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The tame degree and related invariants of non-unique factorizations

Franz Halter-Koch

Abstract. Local tameness and the finiteness of the catenary degree are two crucial finiteness conditions in the theory of non-unique factorizations in monoids and integral domains. In this note, we refine the notion of local tameness and relate the resulting invariants with the usual tame degree and the ω-invariant. Finally we present a simple monoid which fails to be locally tame and yet has nice factorization properties.

1 Introduction and Notations

Our notation and terminology will be consistent with [3]. We briefly recall the key notions and fix the terminology. We denote by $\mathbb{N}$ the set of positive integers, and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m, n \in \mathbb{Z}$, we set $[m, n] = \{x \in \mathbb{Z} \mid m \le x \le n\}$, and we define $\sup \emptyset = 0$.

By a *monoid* we always mean a commutative cancellative semigroup possessing a neutral element. Apart from Section 5 we use multiplicative notation and denote the unit element by $1 \in H$. A monoid F is called free with basis P if every $a \in F$ has a unique representation

$$a = \prod_{p \in P} p^{n_p} \quad \text{with} \quad n_p \in \mathbb{N}_0 \text{ and } n_p = 0 \text{ for almost all } p \in P\,.$$

Let F be a free monoid with basis P.

If $z = u_1 \cdot \ldots \cdot u_n \in F$, where $n \in \mathbb{N}_0$ and $u_1, \ldots, u_n \in P$, then we call $|z| = n$ the *length* of z. For any $z, z' \in F$, let $z_0 = \gcd(z, z')$ be its greatest common divisor, and call $\mathsf{d}(z, z') = \max\{|z_0^{-1}z|, |z_0^{-1}z'|\}$ the *distance* between z and z'.

Let H be a monoid.

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We denote by $H^\times$ the group of invertible elements, by $H_{\mathrm{red}} = H/H^\times$ the *associated reduced monoid*, and we call H *reduced* if $H^\times = \{1\}$ (in this case we have $H = H_{\mathrm{red}}$). We denote by $\mathcal{A}(H)$ the set of atoms (or irreducible elements) of H, and we call H *atomic* if H is generated (as a monoid) by $H^\times \cup \mathcal{A}(H)$. We denote by $\mathsf{Z}(H)$ the free monoid with basis $\mathcal{A}(H_{\mathrm{red}})$ and by $\pi_H \colon \mathsf{Z}(H) \to H_{\mathrm{red}}$ the unique homomorphism satisfying $\pi_H \,|\, \mathcal{A}(H_{\mathrm{red}}) = \mathrm{id}$. We call $\mathsf{Z}(H)$ the *factorization monoid* and π_H the *factorization homomorphism* of H. For $a \in H$, we denote by $\mathsf{Z}(a) = \pi_H^{-1}(aH^\times)$ the *set of factorizations of* a and by $\mathsf{L}(a) = \big\{|z| \bigm| z \in \mathsf{Z}(a)\big\}$ the *set of lengths of* a. If $z,\, z' \in \mathsf{Z}(a)$ and $z \neq z'$, then $\mathsf{d}(z,z') \geq 2$. By definition, we have $\mathsf{L}(a) = \{0\}$ if and only if $a \in H^\times$ and $\mathsf{L}(a) = \{1\}$ if and only if $a \in \mathcal{A}(H)$. If H is atomic, then π_H is surjective, $\mathsf{Z}(a) \neq \emptyset$ for all $a \in H$, and $\min \mathsf{L}(a) \geq 2$ for all $a \in H \setminus (\mathcal{A}(H) \cup H^\times)$. We call H a BF-*monoid* if H is atomic and $\mathsf{L}(a)$ is finite for all $a \in H$.

H is called *factorial* if $|\mathsf{Z}(a)| = 1$ for all $a \in H$. If H is not factorial, then there exist elements $a \in H$ for which $\mathsf{Z}(a)$ becomes arbitrarily large, and it is the goal of the theory of non-unique factorizations to describe and classify the phenomena of non-unique factorizations. This is usually done for atomic monoids, the interesting structures for which the results apply are however integral domains and submonoids of arithmetical interest. The interesting reader should consult the survey articles [4] and [7] for these applications.

Unless otherwise specified, let in the sequel H be an atomic monoid.

All factorization properties $\mathbf{P}$ studied in this note have the following property:

> If $\mathbf{P}$ holds for elements $a_1, \ldots, a_n \in H$, then $\mathbf{P}$ also holds for the elements $a_1H^\times, \ldots, a_nH^\times \in H_{\mathrm{red}}$.

Hence whenever it will be convenient, we shall assume that H is reduced.

2 Invariants of non-unique factorizations

In this section we briefly recall the definition of the invariants to be considered in this paper.

Definition 1. For $b \in H^\times$, we set $\rho(b) = 1$, for $b \notin H^\times$ we set

$$\rho(b) = \frac{\sup \mathsf{L}(b)}{\min \mathsf{L}(b)}\,, \quad \text{and we call} \quad \rho(H) = \sup\{\rho(b) \mid b \in H\} \quad \text{the } \textit{elasticity} \text{ of } H\,.$$

For $k \in \mathbb{N}$ we define $\rho_k(H) = \sup\{\sup \mathsf{L}(b) \mid \min \mathsf{L}(b) \leq k\}$.

