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On a set of asymptotic densities

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Abstract. Let $\mathbb{P} = \{p_1, p_2, \ldots, p_i, \ldots\}$ be the set of prime numbers (or more generally a set of pairwise co-prime elements). Let us denote $A_p^{a,b} = \{p^{an+b}m \mid n \in \mathbb{N} \cup \{0\}; m \in \mathbb{N}, p \,\text{does not divide}\, m\}$, where $a \in \mathbb{N}, b \in \mathbb{N} \cup \{0\}$.

Then for arbitrary finite set B, $B \subset \mathbb{P}$ holds

$$d\left(\bigcap_{p_i \in B} A_{p_i}^{a_i,b_i}\right) = \prod_{p_i \in B} d\left(A_{p_i}^{a_i,b_i}\right),$$

and

$$d\left(A_{p_i}^{a_i,b_i}\right) = \frac{\frac{1}{p_i^{b_i}}\left(1 - \frac{1}{p_i}\right)}{1 - \frac{1}{p_i^{a_i}}}.$$

If we denote

$$A = \left\{\frac{\frac{1}{p^b}\left(1 - \frac{1}{p}\right)}{1 - \frac{1}{p^a}} \mid p \in \mathbb{P}, a \in \mathbb{N}, b \in \mathbb{N} \cup \{0\}\right\},$$

where $\mathbb{P}$ is the set of all prime numbers, then for closure of set A holds

$$\operatorname{cl} A = A \cup B \cup \{0, 1\},$$

where $B = \left\{\frac{1}{p^b}\left(1 - \frac{1}{p}\right) \mid p \in \mathbb{P}, b \in \mathbb{N} \cup \{0\}\right\}$.

1 Introduction

Theorems 1, 2 and 3 introduced in this paper are generalizations of some results from [1] and [2] concerned in sets of natural numbers in form $p^{an+b}m$. In this paper asymptotic densities of sets of natural numbers in form $p^{a_n}m$, where $p, m \in \mathbb{N}$,

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$p > 1$, p does not divide m, and $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of non-negative integers are studied.

The denotation

$$A_p^{a_n} = \{p^{a_n} m \mid m, n \in \mathbb{N}, p \,\text{does not divide}\, m\}$$

is used.

The above mentioned sets $A_p^{a_n}$ are intresting because of one of their properties: If we take two co-prime numbers p, and q, then for the asymptotic density of intersection of sets $A_p^{a_n}$, and $A_q^{b_n}$ holds

$$d(A_p^{a_n} \cap A_q^{b_n}) = d(A_p^{a_n}) d(A_q^{b_n}).$$

Moreover, if we take arbitrary finite number of pairwise co-prime numbers $p_1, p_2 \ldots, p_k$, and arbitrary increasing sequences of non-negative integers $\{a_1(n)\}_{n=1}^{\infty}$, $\{a_2(n)\}_{n=1}^{\infty}, \ldots, \{a_k(n)\}_{n=1}^{\infty}$, then for the asymptotic density of intersection of sets $A_{p_j}^{a_j(n)}$, $j = 1, 2, \ldots, k$ holds

$$d(\bigcap_{j=1}^{k} A_{p_j}^{a_j(n)}) = \prod_{j=1}^{k} d(A_{p_j}^{a_j(n)}).$$

Theorem 4 describes the closure of set of asymptotic densities of sets A_p^{an+b}, where p is prime number, $a \in \mathbb{N}$, and $b \in \mathbb{N} \cup \{0\}$.

2 Asymptotic densities of sets $A_p^{a_n}$

At first the asymptotic densities of sets $A_p^{a_n}$ are determined.

Theorem 1. *Let $p \in \mathbb{N}$, $p > 1$, and let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of non-negative integers. If we denote*

$$A_p^{a_n} = \{p^{a_n} m \mid m, n \in \mathbb{N}, p \,\text{does not divide}\, m\}$$

and $r_p^{a_n} = \sum_{j=1}^{\infty} \frac{1}{p^{a_j}}$, then

$$d(A_p^{a_n}) = \left(1 - \frac{1}{p}\right) r_p^{a_n}.$$

Proof. Let us denote $C_j = \{p^{a_j} m \mid m \in \mathbb{N}\}$ for every $j \in \mathbb{N}$. We can see that the set C_j contains natural numbers in form $p^s m$, where $s \geq a_j$.

