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Abstract. For infinite discrete additive semigroups $X \subset [0,\infty)$ we study normed algebras of arithmetic functions $g\colon X \to \mathbb{C}$ endowed with the linear operations and the convolution. In particular, we investigate the problem of scaling the mean deviation of related multiplicative functions for $X = \log \mathbb{N}$. This involves an extension of Banach algebras of arithmetic functions by introducing weight functions and proving a weighted inversion theorem of Wiener type in the frame of Gelfand's theory of commutative Banach algebras.

1 Introduction

In this note we present weighted inversion theorems for arithmetic functions in the frame of Gelfand's theory of commutative Banach algebras. In particular, we derive a weighted Wiener type inversion theorem for power series and give applications to the theory of multiplicative arithmetic functions.

2 Arithmetic functions on discrete additive semigroups

For a unitary approach to arithmetic functions we consider the class $\mathcal{A}(X)$ of arithmetic functions $g\colon X \to \mathbb{C}$ defined on an infinite discrete additive semigroup $X \subset [0,\infty)$ with $0 \in X$. Endowed with the usual linear operations and the *convolution* defined by

$$\bigl(f * g\bigr)(x) = \sum_{\substack{y,z \in X \\ x = y+z}} f(y)\, g(z) \qquad (x \in X), \tag{1}$$

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$\mathcal{A}(X)$ forms a unital commutative complex algebra. The *unity* $\varepsilon \in \mathcal{A}(X)$ is given by $\varepsilon(0) = 1$ and $\varepsilon(x) = 0$ for $x \neq 0$. The *multiplicative group* of $\mathcal{A}(X)$, i.e. the group of invertible functions under the convolution, is

$$\mathcal{A}^*(X) = \{g \in \mathcal{A}(X) : g(0) \neq 0\}. \tag{2}$$

Indeed, for $g \in \mathcal{A}(X)$ given, we have to show the existence of $f \in \mathcal{A}(X)$ satisfying $f * g = \varepsilon$. From (1) we obtain that $f(0)g(0) = \varepsilon(0) = 1$ so that necessarily $g(0) \neq 0$, and for $0 < x \in X$ we see that

$$f(x)\, g(0) = - \sum_{\substack{y,z \in X, y<x \\ x=y+z}} f(y)\, g(z)$$

defines f recursively, if $g(0) \neq 0$. As usual we write g^{-1} for the *inverse* of $g \in \mathcal{A}^*(X)$, i.e., g^{-1} satisfies $g * g^{-1} = \varepsilon$.

With every $g \in \mathcal{A}(X)$ we associate the *general Dirichlet series*

$$\widetilde{g}(s) = \sum_{x \in X} g(x)\, e^{-xs} \qquad (s \in \mathbb{C}). \tag{3}$$

Endowed with the linear operations and the multiplication defined by

$$\widetilde{f}(s) \cdot \widetilde{g}(s) := (f * g)^{\sim}(s)$$

the series (3) form an algebra $\widetilde{\mathcal{A}}(X)$ that is isomorphic to $\mathcal{A}(X)$. Note that this definition is suggested by formal multiplication of the series and by arranging the resulting product series as general Dirichlet series again, regardless of convergence.

If both $\widetilde{f}(s)$ and $\widetilde{g}(s)$ converge absolutely, then $(f*g)^{\sim}(s) = \widetilde{f}(s) \cdot \widetilde{g}(s)$ converges absolutely. If a Dirichlet series $\widetilde{g}(s)$ converges absolutely at $s_0 \in \mathbb{C}$, then the absolute convergence is uniform in the *closed* half plane $\operatorname{Re} s \geq \operatorname{Re} s_0$. Since the absolute convergence of $\widetilde{g}(s)$ in an open half plane $\operatorname{Re} s > \operatorname{Re} s_0$ implies that of the formal derivative

$$\widetilde{g}\,'(s) = - \sum_{x \in X} x\, g(x)\, e^{-xs}, \tag{4}$$

$\widetilde{g}(s)$ represents a holomorphic function for $\operatorname{Re} s > \operatorname{Re} s_0$. Further, for any $g \in \mathcal{A}(X)$ there is a number $\alpha \in \mathbb{R}$ or $\alpha \in \{-\infty, \infty\}$, called the *abscissa of absolute convergence* of $\widetilde{g}(s)$, such that $\widetilde{g}(s)$ converges absolutely for $\operatorname{Re} s > \alpha$ and does not converge absolutely for $\operatorname{Re} s < \alpha$.

For illustration consider the following examples.