The elasticity is among the best investigated arithmetical invariants of non-unique factorizations, see [1], [3, Ch. 1.4 and Ch. 6.3], and [2] for some recent results. In particular, if $H \neq H^\times$, then

$$\rho(H) = \sup\Big\{\frac{\rho_k(H)}{k} \Bigm| k \in \mathbb{N}\Big\} = \lim_{k\to\infty} \frac{\rho_k(H)}{k}\,,$$

and if H is finitely generated, then there is some $a \in H$ such that $\rho(H) = \rho(a) \in \mathbb{Q}$ (see [3, Proposition 1.4.2 and Theorem 3.1.4]).

Definition 2. For $b \in H$, we denote by $\omega(b)$ the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

> For all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in H$ such that $b \,|\, a_1 \cdot \ldots \cdot a_n$, there exists some subset $\Omega \subset [1, n]$ such that $|\Omega| \le N$ and
> $$b \,\Big|\, \prod_{i \in \Omega} a_i \,.$$

We set $\omega(H) = \sup\{\omega(u) \mid u \in \mathcal{A}(H)\} \in \mathbb{N}_0 \cup \{\infty\}$.

For properties of the ω-invariant and its relevance in factorization theory we refer to [3, Ch. 2.8 and Ch. 7.1] and to [5]. The following Proposition 1 gathers the results which will become relevant in the sequel.

Proposition 1. *Let b, $c \in H$.*

1. *$\omega(b)$ is the the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property: For all $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in \mathcal{A}(H)$ such that $b \mid u_1 \cdot \ldots \cdot u_n$, there exists some subset $\Omega \subset [1, n]$ such that $|\Omega| \le N$ and*
$$b \,\Big|\, \prod_{i \in \Omega} u_i \,.$$

2. *$\omega(b) \le \omega(bc) \le \omega(b) + \omega(c)$.*

3. *$\sup \mathsf{L}(b) \le \omega(b)$, and equality holds if every atom dividing b is a prime. In particular, $\omega(b) = 0$ if and only if $b \in H^\times$, $\omega(b) = 1$ if and only if b is a prime, and $\omega(H) = 0$ if and only if $H = H^\times$.*

4. *If $\omega(u) < \infty$ for all $u \in \mathcal{A}(H)$, then $\omega(a) < \infty$ for all $a \in H$, and H is a* BF-*monoid.*

5. *If H is v-noetherian, then $\omega(a) < \infty$ for all $a \in H$.*

Proof. 1. Let $\omega_0(b)$ be the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ satisfying the given condition. Then clearly $\omega_0(b) \le \omega(b)$. Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in H$ be such that $b \,|\, a_1 \cdot \ldots \cdot a_n$. For $i \in [1, n]$, let $a_i = \varepsilon_i u_{i,1} \cdot \ldots \cdot u_{i,l_i}$ with $\varepsilon_i \in H^\times$, $l_i \in \mathbb{N}_0$ and $u_{i,j} \in \mathcal{A}(H)$. Then
$$b \,\Big|\, \prod_{i=1}^{n} \prod_{j=1}^{l_i} u_{i,j} \,,$$

and therefore

$$b \,\Big|\, \prod_{(i,j)\in\Omega} u_{i,j} \text{ for some } \Omega \subset \prod_{i=1}^{n}[1, l_i] \text{ with } |\Omega| \le \omega_0(b)\,.$$

If $\Omega' = \{i \in [1,n] \mid (i,j) \in \Omega \text{for some} j \in [1, l_i]\}$, then $|\Omega'| \le |\Omega| \le \omega_0(b)$ and

$$b \,\Big|\, \prod_{(i,j)\in\Omega} u_{i,j} \,\Big|\, \prod_{i\in\Omega'} a_i\,, \quad \text{whence} \quad \omega(b) \le \omega_0(b)\,. \qquad \square$$

2. [5, Lemma 3.3.1].

3. Let $n \in \mathsf{L}(b)$ and $b = u_1 \cdot \ldots u_n$, where $u_1, \ldots, u_n \in \mathcal{A}(H)$. Then b divides no proper subproduct of $u_1 \cdot \ldots \cdot u_n$ and thus $\omega(b) \ge n$. Hence $\omega(b) \ge \sup \mathsf{L}(b)$.

If $u_1, \ldots, u_n$ are primes, then $\omega(u_i) = 1$ for all $i \in [1,n]$ by definition, hence $\omega(b) \le n$ by 2., and therefore $\omega(b) = n$.

If b is not a prime, then there exist u, $v \in H$ such that $b \,|\, uv$, $b \nmid u$ and $b \nmid v$. Hence $\omega(b) \ge 2$.

4. holds by 2. and 3., and 5. is proved in [5, Theorem 4.2].

Definition 3. For $a \in H$, the *catenary degree* $\mathsf{c}(a)$ denotes the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

> For any two factorizations z, $z' \in \mathsf{Z}(a)$ there exists a finite sequence of factorizations $(z_0, z_1, \ldots, z_k)$ in $\mathsf{Z}(a)$ such that $z_0 = z$, $z_k = z'$ and $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1,k]$ (we say that z and z' can be concatenated by an N-chain).