Similarly, let us denote $D_j = \{p^{a_j+1} m \mid m \in \mathbb{N}\}$ for every $j \in \mathbb{N}$. We can see that the set D_j contains natural numbers in form $p^s m$, where $s \geq a_j + 1$.

We denote the difference of set C_j, and D_j by Q_j. It holds that

$$Q_j = C_j \setminus D_j = \{p^{a_j} m \mid m \in \mathbb{N}, p \,\text{does not divide}\, m\}. \quad (1)$$

From equation (1) follows

$$A_p^{a_n} = \bigcup_{j \in \mathbb{N}} Q_j.$$

Hence, for every $k \in \mathbb{N}$ holds

$$\bigcup_{j=1}^{k} Q_j \subseteq A_p^{a_n} \subseteq C_{k+1} \cup \bigcup_{j=1}^{k} Q_j. \tag{2}$$

We determine asymptotic densities of sets C_j, D_j, and Q_j. Element $p^{a_j}m \in C_j$ fulfills condition $p^{a_j}m \le n$ if and only if $m \le \frac{n}{p^{a_j}}$.

Hence, m is the number of elements of set C_j which are less or equal to n. Thus, from above mentioned follows that[1]

$$C_j(n) = \left[\frac{n}{p^{a_j}}\right].$$

So we obtain the asymptotic density of set C_j

$$d(C_j) = \lim_{n\to\infty} \frac{C_j(n)}{n} = \frac{1}{p^{a_j}}. \tag{3}$$

Similarly,

$$d(D_j) = \frac{1}{p^{a_j+1}}. \tag{4}$$

Since $D_j \subset C_j$, and $Q_j = C_j \setminus D_j$, from equations (3), and (4) we obtain

$$d(Q_j) = d(C_j) - d(D_j) = \frac{1}{p^{a_j}} - \frac{1}{p^{a_j+1}} = \left(1 - \frac{1}{p}\right)\frac{1}{p^{a_j}}. \tag{5}$$

Sets Q_j are pairwise disjoint (one can easily prove that $Q_i \cap Q_j \neq \emptyset$ implies $i = j$). It means that for every $k \in \mathbb{N}$ holds

$$d\left(\bigcup_{j=1}^{k} Q_j\right) = \sum_{j=1}^{k} d(Q_j). \tag{6}$$

From (2) we obtain estimations of lower and upper asymptotic density of set $A_p^{a_n}$

$$d\left(\bigcup_{j=1}^{k} Q_j\right) \le \underline{d}(A_p^{a_n}) \le \overline{d}(A_p^{a_n}) \le d(C_{k+1}) + d\left(\bigcup_{j=1}^{k} Q_j\right).$$

From (6) we obtain

$$\sum_{j=1}^{k} d(Q_j) \le \underline{d}(A_p^{a_n}) \le \overline{d}(A_p^{a_n}) \le d(C_{k+1}) + \sum_{j=1}^{k} d(Q_j),$$

and from (3), and (5) follows

$$\left(1 - \frac{1}{p}\right)\sum_{j=1}^{k} \frac{1}{p^{a_j}} \le \underline{d}(A_p^{a_n}) \le \overline{d}(A_p^{a_n}) \le \frac{1}{p^{a_{k+1}}} + \left(1 - \frac{1}{p}\right)\sum_{j=1}^{k} \frac{1}{p^{a_j}}.$$

[1] We denote the integral part of real number x by $[x]$, and the number of elements of a set A by $A(n)$.

These inequalities hold for every $k \in \mathbb{N}$. With $k \to \infty$ we obtain

$$d(A_p^{a_n}) = \lim_{k\to\infty} \left(1 - \frac{1}{p}\right) \sum_{j=1}^{k} \frac{1}{p^{a_j}} = \left(1 - \frac{1}{p}\right) \sum_{j=1}^{\infty} \frac{1}{p^{a_j}} = \left(1 - \frac{1}{p}\right) r_p^{a_n}.$$