Example 1. The additive semigroup $X = \mathbb{N}_0$ serves as domain for the algebra of arithmetic functions $g \in \mathcal{A}(\mathbb{N}_0)$. Here the *Cauchy convolution* corresponds to the Cauchy product of formal *power series*. After substituting $z = e^{-s}$ and writing $\widetilde{g}(z)$ instead of $\widetilde{g}(s)$, they take the usual form

$$\widetilde{g}(z) = \sum_{n=0}^{\infty} g(n)\, z^n \qquad (z \in \mathbb{C}). \tag{5}$$

Example 2. The additive semigroup $X = \log \mathbb{N}$ with elements $x = \log n$ serves as domain for the algebra $\mathcal{A}(\log \mathbb{N})$ of arithmetic functions $g\colon \mathbb{N} \to \mathbb{C}$. With $g(\log n)$ replaced by $g(n)$ the *Dirichlet convolution* corresponds to the product of ordinary *Dirichlet series*

$$\widetilde{g}(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \qquad (s \in \mathbb{C}). \tag{6}$$

Well-known subalgebras of $\mathcal{A}(X)$ are those referring to the absolute convergence of Dirichlet series $\widetilde{g} \in \widetilde{\mathcal{A}}(X)$, which reflects the mean growth of the generating arithmetic functions $g \in \mathcal{A}(X)$. Let $H = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ be the open right half plane of the complex plane and $\overline{H}$ its closure. The usual classification distinguishes the subalgebras of functions $g \in \mathcal{A}(X)$ with absolutely convergent series $\widetilde{g}(s)$ for $s \in \overline{H} + \varrho$, with $\varrho \in \mathbb{R}$ fixed. Obviously each of these nested subalgebras is isomorphic to that with $\varrho = 0$, under the mapping $g(x) \mapsto g(x)\, e^{-\varrho x}$. A major problem consists in determining its multiplicative group. The result is an inversion theorem of Wiener type, originally proved for Fourier series (cf. Wiener [19]). With the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\} \subset \mathbb{C}$, the most frequent version is that for power series:

Theorem 1. *If the power series $\widetilde{g}(z)$ converges absolutely and is zero-free for all $z \in \overline{U}$, then the power series $\widetilde{f}(z) := 1/\widetilde{g}(z)$ converges absolutely for $z \in \overline{U}$, too.*

3 Weighted Banach algebras

We aim for a finer classification. To this end let $\mathcal{W}(X)$ be the set of admissible *weight functions* $w\colon X \to [1, \infty)$ satisfying both conditions

$$w(0) = 1 \le w(x + y) \le w(x)\, w(y) \quad \text{for all } \ x, y \in X, \tag{7}$$

$$\lim_{k\to\infty} \sqrt[k]{w(kx)} = 1 \quad \text{for every } \ x \in X. \tag{8}$$

For $w \in \mathcal{W}(X)$ we introduce the normed unital complex algebra

$$\mathcal{D}_w(X) = \big\{ g \in \mathcal{A}(X) : \|g\|_w < \infty \big\}$$

of all functions $g \in \mathcal{A}(X)$ having a finite *w-norm*

$$\|g\|_w = \sum_{x\in X} |g(x)|\, w(x).$$

In particular, for the constant weight function $w = 1$, $\mathcal{D}_1(X)$ consists of all $g \in \mathcal{A}(X)$ with absolutely convergent Dirichlet series $\widetilde{g}(s)$ for $s \in \overline{H}$.

For $w \in \mathcal{W}(X)$ we have $\|\varepsilon\|_w = w(0) = 1$, and we infer from (7) that the w-norm is *submultiplicative*, i.e., $\|f * g\|_w \le \|f\|_w\, \|g\|_w$. Further, $\mathcal{D}_w(X)$ considered as metric space is complete relative to the w-norm. Hence $\mathcal{D}_w(X)$ is a *Banach subalgebra* of $\mathcal{D}_1(X)$ for every $w \in \mathcal{W}(X)$. Note that (8) delimits the growth of $w \in \mathcal{W}(X)$. In fact,

$$w(x) \ll e^{\eta x} \qquad (x \in X) \tag{9}$$

holds for every $\eta > 0$. Therefore the absolute convergence of $\widetilde{g}(s)$ in some *open* half plane transfers to the series $(gw)^{\sim}(s)$.

We return to the Examples 1 and 2.

Example 3. Typical examples of admissible weights $w \in \mathcal{W}(\mathbb{N}_0)$ are powers $w(n) = (1+n)^c$ and exponential functions of the form $w(n) = \exp(c\,n^d)$, with $c \geq 0$ and $0 \leq d < 1$. In particular, for $w(n) = (1+n)^k$ with $k \in \mathbb{N}_0$ and $g \in \mathcal{D}_w(\mathbb{N}_0)$, the power series $\widetilde{g}(z)$ in (5) *and its derivatives up to order* k converge absolutely for $|z| \leq 1$.