$\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\}$ is called the *catenary degree* of H.

By the very definition we have $\mathsf{c}(a) = 0$ if and only if a has unique factorization. If $\mathsf{c}(a) > 0$, then $\mathsf{c}(a) \ge 2$. If $\mathsf{c}(a) = 2$, then $|\mathsf{L}(a)| = 1$, and if $\mathsf{c}(a) = 3$, then $\mathsf{L}(a) = [\min \mathsf{L}(a), \max \mathsf{L}(a)]$ is an interval. The invariant $\mathsf{c}(a)$ measures the disconnectedness of the set of factorizations of a (see [3, Ch. 1.6 and Ch. 6.4]).

Definition 4. For $a \in H$ and $x \in \mathsf{Z}(H)$, let $\mathsf{t}(a,x)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

> If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \ne \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists some $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z,z') \le N$.

For subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$, we define

$$\mathsf{t}(H', X) = \sup\{\mathsf{t}(a,x) \mid a \in H'\,, x \in X\}\,,$$

and for $a \in H$ and $x \in \mathsf{Z}(H)$, we set $\mathsf{t}(H',x) = \mathsf{t}(H', \{x\})$ and $\mathsf{t}(a,X) = \mathsf{t}(\{a\}, X)$.

We define $\mathsf{t}(H) = \mathsf{t}(H, \mathcal{A}(H_{\mathrm{red}}))$. The monoid H is called *tame* if $\mathsf{t}(H) < \infty$, and it is called *locally tame* if $\mathsf{t}(H,u) < \infty$ for all $u \in \mathcal{A}(H_{\mathrm{red}})$.

Tameness is a very strong condition. Local tameness turned out to be crucial for the proof of all finiteness results in the theory of non-unique factorization hitherto. For details we refer to [3, Ch. 1.6, Ch. 4 and Ch. 6.5].

In [5], the authors introduced the following invariants and used them for a detailed study of the behavior of the tame degree.

For $k \in \mathbb{N}$ and $b \in H$, define

$$\tau_k(H,b) = \sup\{\min \mathsf{L}(b^{-1}a) \mid a = u_1 \cdot \ldots \cdot u_j \in bH\,,\ \text{where } j \in [0,k]\,,\ u_1,\ldots,u_j \in \mathcal{A}(H) \text{ and } b \nmid u_i^{-1}a \text{ for all } i \in [1,j]\,\} \in \mathbb{N}_0 \cup \{\infty\}\,.$$

$$\tau_k^*(H,b) = \sup\{\tau_k(H,b) \mid k \in \mathbb{N}\} \in \mathbb{N}_0 \cup \{\infty\}\,,$$

$$\tau(H,b) = \sup\{\min \mathsf{L}(b^{-1}a) \mid a \in bH \setminus H^\times\} \in \mathbb{N}_0 \cup \{\infty\}$$

and

$$\tau^*(H,b) = \sup\Big\{\frac{\min \mathsf{L}(b^{-1}a)}{\min \mathsf{L}(a)} \;\Big|\; a \in bH\,, \min \mathsf{L}(a) \le k\Big\} \in \mathbb{R}_{\ge 0} \cup \{\infty\}\,.$$

In this paper we continue these studies. We proceed with a detailed investigation of the τ^*-invariants [which we now denote by $\tau^*_{(k)}(b)$ instead of $\tau^*_{(k)}(H,b)$] in Section 3 und use it to describe the behavior of a refined variant of the tame degree in Section 4. Finally, in Section 5 we present a simple monoid H which fails to be locally tame and yet has catenary degree $\mathsf{c}(H) = 3$.

3 The τ^*-invariant

Definition 5. For $b \in H$ and $k \in \mathbb{N}_0$, we define

$$\tau_k^*(b) = \sup\{\min \mathsf{L}(b^{-1}a) \mid a \in bH\,, \min \mathsf{L}(a) \le k\} \in \mathbb{N}_0 \cup \{\infty\}\,,$$

and we set

$$\tau_\infty^*(b) = \begin{cases} \tau^*_{\omega(b)}(b)\,, & \text{if } \omega(b) < \infty\,, \\ \infty\,, & \text{if } \omega(b) = \infty\,, \end{cases} \qquad \tau^*(b) = \sup\Big\{\frac{\min \mathsf{L}(b^{-1}a)}{\min \mathsf{L}(a)} \;\Big|\; a \in bH \setminus H^\times\Big\}\,.$$

By definition, $\tau^*(b) \in \mathbb{R}_{\ge 0} \cup \{\infty\}$, $\tau_0^*(b) = 0$ and $\tau_1^*(b) \le \tau_2^*(b) \le \ldots$. If $b \in H^\times$, then $\tau^*(b) = 1$, $\tau_\infty^*(b) = 0$, and if H contains a prime, then $\tau_k^*(b) = k$ for all $k \in \mathbb{N}$. If $m \in \mathbb{N}$ and b is a product of m primes, then $\tau_k^*(b) = \max\{0, k-m\}$.