We should note that the sum $r_p^{a_n} = \sum_{j=1}^{\infty} \frac{1}{p^{a_j}}$ is convergent. The sequence $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of non-negative integers, hence $a_j \geq j-1$ holds for every $j \in \mathbb{N}$. It means that

$$\sum_{j=1}^{\infty} \frac{1}{p^{a_j}} \leq \sum_{j=1}^{\infty} \frac{1}{p^{j-1}} = \frac{p}{p-1}. \qquad \square$$

Theorem 2. *If $p, q \in \mathbb{N} \setminus \{1\}$, $\gcd(p,q) = 1$, $\{a_n\}_{n=1}^{\infty}$, and $\{b_n\}_{n=1}^{\infty}$ are increasing sequences of non-negative integers, $r_p^{a_n} = \sum_{j=1}^{\infty} \frac{1}{p^{a_j}}$, and $r_q^{b_n} = \sum_{j=1}^{\infty} \frac{1}{q^{b_j}}$, then*

$$d(A_p^{a_n} \cap A_q^{b_n}) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) r_p^{a_n} r_q^{b_n} = d(A_p^{a_n}) d(A_q^{b_n}).$$

Proof. Set

$$A_p^{a_n} = \{p^{a_n} m \mid m, n \in \mathbb{N}, p \,\text{does not divide}\, m\}$$

and

$$A_q^{b_n} = \{q^{b_n} m \mid m, n \in \mathbb{N}, q \,\text{does not divide}\, m\}.$$

Since $\gcd(p,q) = 1$,

$$A_p^{a_n} \cap A_q^{b_n} = \{p^{a_j} q^{b_i} m \mid i, j, m \in \mathbb{N}; p, q \,\text{does not divide}\, m\}. \tag{7}$$

Let us denote $C_j = \{p^{a_j} m \mid m \in A_q^{b_n}\}$ for every $j \in \mathbb{N}$. We can see that the set C_j contains natural numbers in form $p^s q^{b_i} m$, where $s \geq a_j$, $m, i \in \mathbb{N}$, and q does not divide m.

Similarly, let us denote $D_j = \{p^{a_j+1} m \mid m \in A_q^{b_n}\}$ for every $j \in \mathbb{N}$. We can see that the set D_j contains natural numbers in form $p^s q^{b_i} m$, where $s \geq a_j + 1$, $m, i \in \mathbb{N}$, and q does not divide m.

We denote the difference of set C_j, and D_j by Q_j. It holds that

$$Q_j = C_j \setminus D_j = \{p^{a_j} m \mid m \in A_q^{b_n}; p, q \,\text{does not divide}\, m\}. \tag{8}$$

We can see that the set Q_j contains natural numbers in form $p^{a_j} q^{b_i} m$, where j is fixed, $m, i \in \mathbb{N}$, and neither p nor q does not divide m.

From equations (7), and (8) follows

$$A_p^{a_n} \cap A_q^{b_n} = \bigcup_{j\in\mathbb{N}} Q_j. \tag{9}$$

Hence, for every $k \in \mathbb{N}$ holds

$$\bigcup_{j=1}^{k} Q_j \subseteq A_p^{a_n} \cap A_q^{b_n} \subseteq C_{k+1} \cup \left(\bigcup_{j=1}^{k} Q_j\right). \tag{10}$$

Now, we determine asymptotic densities of sets C_j, D_j, and Q_j. Element $p^{a_j} m \in C_j$ ($m \in A_q^{b_n}$!) fulfills condition $p^{a_j} m \le n$ if and only if $m \le \frac{n}{p^{a_j}}$.

Hence, the number of elements of the set C_j which are less or equal to n is equal to the number of elements $m \in A_q^{b_n}$ which are less or equal to $\frac{n}{p^{a_j}}$. It means that

$$C_j(n) = A_q^{b_n}\left(\left[\frac{n}{p^{a_j}}\right]\right).$$

So we obtain the asymptotic density of the set C_j

$$d(C_j) = \lim_{n\to\infty} \frac{C_j(n)}{n} = \lim_{n\to\infty} \frac{A_q^{b_n}\left(\left[\frac{n}{p^{a_j}}\right]\right)}{n} = \frac{d(A_q^{b_n})}{p^{a_j}}. \tag{11}$$

