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*Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, Vol. 48 (2009), No. 1, 153--156

Persistent URL: `http://dml.cz/dmlcz/137509`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **48** (2009) 153–156

# Congruences in Ordered Sets and LU Compatible Equivalences

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(Received October 8, 2008)

**Abstract**

A concept of equivalence preserving upper and lower bounds in a poset $P$ is introduced. If $P$ is a lattice, this concept coincides with the notion of lattice congruence.

**Key words:** Ordered set, morphism, $LU$ compatible equivalence.

**2000 Mathematics Subject Classification:** 06A06, 06B10

There are various concepts of a congruence relation in ordered sets. All of them define a congruence as an equivalence relation whose classes are convex subsets. However, this concept is too weak, namely the quotient set by such an equivalence need not be an ordered set. Hence, in the definitions additional conditions are usually required. We can mention e.g. the approaches by M. Kolibiar [2, 3], I. Chajda, V. Snášel [1], J. Lihová, A. Haviar [4] and R. Halaš [5], [6]. A natural assumption for a congruence on an ordered set is that if this set is a lattice then the notion of a congruence has to coincide with the lattice one. The aim of our paper is to introduce a concept of $LU$ compatible equivalence in an ordered set satisfying all the foregoing assumptions which, moreover, corresponds to the concept of morphism preserving upper and lower bounds.

Let $A \neq \emptyset$ be a set and let $\leq$ be a partial order on $A$. For a subset $B \subseteq A$, we denote the set of all lower or upper bounds of $B$ in $A$ with respect to $\leq$ by $L_A(B)$ or $U_A(B)$, respectively, i.e.:

$$L_A(B) = \{x \in A;\ x \leq a \text{ for all } a \in B\}$$
$$U_A(B) = \{x \in A;\ a \leq x \text{ for all } a \in B\}.$$

If there is no danger of misunderstanding, the subscript $A$ will be omitted and we will write $U(B)$ or $L(B)$ only.

We adopt the notation $U(B,C) = U(B \cup C)$ and $L(B,C) = L(B \cup C)$. If $B = \{b_1, b_2, \ldots, b_n\}$ is finite, we will write briefly $U(B) = U(b_1, b_2, \ldots, b_n)$, dually for $L(B)$.

Remark that if $B \subseteq C \subseteq A$ then $U(B) \supseteq U(C)$ and $L(B) \supseteq L(C)$.

**Definition 1** [1] An equivalence $\Theta$ on an ordered set $P$ is called a *congruence* if either $\Theta = P \times P$ or it satisfies:

(i) $[a]_\Theta$ is a convex subset of $P$ for all $a \in P$;

(ii) for every $x, y \in [a]_\Theta$ there exist $u, v \in [a]_\Theta$ such that $u \leq x \leq v$ and $u \leq y \leq v$;

(iii) if $u \leq x$, $u \leq y$ and $u \,\Theta\, x$ then there exists $v \in P$ with $x \leq v$, $y \leq v$ and $y \,\Theta\, v$; if $x \leq v$, $y \leq v$ and $v \,\Theta\, y$ then there exists $u \in P$ with $u \leq x$, $u \leq y$ and $u \,\Theta\, x$.

Of course, the identity relation on $P$ is a congruence on $P$.

It was already shown in [1] that the quotient set by a congruence is an ordered set again.

**Proposition 1** [1] *Let $P$ be an ordered set and $\Theta$ be a congruence on $P$. Then the quotient relation defined on $P/\Theta$ by setting $[a]_\Theta \leq_{/\Theta} [b]_\Theta$ iff there exist $x \in [a]_\Theta, y \in [b]_\Theta$ with $x \leq y$ is an order on $P/\Theta$.*

In the following, for any $A \subseteq P$ denote $[A]_\Theta = \{[a]_\Theta;\ a \in A\}$.

**Corollary 1** [1] *Let $P$ be an ordered set and $\Theta$ be an equivalence on $P$. Then $\Theta$ is a congruence on $P$ if and only if*

*(1) $P/\Theta$ is an ordered set (with the order $\leq_{/\Theta}$);*

*(2) $[L_P(x,y)]_\Theta = L_{P/\Theta}([x]_\Theta, [y]_\Theta)$ and $[U_P(x,y)]_\Theta = U_{P/\Theta}([x]_\Theta, [y]_\Theta)$*

*for every $x, y$ of $P$.*

**Definition 2** Let $(P, \leq)$ be a an ordered set. An equivalence $\Theta$ on $P$ is called *$LU-compatible$* if it satisfies the condition (2) of Corollary 1.