Lemma 1. *If $b \in H$, $k \in \mathbb{N}$ and $k \ge \omega(b)$, then*

$$\tau_k^*(b) \le \tau_\infty^*(b) + k - \omega(b)\,.$$

In particular, if $\omega(b) < \infty$ and $\tau_k^*(b) = \infty$ for some $k \in \mathbb{N}$, then $\tau_\infty^*(b) = \infty$.

Proof. Let $b \in H$, $k \ge \omega(b)$ and $a \in bH$ such that $\min \mathsf{L}(a) = l \le k$. If $l \le \omega(b)$, then $\min \mathsf{L}(b^{-1}a) \le \tau^*_{\omega(b)}(b) \le \tau^*_\infty(b) + k - \omega(b)$. Thus suppose that $l > \omega(b)$, and let $a = u_1 \cdot \ldots \cdot u_l$, where $u_1, \ldots, u_l \in \mathcal{A}(H)$. Then (after renumbering if necessary) we have $b \,|\, c = u_1 \cdot \ldots \cdot u_{\omega(b)}$, and $\min \mathsf{L}(b^{-1}c) \le \tau^*_{\omega(b)}(b) = \tau^*_\infty(b)$. Since $b^{-1}a = u_{\omega(b)+1} \cdot \ldots \cdot u_l b^{-1}c$, it follows that

$$\tau^*_k(b) \le \min \mathsf{L}(b^{-1}a) \le \min \mathsf{L}(b^{-1}c) + l - \omega(b) \le \tau^*_\infty(b) + k - \omega(b)\,. \qquad \square$$

Theorem 1. *Let $b \in H$. Then we have*

$$\tau^*(b) = \sup\Big\{ \frac{\tau^*_k(b)}{k} \;\Big|\; k \in \mathbb{N} \Big\}, \quad \tau^*(b) - 1 \le \tau^*_\infty(b) \le \omega(b)\tau^*(b)\,,$$

and if $\tau^*_\infty(b) < \infty$, then

$$\limsup_{k\to\infty} \frac{\tau^*_k(b)}{k} \le 1\,.$$

In particular, $\tau^*_\infty(b) < \infty$ if and only if $\tau^*(b) < \infty$ and $\omega(b) < \infty$.

Proof. If $a \in bH \setminus H^\times$ and $\min \mathsf{L}(a) = l$, then

$$\frac{\min \mathsf{L}(b^{-1}a)}{\min \mathsf{L}(a)} \le \frac{\tau^*_l(b)}{l} \le \sup\Big\{ \frac{\tau^*_k(b)}{k} \;\Big|\; k \in \mathbb{N} \Big\}\,,$$

and therefore

$$\tau^*(b) = \sup\Big\{ \frac{\min \mathsf{L}(b^{-1}a)}{\min \mathsf{L}(a)} \;\Big|\; a \in bH \setminus H^\times \Big\} \le \sup\Big\{ \frac{\tau^*_k(b)}{k} \;\Big|\; k \in \mathbb{N} \Big\}\,.$$

To prove the reverse inequality, let $\mu \in \mathbb{R}$ be such that

$$\mu < \sup\Big\{ \frac{\tau^*_k(b)}{k} \;\Big|\; k \in \mathbb{N} \Big\}, \quad \text{and then we show that} \quad \tau^*(b) > \mu\,.$$

Indeed, there is some $k \in \mathbb{N}$ satisfying $\tau^*_k(b) > \mu k$, and thus there is some $a \in bH$ such that $\min \mathsf{L}(a) \le k$ and $\min \mathsf{L}(b^{-1}a) > \mu k$, which implies that

$$\tau^*(b) \ge \frac{\min \mathsf{L}(b^{-1}a)}{\min \mathsf{L}(a)} > \frac{\mu k}{k} = \mu\,.$$

If $b \in H^\times$ or $\omega(b) = \infty$, then obviously $\tau^*(b) - 1 \le \tau^*_\infty(b) \le \tau^*(b)\omega(b)$. Thus suppose that $b \in H \setminus H^\times$ and $\omega(b) < \infty$. Then $\omega(b) > 0$ and

$$\tau^*_\infty(b) = \omega(b)\, \frac{\tau^*_{\omega(b)}(b)}{\omega(b)} \le \omega(b)\tau^*(b)\,.$$

If $\tau^*_\infty(b) < \infty$, then $\omega(b) < \infty$, and for $k \ge \omega(b)$ Lemma 1 implies

$$\frac{\tau^*_k(b)}{k} \le \frac{\tau^*_\infty(b)}{k} + 1 \le \tau^*_\infty(b) + 1\,.$$

Thus we obtain

$$\tau^*(b) = \sup\Big\{ \frac{\tau^*_k(b)}{k} \;\Big|\; k \in \mathbb{N} \Big\} \le \tau^*_\infty(b) + 1 \quad \text{and} \quad \limsup_{k\to\infty} \frac{\tau^*_k(b)}{k} \le 1\,. \qquad \square$$

Proposition 2. *If b, $c \in H$ and $k \in \mathbb{N}$, then*

$$\tau_k^*(bc) \le \tau^*(b)\tau_k^*(c) \quad \text{and} \quad \tau^*(bc) \le \tau^*(b)\tau^*(c)\,.$$

In particular, if $\tau^*(u) < \infty$ for all $u \in \mathcal{A}(H)$, then $\tau^*(b) < \infty$ for all $b \in H$.