Similarly,

$$d(D_j) = \frac{d(A_q^{b_n})}{p^{a_j+1}}. \tag{12}$$

Since $D_j \subset C_j$, and $Q_j = C_j \setminus D_j$, from equations (11), and (12) we obtain

$$d(Q_j) = d(C_j) - d(D_j) = \frac{d(A_q^{b_n})}{p^{a_j}} - \frac{d(A_q^{b_n})}{p^{a_j+1}} = d(A_q^{b_n})\left(1 - \frac{1}{p}\right)\frac{1}{p^{a_j}}. \tag{13}$$

Sets Q_j are pairwise disjoint (one can easily prove that $Q_i \cap Q_j \ne \emptyset$ implies $i = j$). It means that for every $k \in \mathbb{N}$ holds

$$d\left(\bigcup_{j=1}^{k} Q_j\right) = \sum_{j=1}^{k} d(Q_j). \tag{14}$$

From (10) we obtain estimations of lower and upper asymptotic density of set $A_p^{a_n} \cap A_q^{b_n}$

$$d\left(\bigcup_{j=1}^{k} Q_j\right) \le \underline{d}(A_p^{a_n} \cap A_q^{b_n}) \le \overline{d}(A_p^{a_n} \cap A_q^{b_n}) \le d(C_{k+1}) + d\left(\bigcup_{j=1}^{k} Q_j\right).$$

From (14) follows

$$\sum_{j=1}^{k} d(Q_j) \le \underline{d}(A_p^{a_n} \cap A_q^{b_n}) \le \overline{d}(A_p^{a_n} \cap A_q^{b_n}) \le d(C_{k+1}) + \sum_{j=1}^{k} d(Q_j),$$

and from (11), and (13) we obtain

$$\left(1 - \frac{1}{p}\right)\sum_{j=1}^{k} \frac{d(A_q^{b_n})}{p^{a_j}} \le \underline{d}(A_p^{a_n} \cap A_q^{b_n}) \le \overline{d}(A_p^{a_n} \cap A_q^{b_n})$$

$$\le \frac{d(A_q^{b_n})}{p^{a_{k+1}}} + \left(1 - \frac{1}{p}\right)\sum_{j=1}^{k} \frac{d(A_q^{b_n})}{p^{a_j}}.$$

These unequalities hold for every $k \in \mathbb{N}$. With $k \to \infty$ we obtain

$$d(A_p^{a_n} \cap A_q^{b_n}) = \lim_{k\to\infty} \left(1 - \frac{1}{p}\right) \sum_{j=1}^{k} \frac{d(A_q^{b_n})}{p^{a_j}} = d(A_q^{b_n}) \left(1 - \frac{1}{p}\right) \sum_{j=1}^{\infty} \frac{1}{p^{a_j}}.$$

And according to Theorem 1 holds

$$d(A_p^{a_n} \cap A_q^{b_n}) = d(A_p^{a_n}) d(A_q^{b_n}). \qquad \square$$

Theorem 3. *Let $P = \{p_1, p_2, \ldots, p_r\}$ be a set of pairwise co-prime natural numbers*[2]*, where $1 \notin P$, and $\{a_1(n)\}_{n=1}^{\infty}, \{a_2(n)\}_{n=1}^{\infty}, \ldots, \{a_r(n)\}_{n=1}^{\infty}$ are increasing sequences of non-negative integers. Then*

$$d\left(\bigcap_{i=1}^{r} A_{p_i}^{a_i(n)}\right) = \prod_{i=1}^{r} d(A_{p_i}^{a_i(n)}) = \prod_{i=1}^{r} \left(\left(1 - \frac{1}{p_i}\right) \sum_{j=1}^{\infty} \frac{1}{p_i^{a_i(j)}}\right).$$

Proof. We can perform the proof of Theorem 3 by induction according to r. The case of $r = 1$ (and $r = 2$) was proved in Theorem 1 (and in Theorem 2). Therefore, we can consider (induction hypothesis) that

$$d\left(\bigcap_{i=1}^{r-1} A_{p_i}^{a_i(n)}\right) = \prod_{i=1}^{r-1} d(A_{p_i}^{a_i(n)}) = \prod_{i=1}^{r-1} \left(\left(1 - \frac{1}{p_i}\right) \sum_{j=1}^{\infty} \frac{1}{p_i^{a_i(j)}}\right). \tag{15}$$