Example 4. For $w \in \mathcal{W}(\log \mathbb{N})$ write $w(n)$ instead of $w(\log n)$. Then $w(n) \ll n^\eta$ for every $\eta > 0$. Typical examples of admissible weights for the ordinary Dirichlet series (6) are the log powers $w(n) = (1 + \log n)^c$ and the functions $w(n) = \exp(c \log^d n)$, with $c \geq 0$ and $0 \leq d < 1$. In particular, for $w(n) = (1 + \log n)^k$ with $k \in \mathbb{N}_0$ and $g \in \mathcal{D}_w(\log \mathbb{N})$ the Dirichlet series $\widetilde{g}(s)$ in (6) *and its derivatives up to order* k converge absolutely for $\operatorname{Re} s \geq 0$.

The problem to determine the multiplicative group of $\mathcal{D}_w(X)$ for weight functions $w \in \mathcal{W}(X)$ is answered by the following theorem (cf. Lucht and Reifenrath [12]).

Theorem 2. *If* $X \subset [0, \infty)$ *is an infinite discrete additive semigroup with* $0 \in X$ *and* $w \in \mathcal{W}(X)$, *then the multiplicative group of the Banach algebra* $\mathcal{D}_w(X)$ *is*

$$\mathcal{D}_w^*(X) = \bigl\{ g \in \mathcal{D}_w(X) : 0 \notin \overline{\widetilde{g}(H)} \bigr\}.$$

The inversion condition $0 \notin \overline{\widetilde{g}(H)}$ is equivalent to $\inf \{ |\widetilde{g}(s)| : \operatorname{Re} s \geq 0 \} > 0$. We remark that the corresponding Lévy extension replacing the inversion by a holomorphic function defined on some region $\Omega \subset \mathbb{C}$ is also true (cf. [12]).

In particular, Wiener's inversion theorem 1 for power series $\widetilde{g}(z)$ according to (5) occurs as the special case $X = \mathbb{N}_0$, $w = 1$ of Theorem 2 (cf. Lucht [10]):

Theorem 3. *For* $w \in \mathcal{W}(\mathbb{N}_0)$ *the multiplicative group of the Banach algebra* $\mathcal{D}_w(\mathbb{N}_0)$ *is*

$$\mathcal{D}_w^*(\mathbb{N}_0) = \bigl\{ g \in \mathcal{D}_w(\mathbb{N}_0) : \widetilde{g}(z) \neq 0 \ \text{ for } \ z \in \overline{U} \bigr\}.$$

Note that the inversion condition is equivalent to $0 \notin \overline{\widetilde{g}(U)}$, because $\overline{U}$ is compact.

The weighted inversion theorem for ordinary Dirichlet series $\widetilde{g}(s)$ according to (6) follows from Theorem 2 for $X = \log \mathbb{N}$ (cf. [12]). In the special case $w = 1$ it was proved in 1957 by Hewitt and Williamson [7] and, independently, by Edwards [2].

Theorem 4. *For* $w \in \mathcal{W}(\log \mathbb{N})$ *the multiplicative group of the Banach algebra* $\mathcal{D}_w(\log \mathbb{N})$ *is*

$$\mathcal{D}_w^*(\log \mathbb{N}) = \bigl\{ g \in \mathcal{D}_w(\log \mathbb{N}) : 0 \notin \overline{\widetilde{g}(H)} \bigr\}. \tag{10}$$

In the next section we confine to a short direct proof of the weighted inversion Theorem 3 for power series and explain the major difficulty of the proof of Theorem 4 for Dirichlet series. This requires some tools from Gelfand's theory of commutative Banach algebras (Gelfand [4], see, for instance, Rudin [15, Chapter 18]).

4 Functional analytic tools and proof of Theorem 3

Let A be a commutative complex algebra with unity e and finite *norm* $\|\,.\,\|$, which makes A into a metric space. Recall that A is a *normed complex algebra*, if the norm is *submultiplicative*, i.e. $\|x \cdot y\| \le \|x\|\,\|y\|$ for all $x, y \in A$. It is a *Banach algebra,* if the metric space A is also complete relative to this norm. Obviously we have $\|e\| \ge 1$, and we shall assume that $\|e\| = 1$.

Gelfand's theory associates with A the space $\Delta(A)$ of homomorphisms of A onto the complex field, or, in other words, the non-trivial multiplicative linear functionals $h\colon A \to \mathbb{C}$. The following general theorem relates the norm on $\Delta(A)$ to that on A and characterizes the invertible elements of A (cf., for instance, Rudin [15, Theorem 18.17]).

Theorem 5. *For all $a \in A$ and $h \in \Delta(A)$ we have $|h(a)| \le \|a\|$. An element $a \in A$ is invertible, if and only if $h(a) \ne 0$ for all $h \in \Delta(A)$.*

To identify the invertible elements of A therefore suggests to determine all non-trivial multiplicative linear functionals of A.