**Lemma 1** *If $\Theta$ is an $LU$ compatible equivalence then the following holds:*

*(1) each block is directed,*

*(2) the condition (iii) of definition 1 is satisfied,*

*(3) $\leq_{/\Theta}$ is transitive.*

**Proof** (1) Let $a, b \in [x]_\Theta$. Then $[a]_\Theta = [x]_\Theta = [b]_\Theta$ whence

$$[L(a,b)]_\Theta = L([a]_\Theta, [b]_\Theta) = L([x]_\Theta) \neq \emptyset.$$

(2) Let $u \leq x$, $u \leq y$ and $u \; \Theta \; x$, i.e. $[y]_\Theta = [v]_\Theta$ for some $v \in U(x,y)$. Since $\Theta$ is an LU-compatible equivalence, we have

$$[U(x,y)]_\Theta = U([x]_\Theta, [y]_\Theta) = U([u]_\Theta, [y]_\Theta) = [U(u,y)]_\Theta = [U(y)]_\Theta.$$

This shows the first part of (iii) of Definition 1. The rest can be shown analogously.

(3) The proof is analogous to that used in [1]. □

**Theorem 1** *Let $(P, \leq)$ be a an ordered set and $\Theta$ be an LU compatible equivalence. Then $(P/\Theta, \leq_{/\Theta})$ is an ordered set if and only if each block of $\Theta$ is convex.*

**Proof** It is easy to show that $\leq_{/\Theta}$ is reflexive. The transitivity of $\leq_{/\Theta}$ follows directly from Lemma 1.

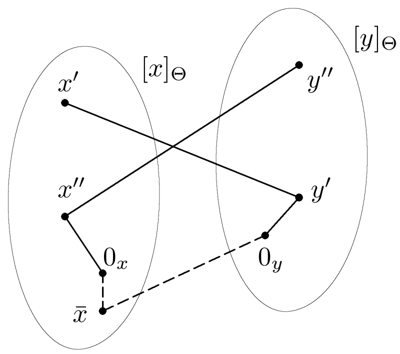
We show that $\leq_{/\Theta}$ is antisymmetric. Let $[x]_\Theta \leq_{/\Theta} [y]_\Theta$ and $[y]_\Theta \leq_{/\Theta} [x]_\Theta$. Then there exist $x', x'' \in [x]_\Theta$ and $y', y'' \in [y]_\Theta$ such that $x' \geq y'$ and $x'' \leq y''$. From Lemma 1 there exist $\bar{y} \in [y]_\Theta$ and $\bar{x} \in [x]_\Theta$ such that $y', y'' \leq \bar{y}$, $\bar{x} \leq x''$ and $\bar{x} \leq y'$. We have $\bar{x} \leq y' \leq x'$. Applying convexity we conclude $[x]_\Theta = [y]_\Theta$.

Conversely, let $(P/\Theta, \leq_{/\Theta})$ be a an ordered set and assume $x \leq y \leq z$ with $[x]_\Theta = [z]_\Theta$. Then $[x]_\Theta \leq_{/\Theta} [y]_\Theta$, $[y]_\Theta \leq_{/\Theta} [z]_\Theta = [x]_\Theta$, thus due to antisymmetry of $\leq_{/\Theta}$ we have $[x]_\Theta = [y]_\Theta$. □

**Theorem 2** *Let $(P, \leq)$ be a ordered set and let $\Theta$ be an LU compatible equivalence. If every equivalence class of $\Theta$ has the least element, then $(P/\Theta, \leq_{/\Theta})$ is an ordered set.*

**Proof** It is easy to see that $\leq_{/\Theta}$ is reflexive, transitivity of $\leq_{/\Theta}$ follows directly by Lemma 1.

To prove its antisymmetry, denote by $0_x$ the least element of an arbitrary block $[x]_\Theta$. Let $[x]_\Theta \leq_{/\Theta} [y]_\Theta$ and $[y]_\Theta \leq_{/\Theta} [x]_\Theta$. Then there exist $x', x'' \in [x]_\Theta$ and $y', y'' \in [y]_\Theta$ such that $x' \geq y'$ and $x'' \leq y''$.



Now $0_x \leq y''$ and $0_y \leq y''$, hence by Lemma 1 there exists $\bar{x} \in [x]_\Theta$ with $\bar{x} \leq 0_x$ and $\bar{x} \leq 0_y$. Since $0_x$ is the least element of $[x]_\Theta$, we have $\bar{x} = 0_x$. This shows $0_x \leq 0_y$. Analogously we prove $0_y \leq 0_x$, consequently $0_x = 0_y$, which finally yields $[x]_\Theta = [y]_\Theta$. □

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