Proof. If $\tau_k^*(c) = \infty$, there is nothing to do. Thus assume that $\tau_k^*(c) = t \in \mathbb{N}$. If $a \in bcH$ and $\min \mathsf{L}(a) \le k$, then $a \in cH$, hence $\min \mathsf{L}(c^{-1}a) \le t$, and

$$\min \mathsf{L}\big((bc)^{-1}a\big) \le \sup\{\min \mathsf{L}(b^{-1}a') \mid a' \in bH\,, \min \mathsf{L}(a') \le t\} \le \tau_t^*(b)\,.$$

Therefore we obtain

$$\tau_k^*(bc) \le \tau_t^*(b) = \frac{\tau_t^*(b)}{t}\, t \le \tau^*(b)\tau_k^*(c)$$

and

$$\tau^*(bc) = \sup\Big\{\frac{\tau_k^*(bc)}{k} \;\Big|\; k \in \mathbb{N}\Big\} \le \tau^*(b)\,\sup\Big\{\frac{\tau_k^*(c)}{k} \;\Big|\; k \in \mathbb{N}\Big\} = \tau^*(b)\tau^*(c)\,. \qquad \square$$

Proposition 3. *If $k \in \mathbb{N}$, $b \in H$ and $m \in \mathsf{L}(b)$, then*

$$\tau_k^*(b) \le \rho_k(H) - m \quad \text{and} \quad \tau^*(b) \le \rho(H)\,.$$

Proof. Let $k \in \mathbb{N}$, $b \in H$, $m \in \mathsf{L}(b)$ and $a \in bH$ be such that $\min \mathsf{L}(a) \le k$. Then $\min \mathsf{L}(b^{-1}a) + m \le \max \mathsf{L}(a) \le \rho_k(H)$ and therefore $\tau_k^*(b) \le \rho_k(H) - m$. By Theorem 1, it follows that

$$\tau^*(b) = \sup\Big\{\frac{\tau_k^*(b)}{k} \;\Big|\; k \in \mathbb{N}\Big\} \le \sup\Big\{\frac{\rho_k(H)}{k} \;\Big|\; k \in \mathbb{N}\Big\} = \rho(H)\,. \qquad \square$$

4 The tame degrees

Definition 6. For $b \in H$ and $k \in \mathbb{N}$, we denote by $\mathsf{t}_k(b)$ the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

> For every $a \in bH$, $z \in \mathsf{Z}(a)$ with $|z| \le k$ and $y \in \mathsf{Z}(b)$, there exists some $z' \in \mathsf{Z}(a) \cap y\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le N$.

We call $\mathsf{t}(b) = \sup\{\mathsf{t}_k(b) \mid k \in \mathbb{N}\}$ the *tame degree* of b.

According to Definition 4 we have

$$\mathsf{t}(b) = \mathsf{t}(H, \mathsf{Z}(b))\,, \quad \mathsf{t}(H) = \sup\{\mathsf{t}(u) \mid u \in \mathcal{A}(H)\}\,,$$

and H is locally tame if and only if $\mathsf{t}(u) < \infty$ for all $u \in \mathcal{A}(H)$.

By the very definition, it follows that $0 = \mathsf{t}_1(b) \le \mathsf{t}_2(b) \le \ldots \le \mathsf{t}_k(b)$ for all $k \in \mathbb{N}$, and if $\mathsf{t}_k(b) > 0$ for some $k \in \mathbb{N}$, then $\mathsf{t}_k(b) \ge 2$.

Lemma 2. *Let $b \in H$.*

1. $\mathsf{t}(b) = 0$ *if and only if every atom dividing b is a prime.*

2. *If $\omega(b) < \infty$, then $\mathsf{t}(b) = \mathsf{t}_{\omega(b)}(b)$.*

Proof. 1. See [3, Lemma 1.6.5.2].

2. We may assume that H is reduced. By definition, we have $\mathsf{t}(b) \ge \mathsf{t}_{\omega(b)}(b)$. To prove the reverse inequality, let $a \in bH$, $y \in \mathsf{Z}(b)$ and $z = u_1 \cdot \ldots \cdot u_n \in \mathsf{Z}(a)$, where $n \in \mathbb{N}_0$ and $u_1, \ldots, u_n \in \mathcal{A}(H)$. Since $a = u_1 \cdot \ldots \cdot u_n \in bH$, there exists (after renumbering if necessary) some $m \in \mathbb{N}_0$ such that $m \le \min\{n, \omega(b)\}$, $c = u_1 \cdot \ldots \cdot u_m \in bH$ and $z_0 = u_1 \cdot \ldots \cdot u_m \in \mathsf{Z}(c)$. Therefore there exists some $z_0' \in \mathsf{Z}(c) \cap y\mathsf{Z}(H)$ such that $\mathsf{d}(z_0, z_0') \le \mathsf{t}_m(b) \le \mathsf{t}_{\omega(b)}(b)$. Now we obtain $z' = z_0' u_{m+1} \cdot \ldots \cdot u_n \in \mathsf{Z}(a) \cap y\mathsf{Z}(H)$ and $\mathsf{d}(z, z') = \mathsf{d}(z_0, z_0') \le \mathsf{t}_{\omega(b)}(b)$. □

Proposition 4. *If b, $c \in H$ and $k \in \mathbb{N}$, then*

$$\mathsf{t}_k(bc) \le 2\mathsf{t}(b) + \mathsf{t}_k(c)\,,$$

In particular, it follows that $\mathsf{t}(bc) \le 2\mathsf{t}(b) + \mathsf{t}(c)$ for all b, $c \in H$, and if H is locally tame, then $\mathsf{t}(a) < \infty$ for all $a \in H$.