Proof. [Proof of Theorem 3] For application of Theorem 5 to the Banach algebra $\mathcal{D}_w(\mathbb{N}_0)$ endowed with the linear operations, the Cauchy convolution and the norm $\|\,.\,\|_w$ with weight functions $w \in \mathcal{W}(\mathbb{N}_0)$ we determine all non-trivial multiplicative linear functionals $h \in \Delta\big(\mathcal{D}_w(\mathbb{N}_0)\big)$. Let $\varepsilon_k \in \mathcal{D}_w(\mathbb{N}_0)$ be defined for $k \in \mathbb{N}_0$ by $\varepsilon_k(n) = \delta_{kn}$ for all $n \in \mathbb{N}_0$, where δ is the Kronecker symbol. Then $\varepsilon_0 = \varepsilon$, and $\varepsilon_k = \varepsilon_1^k := \varepsilon_1 * \cdots * \varepsilon_1$ with k factors ε_1 satisfies

$$\|\varepsilon_k\|_w = w(k) \qquad (k \in \mathbb{N}_0).$$

Every $g \in \mathcal{D}_w(\mathbb{N}_0)$ has the representation

$$g = \sum_{k=0}^{\infty} g(k)\,\varepsilon_k. \tag{11}$$

Given $h \in \Delta\big(\mathcal{D}_w(\mathbb{N}_0)\big)$, we have $z := h(\varepsilon_1) \in \mathbb{C}$ and $h(\varepsilon_k) = h^k(\varepsilon_1) = z^k$. Theorem 5 yields

$$|z|^k = |h(\varepsilon_k)| \le \|\varepsilon_k\|_w = w(k) \qquad (k \in \mathbb{N})$$

so that $|z| \le \sqrt[k]{w(k)}$ for all $k \in \mathbb{N}$. By (8) this is equivalent to $|z| \le 1$. Applying the continuous function h to (11) yields

$$h(g) = \widetilde{g}(z) \qquad \big(g \in \mathcal{D}_w(\mathbb{N}_0)\big). \tag{12}$$

Now Theorem 5 asserts that g is invertible in $\mathcal{D}_w(\mathbb{N}_0)$, if and only if $\widetilde{g}(z)$ does not vanish at any point $z \in \overline{U}$, as stated in Theorem 3. □

Usually inversion theorems of Wiener type are formulated and proved in terms of generating series. The preceding version shows explicitly the significant role of the structure of the underlying semigroup X. Here the simplicity of the proof essentially relies on the fact that the additive semigroup $\mathbb{N}_0$ is generated by the singleton $\{1\}$, which entails the representations (11) and (12) for functions $g \in \mathcal{D}_w(\mathbb{N}_0)$ and their image under h.

In contrast, the additive semigroup $\log\mathbb{N}$ occurring in Theorem 4 is generated by the infinite set $\log\mathbb{P} = \{\log p : p \text{ prime}\}$. Since the specific functionals $h_s \in \Delta := \Delta\big(\mathcal{D}_w(\log\mathbb{N})\big)$ defined by $h_s(g) = \widetilde{g}(s)$ for $s \in \overline{H}$ form a sparse subclass of Δ only, the crucial part of the proof of Theorem 4 consists in verifying that this subclass is *dense* in Δ, i.e., for all $h \in \Delta$, $g \in \mathcal{D}_w(\log\mathbb{N})$ and $\epsilon > 0$ there exists an $s \in \overline{H}$ such that $|h(g) - h_s(g)| < \epsilon$.

5 Weighted inversion of multiplicative functions

Returning to the usual multiplicative notation we replace the additive semigroup $X = \log\mathbb{N}$ in Example 2 with the multiplicative semigroup $\mathbb{N}$. Then the class of arithmetic functions $g\colon \mathbb{N} \to \mathbb{C}$ is a unital commutative complex algebra $\mathcal{B} = \mathcal{B}(\mathbb{N})$ under the linear operations and the Dirichlet convolution $*$,

$$f * g(n) = \sum_{dm=n} f(d)\, g(m) \qquad (n \in \mathbb{N}).$$

The unity $\varepsilon \in \mathcal{B}$ is given by $\varepsilon(n) = \delta_{1n}$ for $n \in \mathbb{N}$, and $\mathcal{B}^* = \{g \in \mathcal{B} : g(1) \neq 0\}$ is the multiplicative group of $\mathcal{B}$. For instance, the *constant function 1*, the *Möbius function* $\mu = 1^{-1}$, the *identity* I with $I(n) = n$ belong to $\mathcal{B}^*$, and the *logarithm* log belongs to $\mathcal{B} \setminus \mathcal{B}^*$.