Since $p_1, p_2, \ldots, p_r$ are pairwise co-prime numbers

$$\bigcap_{i=1}^{r} A_{p_i}^{a_i(n)} = \{p_r^{a_r(j_r)} p_{r-1}^{a_{r-1}(j_{r-1})} \ldots p_1^{a_1(j_1)}.m \mid m \in \mathbb{N}, \quad j_i \in \mathbb{N}, p_i \text{ does not divide } m, i = 1, 2, \ldots, r\}, \tag{16}$$

and

$$\bigcap_{i=1}^{r-1} A_{p_i}^{a_i(n)} = \{p_{r-1}^{a_{r-1}(j_{r-1})} \ldots p_1^{a_1(j_1)}.m \mid m \in \mathbb{N}, \quad j_i \in \mathbb{N}, p_i \text{ does not divide } m, i = 1, 2, \ldots, r-1\}. \tag{17}$$

For simplicity, let us denote

$$A = \bigcap_{i=1}^{r} A_{p_i}^{a_i(n)}, \quad \text{and} \quad A^* = \bigcap_{i=1}^{r-1} A_{p_i}^{a_i(n)}.$$

Further let us denote

$$C_j = \{p_r^{a_r(j)}.m \mid m \in A^*\},$$
$$D_j = \{p_r^{a_r(j)+1}.m \mid m \in A^*\},$$
$$Q_j = C_j \setminus D_j.$$

[2]For each $i, j \in \mathbb{N}$, $i \neq j$ holds $\gcd(p_i, p_j) = 1$.

The same way as in previous proofs we can prove following equations

$$d(C_j) = \frac{d(A^*)}{p_r^{a_r(j)}},$$
$$d(D_j) = \frac{d(A^*)}{p_k^{a_r(j)+1}},$$
$$d(Q_j) = d(C_j) - d(D_j) = \frac{d(A^*)}{p_r^{a_r(j)}}\left(1 - \frac{1}{p_r}\right),$$
$$d\left(\bigcup_{j=1}^{k} Q_j\right) = \sum_{j=1}^{k} d(Q_j).$$

Furthermore, we can prove this relations, following from (16), (17), and holding for every $k \in \mathbb{N}$

$$\bigcup_{j=1}^{k} Q_j \subseteq A = \bigcap_{i=1}^{r} A_{p_i}^{a_i(n)} \subseteq C_{k+1} \cup \bigcup_{j=1}^{k} Q_j,$$

and estimations

$$\sum_{j=1}^{k} d(Q_j) \le \underline{d}(A) \le \overline{d}(A) \le d(C_{k+1}) + \sum_{j=1}^{k} d(Q_j),$$
$$\left(1 - \frac{1}{p_r}\right)\sum_{j=1}^{k}\frac{d(A^*)}{p_r^{a_r(j)}} \le \underline{d}(A) \le \overline{d}(A) \le \frac{d(A^*)}{p_r^{a_r(k+1)}} + \left(1 - \frac{1}{p_r}\right)\sum_{j=1}^{k}\frac{d(A^*)}{p_r^{a_r(j)}}.$$

With $k \to \infty$ we obtain

$$\begin{aligned} d(A) &= d(A^*)\left(1 - \frac{1}{p_r}\right)\sum_{j=1}^{\infty}\frac{1}{p_r^{a_r(j)}} = \\ &= d(A^*)d(A_{p_r}^{a_r(n)}) = \\ &= d\left(\bigcap_{i=1}^{r-1} A_{p_i}^{a_i(n)}\right) d(A_{p_r}^{a_r(n)}). \end{aligned}$$

Finally, according to (15), and Theorem 1

$$\begin{aligned} d(A) &= d(A_{p_r}^{a_r(n)})\prod_{i=1}^{r-1} d(A_{p_i}^{a_i(n)}) = \\ &= \prod_{i=1}^{r} d(A_{p_i}^{a_i(n)}) = \\ &= \prod_{i=1}^{r}\left(\left(1 - \frac{1}{p_i}\right)\sum_{j=1}^{\infty}\frac{1}{p_i^{a_i(j)}}\right). \end{aligned}$$

□

As a special case we can consider sets $A_p^{a_n}$, where $a_n = a(n-1)+b$ is an increasing arithmetical sequence of non-negative integers, p is a prime number, and $a \in \mathbb{N}, b \in \mathbb{N} \cup \{0\}$. For simplicity, we denote them by $A_p^{a,b}$, i.e.