Proof. Let $a \in bcH$, $y \in \mathsf{Z}(bc)$ and $z \in \mathsf{Z}(a)$ with $|z| \le k$. We must prove that there exists some $z' \in \mathsf{Z}(a) \cap y\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le 2\mathsf{t}(b) + \mathsf{t}_k(c)$.

Let $x \in \mathsf{Z}(b)$ be arbitrary. Since $b \,|\, bc$, there exists some $y_1 \in \mathsf{Z}(bc) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(y, y_1) \le \mathsf{t}(b)$. Then $x^{-1}y_1 \in \mathsf{Z}(c)$, and since $c \,|\, a$, there exists some $z_1 \in \mathsf{Z}(a) \cap x^{-1}y_1\mathsf{Z}(H)$ such that $\mathsf{d}(z, z_1) \le \mathsf{t}_k(c)$. Now we have $b \mid c^{-1}a$ and $xy_1^{-1}z_1 \in \mathsf{Z}(c^{-1}a)$, and therefore there exists some $w \in \mathsf{Z}(c^{-1}a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(w, xy_1^{-1}z_1) \le \mathsf{t}(b)$. With $z' = yx^{-1}w \in \mathsf{Z}(a)$ we obtain

$$\begin{aligned}\mathsf{d}(z, z') &\le \mathsf{d}(z, z_1) + \mathsf{d}\big(x^{-1}y_1(xy_1^{-1}z_1), x^{-1}y_1w\big) + \mathsf{d}(x^{-1}y_1w, x^{-1}yw)\\ &= \mathsf{d}(z, z_1) + \mathsf{d}(xy_1^{-1}z_1, w) + \mathsf{d}(y, y_1) \le \mathsf{t}_k(c) + 2\mathsf{t}(b)\,.\end{aligned}$$ □

Theorem 2. *If $b \in H$ and there is some atom dividing b which is not a prime, then*

$$2 \le \omega(b) \le \mathsf{t}(b) + \min \mathsf{L}(b) - 1\,.$$

In particular, if H is locally tame, then H is a BF-*monoid.*

Proof. We may assume that H is reduced.

By Proposition 1 we have $\omega(b) \ge 2$, and it suffices to prove that, for all $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in \mathcal{A}(H)$ with $b \,|\, u_1 \cdot \ldots \cdot u_n$, there exists some $\Omega \subset [1, n]$ such that $|\Omega| \le \mathsf{t}(b) + \min \mathsf{L}(b) - 1$ and

$$b \;\Big|\; \prod_{i \in \Omega} u_i\,.$$

Let $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in \mathcal{A}(H)$ such that $b \mid a = u_1 \cdot \ldots \cdot u_n$. Consider the factorization $z = u_1 \cdot \ldots \cdot u_n \in \mathsf{Z}(a)$, and let $y \in \mathsf{Z}(b)$ be such that $|y| = \min \mathsf{L}(b)$. Then there exists some $z' \in \mathsf{Z}(a) \cap y\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le \mathsf{t}(b)$.

If $z \in y\mathsf{Z}(H)$, then (after renumbering if necessary) we obtain $y = u_1 \cdot \ldots \cdot u_d$, hence $b \,|\, u_1 \cdot \ldots \cdot u_d$ and $d = \min \mathsf{L}(b) \le \mathsf{t}(b) + \min \mathsf{L}(b) - 1$, since $\mathsf{t}(b) \ge 1$ by Lemma 2.

If $z \notin y\mathsf{Z}(H)$, then (after renumbering if necessary) we may assume that

$$z' = y u_1 \cdot \ldots \cdot u_d v_1 \cdot \ldots \cdot v_s \,,$$

where $d \in [0, n]$, $s \in \mathbb{N}_0$, $v_1, \ldots, v_s \in \mathcal{A}(H)$ and $\{u_{d+1}, \ldots, u_n\} \cap \{v_1, \ldots, v_s\} = \emptyset$. It follows that

$$\mathsf{t}(b) \ge \mathsf{d}(z, z') \ge n - d - |\gcd{}_{\mathsf{Z}(H)}\{y, u_{d+1} \cdot \ldots \cdot u_n\}| \ge n - d - (|y| - 1) \,.$$

Since $a = b u_1 \cdot \ldots \cdot u_d v_1 \cdot \ldots \cdot v_s = u_1 \cdot \ldots \cdot u_n$, it follows that $b \,|\, u_{d+1} \cdot \ldots \cdot u_n$, and $n - d \le \mathsf{t}(b) + |y| - 1 = \mathsf{t}(b) + \min \mathsf{L}(b) - 1$. □

Theorem 3. *For all $b \in H$ and $k \in \mathbb{N}$ we have*

$$\begin{aligned} \tau_k^*(b) - k + \min \mathsf{L}(b) \;\le\; \mathsf{t}_k(b) \;\le\; \mathsf{t}(b) \;&\le\; \max\{\omega(b),\, \tau_\infty^*(b) + \sup \mathsf{L}(b)\} \\ &\le \omega(b)[\,\tau^*(b) + 1\,], \quad \textit{and } \mathsf{t}(b) < \infty \textit{ if and only if } \tau_\infty^*(b) < \infty. \end{aligned}$$

Proof. We may assume that H is reduced.