The set $\mathbb{P}$ of primes serves as free multiplicative generator of $\mathbb{N}$. An important subgroup $\mathcal{M}$ of $\mathcal{B}^*$ is that of *multiplicative* functions $g \in \mathcal{B}^*$, i.e., $g(mn) = g(m)\, g(n)$ for all coprime $m, n \in \mathbb{N}$. Obviously $g(1) = 1$ for all $g \in \mathcal{M}$. If $g \in \mathcal{M}$ is *completely multiplicative*, i.e., $g(mn) = g(m)\, g(n)$ holds for all $m, n \in \mathbb{N}$, then $g^{-1} = \mu g$. In particular, $1, \mu, I \in \mathcal{M}$, and 1 and I are completely multiplicative.

Let $\mathbb{P}^\star = \{p^k : p \in \mathbb{P},\ k \in \mathbb{N}\}$ be the set of prime powers with positive integer exponents. For $g \in \mathcal{M}$ and $p \in \mathbb{P}$, we define the function $g_p \in \mathcal{M}$ by

$$g_p(n) = \begin{cases} g(n) & \text{for } n = p^k \in \mathbb{P}^\star \cup \{1\} \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Since $g(n)$ is the product of the $g_p(p^k)$ when n factors as the product of coprime powers p^k, $g \in \mathcal{M}$ can be reconstructed from the functions $g_p \in \mathcal{M}$. We write this formally as

$$g = \mathop{\Large *}_{p\in\mathbb{P}} g_p\,. \tag{14}$$

Conversely, this representation characterizes the multiplicative functions $g \in \mathcal{B}$.

The algebra $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}(\mathbb{N})$ of ordinary Dirichlet series (6) is isomorphic to $\mathcal{B}$. If the Dirichlet series $\widetilde{g}(s)$ of a function $g \in \mathcal{M}$ converges absolutely, then $\widetilde{g}(s)$ has a representation as absolutely convergent *Euler product*

$$\widetilde{g}(s) = \prod_p \widetilde{g}_p(s) \quad \text{with} \quad \widetilde{g}_p(s) = 1 + \frac{g(p)}{p^s} + \frac{g(p^2)}{p^{2s}} + \cdots \tag{15}$$

corresponding to (13) and (14). Conversely, if the series

$$\sum_{p^k \in \mathbb{P}^\star} \frac{g(p^k)}{p^{ks}} = \sum_p \big(\widetilde{g}_p(s) - 1\big) \tag{16}$$

converges absolutely, then $\widetilde{g}(s)$ converges absolutely.

The defining properties (7) and (8) of admissible weight functions $w \in \mathcal{W} = \mathcal{W}(\mathbb{N})$ defined on $\mathbb{N}$ take the form

$$w(1) = 1 \le w(mn) \le w(m)\, w(n) \quad \text{for all } \ m, n \in \mathbb{N}, \tag{17}$$

$$\lim_{k\to\infty} \sqrt[k]{w(n^k)} = 1 \quad \text{for every } \ n \in \mathbb{N}, \tag{18}$$

according to Example 4. The Banach algebra $\mathcal{D}_w(\log \mathbb{N})$, now called $\mathcal{F}_w = \mathcal{F}_w(\mathbb{N})$, consists of all functions $g \in \mathcal{B}$ with finite w-norm

$$\|g\|_w = \sum_{n=1}^{\infty} |g(n)|\, w(n).$$

Theorem 4 yields the multiplicative group

$$\mathcal{F}_w^* = \big\{ g \in \mathcal{F}_w : 0 \notin \overline{\widetilde{g}(H)}\, \big\}.$$

For $w \in \mathcal{W}$ let $g \in \mathcal{M} \cap \mathcal{F}_w^*$. Then the inversion condition takes the simpler form $0 \notin \widetilde{g}_p(\overline{H})$ for all $p \in \mathbb{P}$ or, equivalently,

$$\widetilde{g}_p(s) \neq 0 \quad \text{for all } p \in \mathbb{P} \text{ and } s \in \mathbb{C} \text{ with } \operatorname{Re} s \ge 0. \tag{19}$$

This follows from the Euler product representation (15) of $\widetilde{g}(s)$, because the absolute convergence of the series (16) yields $\widetilde{g}_p(s) \to 1$ as $p \to \infty$, uniformly for $\operatorname{Re} s \ge 0$. Moreover, we see that

$$\sum_p \big(\|g_p\|_w - 1\big) \le \|g\|_w \le \exp\Big(\sum_p \big(\|g_p\|_w - 1\big)\Big).$$

We extend $\mathcal{M} \cap \mathcal{F}_w$ considerably by partly replacing the w-norm with the mean square w-norm (cf. Lucht [10]).