$$A_p^{a,b} = \{p^{a(n-1)+b}m \mid m, n \in \mathbb{N}, p \,\text{does not divide}\, m\},$$

Asymptotic density of $A_p^{a,b}$ is equal (according to Theorem 1) to

$$d(A_p^{a,b}) = \left(1-\frac{1}{p}\right)\sum_{j=1}^{\infty}\frac{1}{p^{a(j-1)+b}} = \frac{\frac{1}{p^b}\left(1-\frac{1}{p}\right)}{1-\frac{1}{p^a}}.$$

Theorem 4. *Let $p_1 < p_2 < \cdots < p_i < \ldots$ be the sequence of all prime numbers,*

$$A = \{d(A_{p_i}^{a,b}) \mid i, a \in \mathbb{N}, b \in \mathbb{N} \cup \{0\}\}$$

and

$$B = \left\{\frac{1}{p_i^b}\left(1-\frac{1}{p_i}\right) \mid i \in \mathbb{N}, b \in \mathbb{N} \cup \{0\}\right\}.$$

Then for the closure of set A holds

$$\operatorname{cl} A = A \cup B \cup \{0, 1\}.$$

Proof. The strategy of this proof is following: It is obvious that $\operatorname{cl} A \subseteq \langle 0, 1\rangle$, and $A \subseteq \operatorname{cl} A$. We choose arbitrary $x_0 \in (0,1), x_0 \notin A, x_0 \notin B$ and we prove that $x_0 \notin \operatorname{cl} A$. Then we prove that $B \subset \operatorname{cl} A$, and $0 \in \operatorname{cl} A, 1 \in \operatorname{cl} A$.

First of all, we are going to prove that there is just a finite number of elements of the set B in an arbitrary interval $(\alpha, \beta) \subseteq (0,1)$, $0 < \alpha < \beta < 1$.

Let us denote

$$k_{b,i} = \frac{1}{p_i^b}\left(1-\frac{1}{p_i}\right). \tag{18}$$

Hence, $B = \{k_{b,i} \mid i \in \mathbb{N}, b \in \mathbb{N} \cup \{0\}\}$. It is obvious that for $b \geq 1$ holds $\lim_{i\to\infty} k_{b,i} = 0$, and $\lim_{i\to\infty} k_{0,i} = 1$. Therefore, for fixed b just a finite number of elements $k_{b,i}$ belongs to the interval (α, β).

Moreover, for arbitrary $\alpha > 0$ exists $b_0 \in \mathbb{N}$ such that for every $b > b_0$ and for every $i \in \mathbb{N}$ holds

$$k_{b,i} = \frac{1}{p_i^b}\left(1-\frac{1}{p_i}\right) < \frac{1}{p_i^b} < \frac{1}{2^b} < \alpha.$$

Hence, only elements $k_{b,i} \in B$ where $b \leq b_0$ belong to interval (α, β). Thus, there is just finite number of elements of the set B in the given interval (α, β).

Let us consider arbitrary $x_0 \in (0,1), x_0 \notin A, x_0 \notin B$. There must exist some interval (α, β), where $0 < \alpha < \beta < 1$, $x_0 \in (\alpha, \beta)$. We know that there exist just a finite number of elements of B in the interval (α, β).

Hence, ($x_0 \notin B$ according to above mentioned assumptions)

$$\exists c_1, c_2 \in B : x_0 \in (c_1, c_2), (c_1, c_2) \cap B = \emptyset. \tag{19}$$

For arbitrary $d \in A$ exists (see (18)) $k_{b,i} \in B$:

$$d = d(A^{a,b}_{p_i}) = \frac{\frac{1}{p_i^b}\left(1 - \frac{1}{p_i}\right)}{1 - \frac{1}{p_i^a}} = \frac{k_{b,i}}{1 - \frac{1}{p_i^a}}. \tag{20}$$

