge 0\,.$$

The dual factors of these specific composite numbers for $t = (n - k)(n + k)$ are identified for small k values in the following table.

k	factors of t
0	primes,
1	twin primes,
2	cousin primes,
3	sexy primes.

References

[1] B. A. Bondarenko. Generalized Pascal triangles and pyramids: Their fractals, graphs, and applications. pp. 22–23, 58. Santa Clara, CA: The Fibonacci Association, 1993.

[2] C. Caldwell. *The Prime Pages.* http://primes.utm.edu/notes/proofs/Wilsons.html.

[3] R. Crandall, C. Pomerance. *Prime Numbers. A Computational Perspective.* Springer-Verlag, New York, 2001.

[4] L. E. Dickson. A New Extension of Dirichlet's Theorem on Prime Numbers. *Messenger Math.* **33**, 1904, p. 155–161.

[5] K. Dilcher, K. B. Stolarsky. A Pascal-Type Triangle Characterizing Twin Primes. *Amer. Math. Monthly* 112, 2005, p. 673–681.

[6] P. G. L. Dirichlet. Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält. *Abhand. Ak. Wiss. Berlin* 48, 1837.

[7] L. Ericksen. Divisibility, iterated digit sums, primality tests. *Math. Slovaca* 59(3), 2009, p. 261–274.

[8] L. Ericksen. Iterated Digit Sums, Recursions and Primality. *Acta Mathematica Universitatis Ostraviensis* 14, 2006, p. 27–35.

[9] L. Ericksen. Primality testing and prime constellations. *Šiauliai Math. Semin.* 3(11), 2008, p. 61–77.

[10] H. Harborth. Über die teilbarkeit im Pascal-Dreieck. *Math.-Phys Semesterber* 22, 1975, p. 13–21.

[11] H. Harborth. Ein primzahlkriterium nach Mann und Shanks. *Arch. Math.* 27(3), 1976, p. 290–294.

[12] H. Harborth. Prime number criteria in Pascal's triangle. *J. London Math. Soc.* (2) 16, 1977 p. 184–190.

[13] R. H. Hudson, K. S. Williams. A divisibility property of binomial coefficients viewed as an elementary sieve. *Internat. J. Math. & Math. Sci.* 4(4), 1981, p. 731–743.

[14] E. E. Kummer. Über die Erganzungssätze zu den allgemeinen Reciprocitätsgesetzen. *J. Reine Angew. Math.* 44, 1852, p. 93–146.

[15] H. B. Mann, D. Shanks. A necessary and sufficient condition for primality and its source. *J. Combinatorial Theory* (A) 13, 1972, p. 131–134.

[16] B. Pascal. Traité du triangle arithmétique, avec quelques autres petits traités sur la mêmes matières. G. Desprez, Paris, 1654, *Oeuvres completes de Blaise Pascal*, T. 3, 1909, p. 433–593.

[17] G. Robin. Grandes valeurs de la fonction somme des diviseurs et hypothese de Riemann. *J. Math. Pures Appl.* 63, 1984, p. 187–213.

[18] A. Schinzel, W. Sierpinski. Sur certaines hypothèses concernant les nombres premiers. *Acta Arith.* 4, 1958, p. 185–208, Erratum 5, 1959, p. 259.

[19] E.W. Weisstein. *Distinct Prime Factors.* http://mathworld.wolfram.com/DistinctPrimeFactors.html.

[20] E.W. Weisstein. *Prime Constellation.* http://mathworld.wolfram.com/PrimeConstellation.html.

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