Theorem 6. *For $w \in \mathcal{W}$ the class*

$$\mathcal{G}_w = \Big\{ g \in \mathcal{M} : \sum_p |g(p)|^2\, w^2(p) < \infty \ \text{ and } \sum_{p,k\ge 2} |g(p^k)|\, w(p^k) < \infty \Big\} \tag{20}$$

is a unital subsemigroup of $\mathcal{M}$ under the Dirichlet convolution, with the multiplicative group

$$\mathcal{G}_w^* = \big\{ g \in \mathcal{G}_w : \widetilde{g}_p(s) \neq 0 \ \text{ for } \ p \in \mathbb{P} \ \text{ and } \ s \in \overline{H} \big\}.$$

Proof. The submultiplicativity (17) of the w-norm combined with the Cauchy-Schwarz inequality entails that $\mathcal{G}_w$ is closed under $*$, and obviously $\varepsilon \in \mathcal{G}_w$. For $f, g \in \mathcal{G}_w^*$ and $p \in \mathbb{P}$ we have $f_p, g_p \in \mathcal{G}_w^*$ and $(f * g)_p\tilde{\ }(s) = \widetilde{f}_p(s)\,\widetilde{g}_p(s) \neq 0$ for $\operatorname{Re} s \geq 0$. Hence $\mathcal{G}_w^*$ is also closed under $*$. It remains to verify that $g \in \mathcal{G}_w^*$ implies $g^{-1} \in \mathcal{G}_w$.

In order to apply Theorem 3 to $\widetilde{g}_p(s)$ with $p \in \mathbb{P}$ fixed, we define a weight function $\omega \in \mathcal{W}(\mathbb{N}_0)$ by $\omega(k) = w(p^k)$ and a function $G \in \mathcal{D}_\omega(\mathbb{N}_0)$ by $G(k) = g_p(p^k)\, p^{-k}$ for $k \in \mathbb{N}_0$. Then the power series $\widetilde{G}(z) = \widetilde{g}_p(s)$ with $z = p^{-s}$ does not vanish for $|z| \leq 1$. Theorem 3 yields $G \in \mathcal{D}_\omega^*(\mathbb{N}_0)$, which is equivalent to $g_p \in \mathcal{G}_w^*$. Therefore $g_p^{-1} \in \mathcal{G}_w$ for each $p \in \mathbb{P}$. We have to transfer this property to g^{-1} and consider the Euler product

$$\widetilde{g}(s) = \prod_{p \leq p_0} \widetilde{g}_p(s) \cdot \prod_{p > p_0} \Big(1 - \frac{g(p)}{p^s}\Big)^{-1} \cdot \prod_{p > p_0} \Big(1 - \frac{g(p)}{p^s}\Big)\, \widetilde{g}_p(s).$$

It corresponds to the decomposition

$$g = \Big(\mathop{\Large *}_{p \leq p_0} g_p\Big) * b * h \tag{21}$$

with p_0 suitably large, and $b, h \in \mathcal{M}$ defined by

$$b(p^k) = \begin{cases} g^k(p) & \text{for } p > p_0\,,\ k \in \mathbb{N}_0 \\ 0 & \text{otherwise,} \end{cases}$$

$$h(p^k) = \begin{cases} g(p^k) - g(p^{k-1})g(p) & \text{for } p > p_0\,,\ k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

We have $h(p) = 0$ for $p \in \mathbb{P}$ and $b(p) = g(p)$ for all $p > p_0$. Now choose p_0 sufficiently large such that for $p > p_0$ both estimates

$$|g(p)|\, w(p) \leq \frac{1}{2} \quad \text{and} \quad \sum_{p, k \geq 2} |h(p^k)|\, w(p^k) \leq \frac{1}{2}$$

hold. Then $b \in \mathcal{G}_w^*$ and $b^{-1} = \mu b \in \mathcal{G}_w^*$, because $b \in \mathcal{M}$ is completely multiplicative. Further $h \in \mathcal{G}_w$. In order to verify that h is invertible within $\mathcal{G}_w$ we conclude from $h^{-1} * h = \varepsilon$ that $h^{-1}(p) = h(p) = 0$ for all $p \in \mathbb{P}$, $h(p^k) = 0$ for all $p \leq p_0$ and $k \in \mathbb{N}$, and

$$h^{-1}(p^k) = -\sum_{2 \leq j \leq k} h(p^j)\, h^{-1}(p^{k-j}) \qquad (p > p_0\,,\ k \geq 2).$$