We are looking for all elements $d \in A$, which belong to the interval (c_1, c_2). Doing so, we solve inequalities

$$c_1 < d < c_2.$$

From (20), and from the fact that $p_i \geq 2$ we obtain

$$c_1 < \frac{k_{b,i}}{1 - \frac{1}{p_i^a}} < c_2,$$

$$\begin{aligned} c_1(1 - \frac{1}{p_i^a}) <& \quad k_{b,i} \quad < c_2(1 - \frac{1}{p_i^a}), \\ c_1(1 - \frac{1}{2^a}) <& \quad k_{b,i} \quad < c_2, \\ \frac{c_1}{2} <& \quad k_{b,i} \quad < c_2, \end{aligned}$$

and from (19)

$$d = d(A^{a,b}_{p_i}) \in (c_1, c_2) \Rightarrow k_{b,i} \in (\frac{c_1}{2}, c_1\rangle. \tag{21}$$

As proved above, there is just a finite number of elements $k_{b,i} \in B$ which satisfy the condition $k_{b,i} \in (\frac{c_1}{2}, c_1\rangle$. Let us denote them (recall that $c_1 \in B$)

$$k_{b_1,i_1} < k_{b_2,i_2} < \cdots < k_{b_r,i_r} = c_1.$$

Hence, (see (20) and (21)),

$$d = d(A^{a,b}_{p_i}) \in (c_1, c_2) \text{ only if } b \in \{b_1, b_2, \ldots, b_r\}, \text{ and } i \in \{i_1, i_2, \ldots, i_r\}. \tag{22}$$

Let us determine for which a the elements $d = d(A^{a,b_j}_{p_{i_j}})$, $j = 1, 2, \ldots, r$ belong to the interval (c_1, c_2)?

We can see that

$$\lim_{a\to\infty} d(A^{a,b_j}_{p_{i_j}}) = \lim_{a\to\infty} \frac{k_{b_j,i_j}}{1 - \frac{1}{p^a_{i_j}}} = k_{b_j,i_j} \text{ for } j = 1, 2, \ldots, r.$$

Thus, for every $j = 1, 2, \ldots, r$ holds:
$\forall \varepsilon > 0 \, \exists a_0(\varepsilon) \in \mathbb{N}:$

$$\forall a > a_0(\varepsilon) : d(A^{a,b_j}_{p_{i_j}}) < k_{b_j,i_j} + \varepsilon \leq k_{b_r,i_r} + \varepsilon = c_1 + \varepsilon. \tag{23}$$

We can choose ε small enough to $x_0 \in (c_1 + \varepsilon, c_2)$ (see (19)). From (22) and (23) follows that the element $d = d(A^{a,b}_{p_i}) \in A$ belongs to the interval $(c_1 + \varepsilon, c_2)$

only if $b \in \{b_1, b_2, \dots, b_r\}$, $i \in \{i_1, i_2, \dots, i_r\}$, and $a \in \{1, 2, \dots, a_0(\varepsilon)\}$. Thus, for every $x_0 \in (0,1), x_0 \notin A, x_0 \notin B$ holds $x_0 \notin \operatorname{cl} A$.

Moreover,

$$\lim_{a\to\infty} d(A_{p_i}^{a,b}) = \lim_{a\to\infty} \frac{\frac{1}{p_i^b}\left(1-\frac{1}{p_i}\right)}{1-\frac{1}{p_i^a}} = \frac{1}{p_i^b}\left(1-\frac{1}{p_i}\right) \in B,$$

$$\lim_{i\to\infty} d(A_{p_i}^{a,0}) = \lim_{i\to\infty} \frac{\frac{1}{p_i^0}\left(1-\frac{1}{p_i}\right)}{1-\frac{1}{p_i^a}} = 1,$$

and

$$\lim_{b\to\infty} d(A_{p_i}^{a,b}) = \lim_{b\to\infty} \frac{\frac{1}{p_i^b}\left(1-\frac{1}{p_i}\right)}{1-\frac{1}{p_i^a}} = 0.$$

Hence, $B \subset \operatorname{cl} A$, $1 \in \operatorname{cl} A$, and $0 \in \operatorname{cl} A$. Thus, $\operatorname{cl} A = A \cup B \cup \{0,1\}$. □

References

[1] Jahoda P., Pěluchová M.: Systems of sets with multiplicative asymptotic density, Math. Slovaca 58(2008), No. 4, 393-404.

[2] Jahoda P.: Notes on the expression of natural numbers as sum of powers, Tatra Mt. Math. Publ. 34 (2005), 1-11.

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