Let $b \in H$, $k \in \mathbb{N}$, $a \in bH$, $y \in \mathsf{Z}(b)$ and $z \in \mathsf{Z}(a)$ such that $|z| \le k$. Then there exists some $z' \in \mathsf{Z}(a) \cap y\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \le \mathsf{t}_k(b)$, and we obtain

$$\begin{aligned} \min \mathsf{L}(b^{-1}a) \le |y^{-1}z'| \le |z'| - \min \mathsf{L}(b) &\le |z| + \mathsf{d}(z, z') - \min \mathsf{L}(b) \\ &\le k + \mathsf{t}_k(b) - \min \mathsf{L}(b) \,. \end{aligned}$$

Therefore it follows that

$$\tau_k^*(b) = \sup\{\min \mathsf{L}(b^{-1}a) \mid a \in bH \,, \min \mathsf{L}(a) \le k\} \le k + \mathsf{t}_k(b) - \min \mathsf{L}(b) \,.$$

This proves the first inequality, and $\mathsf{t}_k(b) \le \mathsf{t}(b)$ holds by definition.

To prove the third inequality, we asume that $a \in bH$, $z = u_1 \cdot \ldots \cdot u_n \in \mathsf{Z}(a)$ and $y = q_1 \cdot \ldots \cdot q_r \in \mathsf{Z}(b)$ with $n, r \in \mathbb{N}_0$ and $u_1, \ldots, u_n, q_1, \ldots, q_r \in \mathcal{A}(H)$. Let $m \in \mathbb{N}_0$ be such that $m \le \min\{n, \omega(b)\}$ and (after renumbering if necessary) $c = u_1 \cdot \ldots \cdot u_m \in bH$. If $l = \min \mathsf{L}(b^{-1}c)$, then there exist $v_1, \ldots, v_l \in \mathcal{A}(H)$ such that $c = q_1 \cdot \ldots \cdot q_r v_1 \cdot \ldots \cdot v_l$, and we consider the factorization

$$z' = y v_1 \cdot \ldots \cdot v_l u_{m+1} \cdot \ldots \cdot u_n \in \mathsf{Z}(a) \cap y\mathsf{Z}(H) \,.$$

Since

$$\begin{aligned} \mathsf{d}(z, z') \le \max\{m, l + r\} &\le \max\{\omega(b), \min \mathsf{L}(b^{-1}c) + \sup \mathsf{L}(b)\} \\ &\le \max\{\omega(b),\, \tau_\infty^*(b) + \sup \mathsf{L}(b)\} \,, \end{aligned}$$

it follows that

$$\mathsf{t}(b) \le \max\{\omega(b),\, \tau_\infty^*(b) + \sup \mathsf{L}(b)\} \,.$$

It remains to prove the last inequality. By Theorem 1 and Proposition 1 it follows that $\tau^*_\infty(b) + \sup \mathsf{L}(b) \le \omega(b)\tau^*(b) + \omega(b) = \omega(b)\,[\,\tau^*(b) + 1\,]$, and therefore

$$\max\{\omega(b),\, \tau^*_\infty(b) + \sup \mathsf{L}(b)\} \le \omega(b)[\,\tau^*(b) + 1\,]\,.$$

If $\mathsf{t}(b) < \infty$, then Theorem 2 implies $\omega(b) < \infty$, and we obtain

$$\tau^*_\infty(b) = \tau^*_{\omega(b)}(b) \le \mathsf{t}(b) + \omega(b) - \min \mathsf{L}(b) < \infty\,.$$

Conversely, if $\tau^*_\infty(b) < \infty$, then $\sup \mathsf{L}(b) \le \omega(b) < \infty$ and thus also $\mathsf{t}(b) < \infty$. □

5 An example

Usually, finiteness results in factorization theory are proved by showing local tameness first. The following example however, already considered in [6, Example 6.11], indicates that even a monoid with catenary degree 3 may fail to be locally tame.

Example 1. The additive monoid

$$H = \{(a, b, c) \in \mathbb{N}_0^3 \mid a > 0 \text{ or } b = c\} \subset \mathbb{N}_0^3$$

is v-noetherian and not locally tame, but yet it satisfies $\mathsf{c}(H) = 3$.