From

$$\begin{aligned} \Sigma &:= \sum_{\substack{p^k \leq x \\ k \geq 2}} |h^{-1}(p^k)|\, w(p^k) \\ &\leq \sum_{\substack{p^k \leq x \\ k \geq 2}} \sum_{2 \leq j \leq k} |h(p^j)|\, w(p^j) \cdot |h^{-1}(p^{k-j})|\, w(p^{k-j}) \end{aligned}$$

$$= \sum_{\substack{p^k \le x \\ k \ge 2}} |h(p^k)|\, w(p^k) + \sum_{\substack{p^{j+\ell} \le x \\ j,\ell \ge 2}} |h(p^j)|\, w(p^j) \cdot |h^{-1}(p^\ell)|\, w(p^\ell)$$

$$\le \big(1+\Sigma\big) \sum_{\substack{p^k \le x \\ k \ge 2}} |h(p^k)|\, w(p^k) \le \frac{1}{2}\big(1+\Sigma\big)$$

we see that $\Sigma \le 1$. Hence $h^{-1} \in \mathcal{G}_w$, and (21) entails that

$$g^{-1} = \Big(\mathop{\ast}_{p \le p_0} g_p^{-1} \Big) * b^{-1} * h^{-1}$$

is a convolution of finitely many elements of $\mathcal{G}_w$ so that $g^{-1} \in \mathcal{G}_w$. □

Note that Theorem 6 does not presume the absolute convergence of $\widetilde{g}(s)$ for $\operatorname{Re} s \ge 0$.

6 Arithmetic applications

A function $g \in \mathcal{A}$ is said to possess a *mean-value* $M(g)$, if the limit

$$M(g) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} g(n)$$

exists. Influenced by the Erdős-Wintner problem, mean-value theorems for multiplicative functions became important in the theory of arithmetic functions. The progress achieved since 1961 is visible in the results of, e.g., Delange [1], Wirsing [20], [21], Halász [5], Elliott [3], and Indlekofer [8]. Elementary and analytic proof techniques often involve the replacement of a multiplicative function f by a somewhat simpler function, say g, and the back transfer of properties from g to f.

In 1961 Delange [1] stated and used an assertion concerning the transfer of mean-values between related multiplicative functions.

Proposition 1. *For $f, g \in \mathcal{M}$ bounded by 1 and satisfying*

$$\sum_p \frac{|f(p) - g(p)|}{p} < \infty, \tag{22}$$

the existence of $M(g)$ yields that of $M(f)$, if $g(2^k) \ne -1$ for some $k \in \mathbb{N}$. Moreover, the Dirichlet series $\widetilde{h}(s)$ of $h = f * g^{-1}$ converges absolutely at $s = 1$ and $M(f) = \widetilde{h}(1)\, M(g)$.

Note that the boundedness of g by 1 combined with $g(2^k) \ne -1$ for some $k \in \mathbb{N}$ implies $\widetilde{g}_p(s) \ne 0$ for all p and $\operatorname{Re} s \ge 1$. The first proof of Proposition 1 was given by Schwarz [16], via Wiener's inversion theorem for power series. After some intermediate improvements concerning possible extensions of the class of multiplicative functions (see [17], [9]), Heppner and Schwarz [6] proved the following relationship theorem.

Proposition 2. *Let*

$$\mathcal{H} = \Big\{ g \in \mathcal{M} : \sum_p \frac{|g(p)|^2}{p^2} < \infty\,, \sum_{p,k\ge 2} \frac{|g(p^k)|}{p^k} < \infty \Big\}.$$

Then, for $f, g \in \mathcal{H}$ satisfying (22), the existence of $M(g)$ implies that of $M(f)$, if $\widetilde{g}_p(s) \neq 0$ for all p and $\operatorname{Re} s \geq 1$.

Note that $\mathcal{H}$ is closed under convolution.

Proposition 2 raises the problem to find a quantitative version. In fact, the solution based on Theorem 3 immediately follows from Theorem 6. For abbreviation we set

$$M(g,x) = \sum_{n\le x} g(n)$$

and state the result in a slightly modified version compared to Propositions 1 and 2 (cf. Lucht [10]):

Theorem 7. *Let $w \in \mathcal{W}$ be defined by $w(n) = (1+\log n)^k$ for $k \in \mathbb{N}_0$ fixed. Suppose that $f \in \mathcal{G}_w$ and $g \in \mathcal{G}_w^*$ satisfy*

$$\sum_p |f(p) - g(p)|\, w(p) < \infty. \tag{23}$$

If there are constants $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$ with $\operatorname{Re}\alpha \geq \beta \geq 0$, $\ell \in \mathbb{N}_0$, and a polynomial $P(x) \in \mathbb{C}[x]$ of degree $\leq k$ such that

$$M(g,x) = x^\alpha P(\log x) + \mathrm{o}\big(x^\beta \log^\ell x\big) \qquad (x\to\infty), \tag{24}$$

then there exists a polynomial $Q(x) \in \mathbb{C}[x]$ of degree $\leq k$ such that

$$M(f,x) = x^\alpha Q(\log x) + \mathrm{o}\big(x^\beta \log^\ell x\big) \qquad (x\to\infty). \tag{25}$$

Moreover, $h = f * g^{-1} \in \mathcal{F}_w \cap \mathcal{M}$, the Dirichlet series $\widetilde{h}(s) = \widetilde{f}(s)/\widetilde{g}(s)$ and its derivatives up to the order k converge absolutely for $\operatorname{Re} s \geq 0$, and

$$Q(t) = \sum_{0\le j\le k} \frac{\widetilde{h}^{\,(j)}(\alpha)}{j!} P^{(j)}(t)\,. \tag{26}$$

Proof. By Theorem 6, $h = f * g^{-1} \in \mathcal{G}_w$. From $h(p) = f(p) - g(p)$ combined with (22) it follows that $h \in \mathcal{F}_w$. By inserting $f = g * h$ into $M(f,x)$ and using (23), we obtain the assertions (24) and (25) by elementary evaluation. □

We may rewrite Theorem 7 with f and g replaced with the quotient functions f/I and g/I, respectively. This is equivalent to a shift by 1 of the argument s in the corresponding Dirichlet series. Then Proposition 2 occurs as the special case $w = 1$ and $\alpha = \beta = \ell = 0$ of Theorem 7.

The next application concerns the transfer of the convergence quality of Dirichlet series between related multiplicative functions (cf. Lucht [10]).

Theorem 8. *Let $w \in \mathcal{W}$ be defined by $w(n) = (1+\log n)^k$ for $k \in \mathbb{N}_0$ fixed. Suppose that $f \in \mathcal{G}_w$ and $g \in \mathcal{G}_w^*$ are w-related in the sense of* (23). *If the Dirichlet series $\widetilde{g}(s)$ and its derivatives up to the order k converge at some point s with $\mathrm{Re}\, s \geq 0$, then $\widetilde{f}^{(j)}(s)$ does so for $0 \leq j \leq k$. Moreover, $h = f * g^{-1} \in \mathcal{F}_w \cap \mathcal{M}$, the Dirichlet series $\widetilde{h}(s) = \widetilde{f}(s)/\widetilde{g}(s)$ and its derivatives up to the order k converge absolutely at s, and*

$$\widetilde{f}^{(k)}(s) = \big(\widetilde{g} \cdot \widetilde{h}\big)^{(k)}(s) = \sum_{j=0}^{k} \binom{k}{j} \widetilde{g}^{(j)}(s)\, \widetilde{h}^{(k-j)}(s).$$

Proof. For every $g \in \mathcal{A}$ the absolute convergence of the series $(gw)^{\sim}(s)$ is equivalent to that of $(g \log^k)^{\sim}(s)$. Hence the assertion follows from Theorem 6. □

Note that Theorem 8 does *not* presume the *absolute* convergence of the series $\widetilde{g}(s)$. We only use the convergence of $(g * h)^{\sim}(s)$ to $\widetilde{g}(s) \cdot \widetilde{h}(s)$ for convergent series $\widetilde{g}(s)$ and absolutely convergent series $\widetilde{h}(s)$.

Finally, we mention an application to Ramanujan expansions of arithmetic functions $g \in \mathcal{B}$. In 1919, for $a, n \in \mathbb{N}$, Ramanujan [14] introduced the sum $c_n(a)$ called *Ramanujan sum* as sum of the ath powers of the nth primitive roots of unity. He used these sums to represent a variety of arithmetic functions g as pointwise convergent series of the form

$$g(a) = \sum_{n=1}^{\infty} \widehat{g}(n)\, c_n(a) \qquad (a \in \mathbb{N}) \tag{27}$$

with certain coefficients $\widehat{g}(n)$. Ramanujan's paper initiated the development of the Fourier analysis of arithmetic functions, which essentially covers arithmetic functions that possess a non-zero mean-value (see, e.g., Schwarz and Spilker [18]). Therefore some of Ramanujan's examples remained mysterious (cf. Knopfmacher [13]), e.g., the expansion (27) of the divisor function $d = 1 * 1$ with $\widehat{d}(n) = -\frac{\log n}{n}$.

A natural explanation of such expansions relies on the close relation of the Ramanujan sums c_n and the Möbius function μ. Namely, observe that the convolution

$$\eta_a(n) = \sum_{d|n} c_d(a) = \begin{cases} n & \text{if } n \mid a \\ 0 & \text{otherwise} \end{cases} \qquad (n \in \mathbb{N})$$

defines a function $\eta_a \in \mathcal{M}$ with finite support $\{n \in \mathbb{N} : \eta_a(n) \neq 0\}$. Note that the definition of η_a is equivalent to $c_{\cdot}(a) = \mu * \eta_a$. This offers an alternative approach (cf. Lucht [11]) to Ramanujan expansions for multiplicative functions via Theorem 6.

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