Proof. We show first that H is v-noetherian. For this we consider the noetherian domain $R = \mathbb{Z}[X^2, X^3]$, its multiplicative monoid $R^\bullet = R \setminus \{0\}$ and the monoid

$$\widetilde{H} = \{X^{2a}(1+X)^b(1-X)^c \mid a,\, b,\, c \in \mathbb{N}_0\,, a > 0 \text{ or } b = c\} \subset R^\bullet\,.$$

$\widetilde{H} \times \{\pm 1\}$ is a divisor-closed (hence saturated) submonoid of $R^\bullet$. Since R is noetherian, the monoid $R^\bullet$ is v-noetherian. Hence the monoids $\widetilde{H} \times \{\pm 1\}$ and $\widetilde{H} = (\widetilde{H} \times \{\pm 1\})_{\rm red}$ are also v-noetherian, and the map

$$\Phi\colon H \to \widetilde{H}\,, \quad \text{defined by} \quad \Phi(a, b, c) = X^{2a}(1+X)^b(1-X)^c\,,$$

is an isomorphism. Hence H is v-noetherian, too, and Proposition 1.5 implies that $\omega(\boldsymbol{x}) < \infty$ for all $\boldsymbol{x} \in H$.

For $x,\, y \in \mathbb{N}_0$, we set $\boldsymbol{u}_x = (1, x, 0)$, $\boldsymbol{v}_y = (1, 0, y)$ and $\boldsymbol{w} = (0, 1, 1)$ (observe that $\boldsymbol{u}_0 = \boldsymbol{v}_0$). Then $\mathcal{A}(H) = \{\boldsymbol{u}_x,\, \boldsymbol{v}_y,\, \boldsymbol{w} \mid x,\, y \in \mathbb{N}_0\}$.

The factorizations of an element $\boldsymbol{x} = (a, b, c) \in H$ with $b \le c$ are as follows (we write the factorization monoid $\mathsf{Z}(H)$ multiplicatively).

If $b = c = 0$, then $\mathsf{Z}(\boldsymbol{x}) = \{\boldsymbol{u}_0^a\}$.

If $a = 0$ (and consequently $b = c$), then $\mathsf{Z}(\boldsymbol{x}) = \{\boldsymbol{w}^b\}$.

If $a = 1$, then $\mathsf{Z}(\boldsymbol{x}) = \{\boldsymbol{v}_{c-b}\boldsymbol{w}^b\}$.

If $a \ge 2$, then $\mathsf{Z}(\boldsymbol{x})$ consists of all products

$$\prod_{i=1}^{r} \boldsymbol{u}_{x_i} \prod_{j=1}^{s} \boldsymbol{v}_{y_j} \, \boldsymbol{w}^t \,,$$

where r, s, t, $x_1, \ldots, x_r$, $y_1, \ldots, y_s \in \mathbb{N}_0$, $a = r+s$, $b = x_1 + \ldots + x_r + t$, $c = y_1 + \ldots + y_s + t$ and

$$\Big| \prod_{i=1}^{r} \boldsymbol{u}_{x_i} \prod_{j=1}^{s} \boldsymbol{v}_{y_j} \, \boldsymbol{w}^t \Big| = r + s + t = a + t \,.$$

Hence it follows that $\mathsf{L}(\boldsymbol{x}) \subset [a, a+b]$, and for every $j \in [0, b]$, there is the factorization $\mathsf{z}_j = \boldsymbol{u}_0^{a-2} \boldsymbol{u}_{b-j} \boldsymbol{v}_{c-j} \boldsymbol{w}^j \in \mathsf{Z}(\boldsymbol{x})$ satisfying $|\mathsf{z}_j| = a + j$, showing that $\mathsf{L}(\boldsymbol{x}) = [a, a+b]$.

For x, x', y, $y' \in \mathbb{N}$ the relations

$$\boldsymbol{u}_x + \boldsymbol{u}_{x'} = \boldsymbol{v}_0 + \boldsymbol{u}_{x+x'} \,, \quad \boldsymbol{v}_y + \boldsymbol{v}_{y'} = \boldsymbol{v}_0 + \boldsymbol{v}_{y+y'} \quad \text{and} \quad \boldsymbol{u}_x + \boldsymbol{v}_y = \boldsymbol{u}_{x-1} + \boldsymbol{v}_{y-1} + \boldsymbol{w}$$

show that any two factorizations of an element $\boldsymbol{x} \in H$ can be concatenated by a 3-chain. Hence $\mathsf{c}(H) = 3$.

For $x \in \mathbb{N}$, we consider the elements $\boldsymbol{a}_x = (2, x, x)$ and $\boldsymbol{b} = (1, 0, 0)$. Then $\boldsymbol{a}_x \in \boldsymbol{b} + H$,

$$\mathsf{Z}(-\boldsymbol{b} + \boldsymbol{a}_x) = \{\boldsymbol{u}_0 \boldsymbol{w}^x\} \quad \text{and} \quad \boldsymbol{u}_x \boldsymbol{v}_x \in \mathsf{Z}(\boldsymbol{a}_x) \,, \quad \text{whence} \quad \tau_2^*(\boldsymbol{b}) = \infty \,.$$

Hence it follows $\tau^*(\boldsymbol{b}) = \infty$ by Theorem 1, and $\tau_\infty^*(\boldsymbol{b}) = \infty$ by Lemma 1. In particular, H is not locally tame by Proposition 4 and Theorem 3.

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